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Research Article

On Complex Singularity Analysis for Some Linear Partial Differential Equations in $\mathbb{C}^3$

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We investigate the existence of local holomorphic solutions Y of linear partial differential equations in three complex variables whose coefficients are holomorphic on some polydisc in $\mathbb{C}^2$ outside some singular set Θ. The coefficients are written as linear combinations of powers of a solution X of some first-order nonlinear partial differential equation following an idea, we have initiated in a previous work (Malek and Stenger 2011). The solutions Y are shown to develop singularities along Θ with estimates of exponential type depending on the growth's rate of X near the singular set. We construct these solutions with the help of series of functions with infinitely many variables which involve derivatives of all orders of X in one variable. Convergence and bounds estimates of these series are studied using a majorant series method which leads to an auxiliary functional equation that contains differential operators in infinitely many variables. Using a fixed point argument, we show that these functional equations actually have solutions in some Banach spaces of formal power series.

1. Introduction

In this paper, we study a family of linear partial differential equations of the form

$$\begin{aligned}\partial_w^S Y(t,z,w) = \sum_{k\in\mathcal{S}} \Big(& a_{1,k}(t,z,w)\,\partial_t\partial_w^k Y(t,z,w) \\ & + a_{2,k}(t,z,w)\,\partial_z\partial_w^k Y(t,z,w) \\ & + a_{3,k}(t,z,w)\,\partial_w^k Y(t,z,w)\Big)\end{aligned} \quad (1)$$

for given initial data $\partial_w^j Y(t,z,0) = \varphi_j(t,z)$, $0 \le j \le S-1$, where $\mathcal{S}$ is a subset of $\mathbb{N}^2$ and S is an integer which satisfies the constraints (175). The coefficients $a_{m,k}(t,z,w)$ are holomorphic functions on some domain $(D(0,r)^2 \setminus \Theta) \times D(0,\overline{w})$ where Θ is some singular set of $D(0,r)^2$ (where $D(0,\delta)$ denotes the disc centered at 0 in $\mathbb{C}$ with radius $\delta > 0$) and the initial data $\varphi_j(t,z)$ are assumed to be holomorphic functions on the polydisc $D(0,r)^2$.

In order to avoid cumbersome statements and tedious computations, the authors have chosen to restrict their study to (1) that involves at most first-order derivatives with respect to t and z but the method proposed in this work can also be extended to higher order derivatives too.

In this work, we plan to construct holomorphic solutions of the problem (1) on $(D(0,r)^2 \setminus \Theta) \times D(0,\overline{w})$ and we will give precise growth rate for these solutions near the singular set Θ of the coefficients $a_{m,k}(t,z,w)$ (Theorem 21).

There exists a huge literature on the study of complex singularities and analytic continuation of solutions to linear partial differential equations starting from the fundamental contributions of Leray in [1]. Many important results are known for singular initial data and concern equations with bounded holomorphic coefficients. In that context, the singularities of the solution are generally contained in characteristic hypersurfaces issued from the singular locus of the initial conditions. For meromorphic initial data, we may refer to [2–5] and for more general ramified multivalued initial data, we may cite [6–9]. In our framework, the initial data are assumed to be nonsingular and the coefficients of the equation now carry the singularities. To the best knowledge of the authors, few results have been worked out in that case. For instance, the research of the so-called *Fuchsian singularities* in the context of partial differential equations is widely developed; we provide [10–13] as examples of references in this direction. It turns out that the situation we consider is actually close to a singular perturbation problem since the nature of the equation changes nearby the singular locus of it coefficients.

This work is a continuation of our previous study [14]. In [14], the authors focused on linear partial differential equations in $\mathbb{C}^2$. They have constructed local holomorphic solutions with a careful study of their asymptotic behaviour near the singular locus of the initial data. These initial data were chosen to be polynomial in t, z and a function $u(t)$ satisfying some nonlinear differential equation of first order on some punctured disc $D(t_0, r) \setminus \{t_0\} \subset \mathbb{C}$ and owning an isolated singularity at t_0 which is either a pole or an algebraic branch point according to a result of Painlevé. Inspired by the classical *tanh method* introduced in [15], they have considered formal series solutions of the form

$$u(t, z) = \sum_{l \geq 0} u_l(t, z)(u(t))^l, \tag{2}$$

where u_l are holomorphic functions on $D(t_0, r) \times D$ where $D \subset \mathbb{C}$ is a small disc centered at 0. They have given suitable conditions for these series to be well defined and holomorphic for t in a sector S with vertex t_0 and moreover as t tends to t_0 the solutions $u(t, z)$ are shown to carry at most exponential bounds estimates of the form $C\exp(M|t - t_0|^{-\mu})$ for some constants $C, M, \mu > 0$.

In this work, the coefficients $a_{m,k}(t, z, w)$ are constructed as polynomials in some function $X(t, z)$ with holomorphic coefficients in (t, z, w), where $X(t, z)$ is now assumed to solve some nonlinear partial differential equation of first order and is asked to be holomorphic on a domain $D(0, r)^2 \setminus \Theta$ and to be singular along the set Θ. The class of functions in which one can choose the coefficients $a_{m,k}(t, z, w)$ is quite large since it contains meromorphic and multivalued holomorphic functions in (t, z) (see the example of Section 2.1).

In our setting, one cannot achieve the goal only dealing with formal expansions involving the function $X(t, z)$ like (2) since the derivatives of $X(t, z)$ with respect to t or z cannot be expressed only in terms of $X(t, z)$. In order to get suitable recursion formulas, it turns out that we need to deal with series expansions that take into account all the derivatives of $X(t, z)$ with respect to z. For this reason, the construction of the solutions will follow the one introduced in a recent work of Tahara and will involve Banach spaces of holomorphic functions with infinitely many variables.

In [16], Tahara introduced a new equivalence problem connecting two given nonlinear partial differential equations of first order in the complex domain. He showed that the equivalence maps have to satisfy the so-called coupling equations which are nonlinear partial differential equations of first order but with infinitely many variables. It is worthwhile saying that within the framework of mathematical physics, spaces of functions of infinitely many variables play a fundamental role in the study of nonlinear integrable partial differential equations known as solitons equations as described in the theory of Sato. See [17] for an introduction.

The layout of the paper is a follows. In a first step described in Section 2.2, we construct formal series of the form

$$U(t, z, w) = \sum_{\alpha \geq 0} \phi_\alpha \left(t, z, \left(\frac{\partial_z^h X(t, z)}{h! \gamma^h}\right)_{0 \leq h \leq \alpha}\right) \frac{w^\alpha}{\alpha!}, \tag{3}$$

solutions of some auxiliary nonhomogeneous integrodifferential equation (17) with polynomial coefficients in $X(t, z)$. The coefficients ϕ_α, $\alpha \geq 0$, are holomorphic functions on some polydisc in $\mathbb{C}^{\alpha+3}$ that satisfy some differential recursion (Proposition 2).

In Section 2.3, we establish a sequence of inequalities for the modulus of the differentials of arbitrary order of the functions ϕ_α denoted by $\varphi_{\alpha,n_0,n_1,(l_h)_{0\leq h\leq\alpha}}$ for all nonnegative integers α, n_0, n_1, l_h with $0 \leq h \leq \alpha$ (Proposition 3). In the next section, we construct a sequence of coefficients $\psi_{\alpha,n_0,n_1,(l_h)_{0\leq h\leq\alpha}}$ which is larger than the latter sequence

$$\varphi_{\alpha,n_0,n_1,(l_h)_{0\leq h\leq\alpha}} \leq \psi_{\alpha,n_0,n_1,(l_h)_{0\leq h\leq\alpha}} \tag{4}$$

for any nonnegative integers α, n_0, n_1, l_h with $0 \leq h \leq \alpha$ and whose generating formal series satisfies some integrodifferential functional equation (51) that involves differential operators with infinitely many variables (Propositions 5 and 6). The idea of considering recursions over the complete family of derivatives and the use of majorant series which lead to auxiliary Cauchy problems were already applied in former papers by the authors of this work; see [14, 18–21].

In Section 3, we solve the functional equation (51) by applying a fixed point argument in some Banach space of formal series with infinitely many variables (Proposition 19). The definition of these Banach spaces (Definition 7) is inspired from formal series spaces introduced in our previous work [14]. The core of the proof is based on continuity properties of linear integrodifferential operators in infinitely many variables explained in Section 3.1 and constitutes the most technical part of the paper.

Finally, in Section 4, we prove the main result of our work. Namely, we construct analytic functions $Y(t, z, w)$, solutions of (1) for the prescribed initial data, defined on sets $K \times D(0, \overline{w})$ for any compact set $K \subset D(0, r)^2 \setminus \Theta$ with precise bounds of exponential type in terms of the maximum value of $|X(t, z)|$ over K (Theorem 21). The proof puts together all the constructions performed in the previous sections. More precisely, for some specific choice of the nonhomogeneous term in (17), a formal solution (3) of (17) gives rise to a formal solution $Y(t, z, w)$ of (1) with the given initial data that can be written as the sum of the integral $\partial_w^{-S} U(t, z, w)$ and a polynomial in w having the initial data φ_j as coefficients. Owing to the fact that the generating series of the sequence $\psi_{\alpha,n_0,n_1,(l_h)_{0\leq h\leq\alpha}}$, solution of (51), belongs to the Banach spaces mentioned above, we get estimates for the holomorphic functions ϕ_α with precise bounds of exponential type in terms of the radii of the polydiscs where they are defined; see (196). As a result, the formal solution $U(t, z, w)$ is actually convergent for w near the origin and for (t,z) belonging to any compact set of $D(0,r)\setminus\Theta$. Moreover, exponential bounds are achieved; see (197). The same properties then hold for $Y(t,z,w)$.

2. Formal Series Solutions of Linear Integrodifferential Equations

2.1. Some Nonlinear Partial Differential Equation. We consider the following nonlinear partial differential equation:

$$\partial_t X(t,z) = a(t,z)\,\partial_z X(t,z) + \sum_{p=0}^{d} a_p(t,z)\,X^p(t,z), \quad (5)$$

where $d \geq 2$ is some integer and the coefficients $a(t,z)$, $a_p(t,z)$ are holomorphic functions on some polydisc $D(0,R')^2 \subset \mathbb{C}^2$ such that $a_d(t,z)$ is not identically equal to zero on $D(0,R')^2$.

Notice that (5) can be solved by using the classical method of characteristics which is described in some classical textbooks like [22, page 118] or [23, page 100]. However, the solutions of (5) cannot in general be expressed in closed form. Nevertheless, we can mention some general results concerning qualitative properties of holomorphic solutions to (5) and even to more general first-order partial differential equations of the form

$$\partial_t u(t,x) = F(t,x,u(t,x),\partial_x u(t,x)) \quad (6)$$

for $(t,x) \in \mathbb{C}\times\mathbb{C}^n$ where F is some holomorphic function and $n \geq 1$ an integer. For the construction of holomorphic functions to (6) with singularities located on some specific hypersurfaces (like $\{t=0\}$), see [24, 25]. For the existence of local multivalued holomorphic solutions ramified around some singular sets, we may refer to [26, 27]. Concerning the study of the analytic continuation of singular solutions bounded on some hypersurface, we cite [28] and with prescribed upper estimates, we quote [29, 30].

In this work, we make the assumption that (5) has a holomorphic solution $X(t,z)$ on $D(0,R')^2\setminus\Theta$ where Θ is some set of $D(0,R')^2$ (Θ will be called a singular set in the sequel).

In the next example, we show that a large class of functions can be obtained as solutions of equations of the form (5).

Example 1. Let $n \geq 1$ be an integer and let $g : D(0,R')^2 \to \mathbb{C}$ be a holomorphic function which is not identically equal to zero. We consider

$$a_{n+1}(t,z) = \frac{1}{n}\left(\partial_z g(t,z) - \partial_t g(t,z)\right) \quad (7)$$

which defines a holomorphic function on $D(0,R')^2$. Then, the function $X(t,z) = 1/(g(t,z))^{1/n}$ is a holomorphic solution of the equation

$$\partial_t X(t,z) = \partial_z X(t,z) + a_{n+1}(t,z)\,X^{n+1}(t,z) \quad (8)$$

on $D(0,R')^2\setminus\Theta$ where Θ is the singular set defined by $\Theta = \{(t,z) \in D(0,R')^2 / g(t,z) \in L_\theta\}$ and L_θ is some half-line $\mathbb{R}_+e^{i\theta}$ with $\theta\in\mathbb{R}$ depending on the choice of the determination of the logarithm.

2.2. Composition Series. Let X be as in the previous subsection. In the following, we choose a compact subset K_0 with nonempty interior of $D(0,R)^2\setminus\Theta$ for some $R < R'$ and we consider a real number $\rho > 1$ such that

$$\sup_{(t,z)\in K_0} |X(t,z)| \leq \frac{\rho}{2}. \quad (9)$$

Let $K \subsetneq K_0$ be a compact set with nonempty interior $\mathrm{Int}(K)$. From the Cauchy formula, there exists a real number $\nu > 0$ such that

$$\sup_{(t,z)\in\mathrm{Int}(K)} \frac{\left|\partial_z^h X(t,z)\right|}{h!\nu^h} \leq \frac{\rho}{2} \quad (10)$$

for all integers $h \geq 0$. For all integers $\alpha \geq 0$, we denote $I(\alpha) = \{0,\ldots,\alpha\}$. We consider a sequence of functions $\phi_\alpha(v_0,v_1,(u_h)_{h\in I(\alpha)})$ which are holomorphic and bounded on the polydisc $D(0,R)^2\Pi_{h\in I(\alpha)}D(0,\rho)$, for all $\alpha\geq 0$.

We define the formal series in the w variable as

$$U(t,z,w) = \sum_{\alpha\geq 0}\phi_\alpha\left(t,z,\left(\frac{\partial_z^h X(t,z)}{h!\nu^h}\right)_{h\in I(\alpha)}\right)\frac{w^\alpha}{\alpha!}. \quad (11)$$

For all $\alpha \geq 0$, we consider a holomorphic and bounded function $\widetilde{\omega}_\alpha(v_0,v_1,(u_h)_{h\in I(\alpha)})$ on the product $D(0,R')^2\,\Pi_{h\in I(\alpha)}D(0,\rho)$. We define the formal series

$$\widetilde{\omega}(t,z,w) = \sum_{\alpha\geq 0}\widetilde{\omega}_\alpha\left(t,z,\left(\frac{\partial_z^h X(t,z)}{h!\nu^h}\right)_{h\in I(\alpha)}\right)\frac{w^\alpha}{\alpha!}. \quad (12)$$

Let $\mathcal{S}$ be a finite subset of $\mathbb{N}$ and let $S\geq 1$ be an integer which satisfies the property that

$$S > k \quad (13)$$

for all $k\in\mathcal{S}$. For all $k\in\mathcal{S}$, $m=1,2,3$, and all integers $\alpha\geq 0$, we define a function $b_{m,k,\alpha}(t,z,u_0)$ which is holomorphic on $D(0,R')^2\times\mathbb{C}$ and satisfies estimates of the following form. There exist two constants $D_{m,k}>0$, $\widehat{D}_{m,k}>0$ and an integer $d_{m,k}\geq 0$ such that

$$\sup_{|t|<R',|z|<R',|u_0|\leq\rho}\left|b_{m,k,\alpha}(t,z,u_0)\right| \leq D_{m,k}\rho^{d_{m,k}}\widehat{D}_{m,k}^{\alpha}\alpha! \quad (14)$$

for all $\alpha\geq 0$, with all $\rho\geq 1$. In particular, each function $u_0 \mapsto b_{m,k,\alpha}(t,z,u_0)$ is a polynomial of degree at most $d_{m,k}$ for all $(t,z)\in D(0,R')^2$. Finally, for all $k\in\mathcal{S}$, $m=1,2,3$, we consider the series

$$b_{m,k}(t,z,u_0,w) = \sum_{\alpha\geq 0} b_{m,k,\alpha}(t,z,u_0)\frac{w^\alpha}{\alpha!} \quad (15)$$

which define holomorphic functions on $D(0,R')^2\times\mathbb{C}\times D(0,\overline{w})$, for any $0<\overline{w}\leq 1/\widehat{D}_{m,k}$.

Proposition 2. *Assume that the sequence of functions $(\phi_\alpha)_{\alpha\geq 0}$ satisfies the following recursion:*

$$
\begin{aligned}
&\frac{\phi_\alpha\left(v_0, v_1, (u_h)_{h\in I(\alpha)}\right)}{\alpha!} \\
&= \sum_{k\in\mathcal{S}} \sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k} \frac{b_{1,k,\alpha_1}(v_0, v_1, u_0)}{\alpha_1!} \\
&\quad \times \Bigg(\frac{\partial_{v_0}\phi_{\alpha_2+k-S}\left(v_0, v_1, (u_h)_{h\in I(\alpha_2+k-S)}\right)}{\alpha_2!} \\
&\quad + \sum_{j\in I(\alpha_2+k-S)} \Bigg(\sum_{l_1+l_2=j} \frac{\partial_{v_1}^{l_1} a(v_0, v_1)}{l_1! \nu^{l_1}} (l_2+1)\nu u_{l_2+1} \\
&\quad + \sum_{p=0}^{d} \sum_{j_0+\cdots+j_p=j} \frac{\partial_{v_1}^{j_0} a_p(v_0, v_1)}{j_0! \nu^{j_0}} \times \Pi_{l=1}^{p} u_{j_l} \Bigg) \\
&\quad \times \frac{\partial_{u_j}\phi_{\alpha_2+k-S}\left(v_0, v_1, (u_h)_{h\in I(\alpha_2+k-S)}\right)}{\alpha_2!} \Bigg) \\
&\quad + \sum_{k\in\mathcal{S}} \sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k} \frac{b_{2,k,\alpha_1}(v_0, v_1, u_0)}{\alpha_1!} \\
&\quad \times \Bigg(\frac{\partial_{v_1}\phi_{\alpha_2+k-S}\left(v_0, v_1, (u_h)_{h\in I(\alpha_2+k-S)}\right)}{\alpha_2!} \\
&\quad + \sum_{j\in I(\alpha_2+k-S)} (j+1)\nu u_{j+1} \times \frac{\partial_{u_j}\phi_{\alpha_2+k-S}\left(v_0, v_1, (u_h)_{h\in I(\alpha_2+k-S)}\right)}{\alpha_2!} \Bigg) \\
&\quad + \sum_{k\in\mathcal{S}} \sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k} \frac{b_{3,k,\alpha_1}(v_0, v_1, u_0)}{\alpha_1!} \times \frac{\phi_{\alpha_2+k-S}\left(v_0, v_1, (u_h)_{h\in I(\alpha_2+k-S)}\right)}{\alpha_2!} \\
&\quad + \frac{\tilde{\omega}_\alpha\left(v_0, v_1, (u_h)_{h\in I(\alpha)}\right)}{\alpha!}
\end{aligned}
\tag{16}
$$

for all $\alpha \geq 0$, all $v_0, v_1 \in D(0,R)$, all $u_h \in D(0,\rho)$, for $h \in I(\alpha)$. Then, the formal series $U(t,z,w)$ satisfies the following integrodifferential equation:

$$
\begin{aligned}
U(t,z,w) = \sum_{k\in\mathcal{S}} \Big(& b_{1,k}(t,z,X(t,z),w)\,\partial_t \partial_w^{-S+k} U(t,z,w) \\
& + b_{2,k}(t,z,X(t,z),w)\,\partial_z \partial_w^{-S+k} U(t,z,w) \\
& + b_{3,k}(t,z,X(t,z),w)\,\partial_w^{-S+k} U(t,z,w) \Big) \\
& + \tilde{\omega}(t,z,w)
\end{aligned}
\tag{17}
$$

for all $(t,z) \in \operatorname{Int}(K)$, where ∂_w^{-m} denotes the m-iterate of the usual integration operator $\int_0^w [\cdot] ds$.

Proof. We have that

$$
\begin{aligned}
& b_{3,k}(t,z,X(t,z),w)\,\partial_w^{-S+k} U(t,z,w) \\
&= \sum_{\alpha\geq 0} \Bigg(\sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k} \alpha! \frac{b_{3,k,\alpha_1}(t,z,X(t,z))}{\alpha_1!} \times \frac{\phi_{\alpha_2+k-S}\left(t,z,\left(\partial_z^h X(t,z)/h!\nu^h\right)_{h\in I(\alpha_2+k-S)}\right)}{\alpha_2!} \Bigg) \\
&\quad \times \frac{w^\alpha}{\alpha!},
\end{aligned}
\tag{18}
$$

and we also see that

$$
\begin{aligned}
& b_{2,k}(t,z,X(t,z),w)\,\partial_z \partial_w^{-S+k} U(t,z,w) \\
&= \sum_{\alpha\geq 0} \Bigg(\sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k} \alpha! \frac{b_{2,k,\alpha_1}(t,z,X(t,z))}{\alpha_1!} \times \frac{\partial_z\left(\phi_{\alpha_2+k-S}\left(t,z,\left(\partial_z^h X(t,z)/h!\nu^h\right)_{h\in I(\alpha_2+k-S)}\right)\right)}{\alpha_2!} \Bigg) \\
&\quad \times \frac{w^\alpha}{\alpha!}
\end{aligned}
\tag{19}
$$

with

$$
\begin{aligned}
& \partial_z \left(\phi_{\alpha_2+k-S}\left(t,z,\left(\frac{\partial_z^h X(t,z)}{h!\nu^h}\right)_{h\in I(\alpha_2+k-S)} \right) \right) \\
&= \left(\partial_{v_1}\phi_{\alpha_2+k-S}\right)\left(t,z,\left(\frac{\partial_z^h X(t,z)}{h!\nu^h}\right)_{h\in I(\alpha_2+k-S)} \right) \\
&\quad + \sum_{j\in I(\alpha_2+k-S)} (j+1)\nu \frac{\partial_z^{j+1} X(t,z)}{(j+1)!\nu^{j+1}} \left(\partial_{u_j}\phi_{\alpha_2+k-S}\right) \\
&\quad \times \left(t,z,\left(\frac{\partial_z^h X(t,z)}{h!\nu^h}\right)_{h\in I(\alpha_2+k-S)} \right),
\end{aligned}
\tag{20}
$$

for all $(t,z) \in \mathrm{Int}(K)$. We also get that

$$b_{1,k}(t,z,X(t,z),w)\,\partial_t\partial_w^{-S+k}U(t,z,w) = \sum_{\alpha\geq 0}\left(\sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k}\alpha!\frac{b_{1,k,\alpha_1}(t,z,X(t,z))}{\alpha_1!} \times \frac{\partial_t\left(\phi_{\alpha_2+k-S}\left(t,z,\left(\partial_z^h X(t,z)/h!\nu^h\right)_{h\in I(\alpha_2+k-S)}\right)\right)}{\alpha_2!}\right) \times \frac{w^\alpha}{\alpha!} \tag{21}$$

with

$$\begin{aligned}\partial_t&\left(\phi_{\alpha_2+k-S}\left(t,z,\left(\frac{\partial_z^h X(t,z)}{h!\nu^h}\right)_{h\in I(\alpha_2+k-S)}\right)\right) \\ &= \left(\partial_{v_0}\phi_{\alpha_2+k-S}\right)\left(t,z,\left(\frac{\partial_z^h X(t,z)}{h!\nu^h}\right)_{h\in I(\alpha_2+k-S)}\right) \\ &\quad + \sum_{j\in I(\alpha_2+k-S)}\frac{\partial_t\partial_z^j X(t,z)}{j!\nu^j}\left(\partial_{u_j}\phi_{\alpha_2+k-S}\right) \times \left(t,z,\left(\frac{\partial_z^h X(t,z)}{h!\nu^h}\right)_{h\in I(\alpha_2+k-S)}\right),\end{aligned} \tag{22}$$

for all $(t,z) \in \mathrm{Int}(K)$. Now, from (5) and the classical Schwarz's result on equality of mixed partial derivatives, we get that

$$\frac{\partial_t\partial_z^j X(t,z)}{j!\nu^j} = \frac{\partial_z^j\partial_t X(t,z)}{j!\nu^j} = \frac{1}{j!\nu^j}\partial_z^j\left(a(t,z)\,\partial_z X(t,z) + \sum_{p=0}^{d}a_p(t,z)\,X^p(t,z)\right), \tag{23}$$

and from the Leibniz formula, we can write

$$\begin{aligned}\frac{1}{j!\nu^j}\partial_z^j\left(a(t,z)\,\partial_z X(t,z)\right) &= \sum_{l_1+l_2=j}\frac{\partial_z^{l_1}a(t,z)}{l_1!\nu^{l_1}}(l_2+1)\,\nu\frac{\partial_z^{l_2+1}X(t,z)}{(l_2+1)!\nu^{l_2+1}}, \\ \frac{1}{j!\nu^j}\partial_z^j\left(a_p(t,z)\,X^p(t,z)\right) &= \sum_{j_0+\cdots+j_p=j}\frac{\partial_z^{j_0}a_p(t,z)}{j_0!\nu^{j_0}}\Pi_{l=1}^{p}\frac{\partial_z^{j_l}X(t,z)}{j_l!\nu^{j_l}},\end{aligned} \tag{24}$$

for all $(t,z) \in \mathrm{Int}(K)$. Finally, gathering all the equalities above and using the recursion (16), one gets the integrodifferential equation (17). □

2.3. Recursion for the Derivatives of the Functions ϕ_α, $\alpha \geq 0$. We consider a sequence of functions $\phi_\alpha(v_0,v_1,(u_h)_{h\in I(\alpha)})$, $\alpha \geq 0$, which are holomorphic and bounded on some polydisc $D(0,R)^2\Pi_{h\in I(\alpha)}D(0,\rho)$ for some real numbers $R > 0$ and $\rho > 1$ and which satisfy the equalities (16). We introduce the sequence

$$\varphi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}} = \sup_{|v_0|<R,|v_1|<R,|u_h|<\rho,h\in I(\alpha)}\left|\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_h} \times \phi_\alpha\left(v_0,v_1,(u_h)_{h\in I(\alpha)}\right)\right| \tag{25}$$

for all $n_0, n_1 \geq 0$, all $l_h \geq 0$, $h \in I(\alpha)$, for all $\alpha \geq 0$. We define also the following sequences:

$$\begin{aligned}b_{m,k,\alpha,n_0,n_1,l_0} &= \sup_{|v_0|<R,|v_1|<R,|u_0|<\rho}\left|\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\partial_{u_0}^{l_0}b_{m,k,\alpha}(v_0,v_1,u_0)\right|, \\ \widetilde{\omega}_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}} &= \sup_{|v_0|<R,|v_1|<R,|u_h|<\rho,h\in I(\alpha)}\left|\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_h} \times \widetilde{\omega}_\alpha\left(v_0,v_1,(u_h)_{h\in I(\alpha)}\right)\right|\end{aligned} \tag{26}$$

for $m = 1,2,3$ and $k \in \mathcal{S}$. We put

$$A_j\left(v_0,v_1,(u_h)_{h\in I(\alpha+1)}\right) = \sum_{l_1+l_2=j}\frac{\partial_{v_1}^{l_1}a(v_0,v_1)}{l_1!\nu^{l_1}}(l_2+1)\,\nu u_{l_2+1} + \sum_{p=0}^{d}\sum_{j_0+\cdots+j_p=j}\frac{\partial_{v_1}^{j_0}a_p(v_0,v_1)}{j_0!\nu^{j_0}}\Pi_{l=1}^{p}u_{j_l}, \tag{27}$$

$$B_j\left(v_0, v_1, (u_h)_{h\in I(\alpha+1)}\right) = (j+1)\, v u_{j+1} \tag{28}$$

for all $j \in I(\alpha)$, $v_0, v_1 \in D(0,R')$ and $u_h \in \mathbb{C}$, $h \in I(\alpha)$. We define the sequences

$$\begin{aligned} &A_{j,\alpha,n_0,n_1,(l_h)_{h\in I(\alpha+1)}} \\ &\quad = \sup_{|v_0|<R,|v_1|<R,|u_h|<\rho,h\in I(\alpha)} \Big|\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_h} \\ &\qquad \times A_j\left(v_0, v_1, (u_h)_{h\in I(\alpha+1)}\right)\Big|, \\ &B_{j,\alpha,n_0,n_1,(l_h)_{h\in I(\alpha+1)}} \\ &\quad = \sup_{|v_0|<R,|v_1|<R,|u_h|<\rho,h\in I(\alpha)} \Big|\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_h} \\ &\qquad \times B_j\left(v_0, v_1, (u_h)_{h\in I(\alpha+1)}\right)\Big| \end{aligned} \tag{29}$$

for all $j \in I(\alpha)$, all $n_0, n_1 \ge 0$, all $l_h \ge 0$, $h \in I(\alpha+1)$, for all $\alpha \ge 0$. We also recall the definition of the Kronecker symbol $\delta_{0,l}$ which is equal to 0 if $l \neq 0$ and equal to 1 if $l = 0$.

Proposition 3. *The sequence $\varphi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}}$ satisfies the following inequality:*

$$\begin{aligned} \frac{\varphi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}}}{\alpha!} \le{}& \sum_{k\in\mathcal{S}} \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\ge S-k}} \sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1\\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{1,1}!n_{1,2}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!} \\ &\times \frac{b_{1,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1!}\Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}} \times \frac{\varphi_{\alpha_2+k-S,n_{0,2}+1,n_{1,2},(l_{h,2})_{h\in I(\alpha_2+k-S)}}}{\alpha_2!}\Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,2}} \\ &+ \sum_{j\in I(\alpha_2+k-S)} \sum_{\substack{n_{0,1}+n_{0,2}+n_{0,3}=n_0, n_{1,1}+n_{1,2}+n_{1,3}=n_1\\ l_{h,1}+l_{h,2}+l_{h,3}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{0,3}!n_{1,1}!n_{1,2}!n_{1,3}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!l_{h,3}!} \\ &\times \frac{b_{1,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1!}\Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}} \times A_{j,\alpha_2+k-S+1,n_{0,2},n_{1,2},(l_{h,2})_{h\in I(\alpha_2+k-S+1)}} \\ &\times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S+1)}\delta_{0,l_{h,2}} \times \frac{\varphi_{\alpha_2+k-S,n_{0,3},n_{1,3},(l_{h,3})_{h\in I(\alpha_2+k-S),h\neq j},l_{j,3}+1}}{\alpha_2!} \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,3}} \\ &+ \sum_{k\in\mathcal{S}} \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\ge S-k}} \sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1\\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{1,1}!n_{1,2}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!} \\ &\times \frac{b_{2,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1!}\Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}} \times \frac{\varphi_{\alpha_2+k-S,n_{0,2},n_{1,2}+1,(l_{h,2})_{h\in I(\alpha_2+k-S)}}}{\alpha_2!}\Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,2}} \\ &+ \sum_{j\in I(\alpha_2+k-S)} \sum_{\substack{n_{0,1}+n_{0,2}+n_{0,3}=n_0, n_{1,1}+n_{1,2}+n_{1,3}=n_1\\ l_{h,1}+l_{h,2}+l_{h,3}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{0,3}!n_{1,1}!n_{1,2}!n_{1,3}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!l_{h,3}!} \\ &\times \frac{b_{2,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1!}\Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}} \times B_{j,\alpha_2+k-S+1,n_{0,2},n_{1,2},(l_{h,2})_{h\in I(\alpha_2+k-S+1)}} \\ &\times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S+1)}\delta_{0,l_{h,2}} \times \frac{\varphi_{\alpha_2+k-S,n_{0,3},n_{1,3},(l_{h,3})_{h\in I(\alpha_2+k-S),h\neq j},l_{j,3}+1}}{\alpha_2!} \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,3}} \\ &+ \sum_{k\in\mathcal{S}} \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\ge S-k}} \sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1\\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{1,1}!n_{1,2}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!} \end{aligned}$$

$$\times \frac{b_{3,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1!} \Pi_{h\in I(\alpha)\setminus\{0\}} \delta_{0,l_{h,1}} \times \frac{\varphi_{\alpha_2+k-S,n_{0,2},n_{1,2},(l_{h,2})_{h\in I(\alpha_2+k-S)}}}{\alpha_2!} \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)} \delta_{0,l_{h,2}}$$
$$+ \frac{\widetilde{\omega}_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}}}{\alpha!} \tag{30}$$

for all $\alpha \geq 0$, *all* $n_0, n_1, l_h \geq 0$ *for* $h \in I(\alpha)$.

Proof. In order to get the inequality (30), we apply the differential operator $\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_h}$ on the left and right hand side of the recursion (16) and we use the expansions that are computed below.

From the Leibniz formula, we deduce that

$$\begin{aligned}
&\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_h} \\
&\quad\times \left(b_{3,k,\alpha_1}(v_0,v_1,u_0)\,\phi_{\alpha_2+k-S}\left(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\right)\right) \\
&= \sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1 \\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{1,1}!n_{1,2}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!} \\
&\qquad\times \partial_{v_0}^{n_{0,1}}\partial_{v_1}^{n_{1,1}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,1}}\left(b_{3,k,\alpha_1}(v_0,v_1,u_0)\right) \\
&\qquad\times \partial_{v_0}^{n_{0,2}}\partial_{v_1}^{n_{1,2}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,2}} \\
&\qquad\times \left(\phi_{\alpha_2+k-S}\left(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\right)\right), \\
&\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_h} \\
&\quad\times \left(b_{1,k,\alpha_1}(v_0,v_1,u_0)\,\partial_{v_0}\phi_{\alpha_2+k-S}\left(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\right)\right) \\
&= \sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1 \\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{1,1}!n_{1,2}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!} \\
&\qquad\times \partial_{v_0}^{n_{0,1}}\partial_{v_1}^{n_{1,1}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,1}}\left(b_{1,k,\alpha_1}(v_0,v_1,u_0)\right) \\
&\qquad\times \partial_{v_0}^{n_{0,2}+1}\partial_{v_1}^{n_{1,2}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,2}} \\
&\qquad\times \left(\phi_{\alpha_2+k-S}\left(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\right)\right).
\end{aligned} \tag{31}$$

Moreover, we can write

$$\begin{aligned}
&\partial_{v_0}^{n_{0,1}}\partial_{v_1}^{n_{1,1}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,1}}\left(b_{3,k,\alpha_1}(v_0,v_1,u_0)\right) \\
&\quad= \partial_{v_0}^{n_{0,1}}\partial_{v_1}^{n_{1,1}}\partial_{u_0}^{l_{0,1}}b_{3,k,\alpha_1}(v_0,v_1,u_0) \\
&\quad\times \Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}}
\end{aligned} \tag{32}$$

with

$$\begin{aligned}
&\partial_{v_0}^{n_{0,2}}\partial_{v_1}^{n_{1,2}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,2}} \\
&\quad\times \left(\phi_{\alpha_2+k-S}\left(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\right)\right) \\
&\quad= \partial_{v_0}^{n_{0,2}}\partial_{v_1}^{n_{1,2}}\Pi_{h\in I(\alpha_2+k-S)}\partial_{u_h}^{l_{h,2}} \\
&\quad\times \left(\phi_{\alpha_2+k-S}\left(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\right)\right) \\
&\quad\times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,2}},
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\partial_{v_0}^{n_{0,1}}\partial_{v_1}^{n_{1,1}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,1}}\left(b_{1,k,\alpha_1}(v_0,v_1,u_0)\right) \\
&\quad= \partial_{v_0}^{n_{0,1}}\partial_{v_1}^{n_{1,1}}\partial_{u_0}^{l_{0,1}}b_{1,k,\alpha_1}(v_0,v_1,u_0) \\
&\quad\times \Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}}
\end{aligned} \tag{34}$$

with

$$\begin{aligned}
&\partial_{v_0}^{n_{0,2}+1}\partial_{v_1}^{n_{1,2}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,2}} \\
&\quad\times \left(\phi_{\alpha_2+k-S}\left(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\right)\right) \\
&\quad= \partial_{v_0}^{n_{0,2}+1}\partial_{v_1}^{n_{1,2}}\Pi_{h\in I(\alpha_2+k-S)} \\
&\quad\times \partial_{u_h}^{l_{h,2}}\left(\phi_{\alpha_2+k-S}\left(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\right)\right) \\
&\quad\times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,2}}.
\end{aligned} \tag{35}$$

By construction, we have

$$\begin{aligned}
&A_j\left(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S+1)}\right) \\
&\quad= \sum_{l_1+l_2=j} \frac{\partial_{v_1}^{l_1}a(v_0,v_1)}{l_1!\nu^{l_1}}(l_2+1)\,\nu u_{l_2+1} \\
&\qquad+ \sum_{p=0}^{d}\sum_{j_0+\cdots+j_p=j} \frac{\partial_{v_1}^{j_0}a_p(v_0,v_1)}{j_0!\nu^{j_0}}\Pi_{l=1}^{p}u_{j_l}
\end{aligned} \tag{36}$$

for all $j \in I(\alpha_2 + k - S)$. Again, by the Leibniz formula, we get that

$$\begin{aligned}
&\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_h} \\
&\quad \times \Big(b_{1,k,\alpha_1}(v_0,v_1,u_0)\, A_j\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S+1)}\big) \\
&\quad\quad \times \partial_{u_j}\phi_{\alpha_2+k-S}\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\big)\Big) \\
&= \sum_{\substack{n_{0,1}+n_{0,2}+n_{0,3}=n_0,\, n_{1,1}+n_{1,2}+n_{1,3}=n_1\\ l_{h,1}+l_{h,2}+l_{h,3}=l_h,\, h\in I(\alpha)}} \Big((n_0!n_1!\Pi_{h\in I(\alpha)}l_h!) \\
&\quad \times (n_{0,1}!n_{0,2}!n_{0,3}!n_{1,1}!n_{1,2}!n_{1,3}! \\
&\quad \times \Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!l_{h,3}!)^{-1}\Big) \\
&\quad \times \partial_{v_0}^{n_{0,1}}\partial_{v_1}^{n_{1,1}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,1}} \\
&\quad \times \big(b_{1,k,\alpha_1}(v_0,v_1,u_0)\big) \\
&\quad \times \partial_{v_0}^{n_{0,2}}\partial_{v_1}^{n_{1,2}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,2}} \\
&\quad \times \Big(A_j\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S+1)}\big)\Big) \\
&\quad \times \partial_{v_0}^{n_{0,3}}\partial_{v_1}^{n_{1,3}}\big(\Pi_{h\in I(\alpha),h\neq j}\partial_{u_h}^{l_{h,3}}\big)\partial_{u_j}^{l_{j,3}+1} \\
&\quad \times \phi_{\alpha_2+k-S}\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\big).
\end{aligned} \tag{37}$$

Inside the formula (37), we can rewrite the relations (34) and

$$\begin{aligned}
&\partial_{v_0}^{n_{0,2}}\partial_{v_1}^{n_{1,2}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,2}}A_j\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S+1)}\big) \\
&\quad = \partial_{v_0}^{n_{0,2}}\partial_{v_1}^{n_{1,2}}\Pi_{h\in I(\alpha_2+k-S+1)}\partial_{u_h}^{l_{h,2}} \\
&\quad\quad \times A_j\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S+1)}\big) \\
&\quad\quad \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S+1)}\delta_{0,l_{h,2}}
\end{aligned} \tag{38}$$

with

$$\begin{aligned}
&\partial_{v_0}^{n_{0,3}}\partial_{v_1}^{n_{1,3}}\big(\Pi_{h\in I(\alpha),h\neq j}\partial_{u_h}^{l_{h,3}}\big)\partial_{u_j}^{l_{j,3}+1} \\
&\quad \times \phi_{\alpha_2+k-S}\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\big) \\
&\quad = \partial_{v_0}^{n_{0,3}}\partial_{v_1}^{n_{1,3}}\big(\Pi_{h\in I(\alpha_2+k-S),h\neq j}\partial_{u_h}^{l_{h,3}}\big)\partial_{u_j}^{l_{j,3}+1} \\
&\quad\quad \times \phi_{\alpha_2+k-S}\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\big) \\
&\quad\quad \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,3}}.
\end{aligned} \tag{39}$$

In the same way, one gets the following equalities:

$$\begin{aligned}
&\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_h}\Big(b_{2,k,\alpha_1}(v_0,v_1,u_0)\,\partial_{v_1} \\
&\quad\quad \times \phi_{\alpha_2+k-S}\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\big)\Big) \\
&\quad = \sum_{\substack{n_{0,1}+n_{0,2}=n_0,\, n_{1,1}+n_{1,2}=n_1\\ l_{h,1}+l_{h,2}=l_h,\, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{1,1}!n_{1,2}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!} \\
&\quad\quad \times \partial_{v_0}^{n_{0,1}}\partial_{v_1}^{n_{1,1}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,1}}\big(b_{2,k,\alpha_1}(v_0,v_1,u_0)\big) \\
&\quad\quad \times \partial_{v_0}^{n_{0,2}}\partial_{v_1}^{n_{1,2}+1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,2}} \\
&\quad\quad \times \Big(\phi_{\alpha_2+k-S}\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\big)\Big)
\end{aligned} \tag{40}$$

with the factorizations

$$\begin{aligned}
&\partial_{v_0}^{n_{0,1}}\partial_{v_1}^{n_{1,1}}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,1}}\big(b_{2,k,\alpha_1}(v_0,v_1,u_0)\big) \\
&\quad = \partial_{v_0}^{n_{0,1}}\partial_{v_1}^{n_{1,1}}\partial_{u_0}^{l_{0,1}}b_{2,k,\alpha_1}(v_0,v_1,u_0) \\
&\quad\quad \times \Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}},
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\partial_{v_0}^{n_{0,2}}\partial_{v_1}^{n_{1,2}+1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_{h,2}} \\
&\quad\quad \times \Big(\phi_{\alpha_2+k-S}\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\big)\Big) \\
&\quad = \partial_{v_0}^{n_{0,2}}\partial_{v_1}^{n_{1,2}+1}\Pi_{h\in I(\alpha_2+k-S)}\partial_{u_h}^{l_{h,2}} \\
&\quad\quad \times \Big(\phi_{\alpha_2+k-S}\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\big)\Big) \\
&\quad\quad \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,2}}.
\end{aligned} \tag{42}$$

We recall that

$$B_j\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S+1)}\big) = (j+1)\,\nu u_{j+1} \tag{43}$$

for all $j \in I(\alpha_2 + k - S)$ and we deduce that

$$\begin{aligned}
&\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha)}\partial_{u_h}^{l_h}\Big(b_{2,k,\alpha_1}(v_0,v_1,u_0) \\
&\quad\quad \times B_j\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S+1)}\big) \\
&\quad \times \partial_{u_j}\phi_{\alpha_2+k-S}\big(v_0,v_1,(u_h)_{h\in I(\alpha_2+k-S)}\big)\Big) \\
&= \sum_{\substack{n_{0,1}+n_{0,2}+n_{0,3}=n_0,\, n_{1,1}+n_{1,2}+n_{1,3}=n_1\\ l_{h,1}+l_{h,2}+l_{h,3}=l_h,\, h\in I(\alpha)}} \Big((n_0!n_1!\Pi_{h\in I(\alpha)}l_h!) \\
&\quad \times (n_{0,1}!n_{0,2}!n_{0,3}!n_{1,1}!n_{1,2}!n_{1,3}! \\
&\quad \times \Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!l_{h,3}!)^{-1}\Big)
\end{aligned}$$

$$\times \partial_{v_0}^{n_{0,1}} \partial_{v_1}^{n_{1,1}} \Pi_{h\in I(\alpha)} \partial_{u_h}^{l_{h,1}}$$
$$\times \left(b_{2,k,\alpha_1}\left(v_0, v_1, u_0\right)\right)$$
$$\times \partial_{v_0}^{n_{0,2}} \partial_{v_1}^{n_{1,2}} \Pi_{h\in I(\alpha)} \partial_{u_h}^{l_{h,2}}$$
$$\times \left(B_j\left(v_0, v_1, (u_h)_{h\in I(\alpha_2+k-S+1)}\right)\right)$$
$$\times \partial_{v_0}^{n_{0,3}} \partial_{v_1}^{n_{1,3}} \left(\Pi_{h\in I(\alpha),h\neq j} \partial_{u_h}^{l_{h,3}}\right) \partial_{u_j}^{l_{j,3}+1}$$
$$\times \phi_{\alpha_2+k-S}\left(v_0, v_1, (u_h)_{h\in I(\alpha_2+k-S)}\right). \tag{44}$$

Inside the formula (44), we can rewrite the relations (41) and

$$\begin{aligned}
&\partial_{v_0}^{n_{0,2}} \partial_{v_1}^{n_{1,2}} \Pi_{h\in I(\alpha)} \partial_{u_h}^{l_{h,2}} B_j\left(v_0, v_1, (u_h)_{h\in I(\alpha_2+k-S+1)}\right)\\
&\quad = \partial_{v_0}^{n_{0,2}} \partial_{v_1}^{n_{1,2}} \Pi_{h\in I(\alpha_2+k-S+1)} \partial_{u_h}^{l_{h,2}}\\
&\quad\quad \times B_j\left(v_0, v_1, (u_h)_{h\in I(\alpha_2+k-S+1)}\right)\\
&\quad\quad \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S+1)} \delta_{0,l_{h,2}}
\end{aligned} \tag{45}$$

with the factorization (39). □

2.4. Majorant Series and a Functional Equation with Infinitely Many Variables

Definition 4. One denotes by $\mathbb{G}[[V_0, V_1, (U_h)_{h\geq 0}, W]]$ the vector space of formal series in the variables $V_0, V_1, (U_h)_{h\geq 0}$, and W of the form

$$\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right) = \sum_{\alpha\geq 0} \Psi_\alpha\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right) \frac{W^\alpha}{\alpha!}, \tag{46}$$

where $\Psi_\alpha \in \mathbb{C}[[V_0, V_1, (U_h)_{h\in I(\alpha)}]]$ for all $\alpha \geq 0$.

We keep the notations of the previous section and we introduce the following formal series:

$$\begin{aligned}
&B_{m,k}\left(V_0, V_1, U_0, W\right)\\
&\quad = \sum_{\alpha\geq 0}\left(\sum_{n_0,n_1,l_0\geq 0} b_{m,k,\alpha,n_0,n_1,l_0} \frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!}\frac{U_0^{l_0}}{l_0!}\right)\frac{W^\alpha}{\alpha!},\\
&\widetilde{\Omega}\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)\\
&\quad = \sum_{\alpha\geq 0}\left(\sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha)} \widetilde{\omega}_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}} \frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!}\Pi_{h\in I(\alpha)}\frac{U_h^{l_h}}{l_h!}\right)\\
&\quad\quad \times \frac{W^\alpha}{\alpha!}
\end{aligned} \tag{47}$$

for $m = 1, 2, 3$, all $k \in \mathcal{S}$, and

$$\begin{aligned}
&\mathbf{A}_{j,\alpha}\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right)\\
&\quad = \sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha)} A_{j,\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}} \frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!}\Pi_{h\in I(\alpha)}\frac{U_h^{l_h}}{l_h!},\\
&\mathbf{B}_{j,\alpha}\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right)\\
&\quad = \sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha)} B_{j,\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}} \frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!}\Pi_{h\in I(\alpha)}\frac{U_h^{l_h}}{l_h!}
\end{aligned} \tag{48}$$

for all $\alpha \geq 0$, all $j \in I(\alpha)$. We also introduce the following linear operators acting on $\mathbb{G}[[V_0, V_1, (U_h)_{h\geq 0}, W]]$. Let

$$\begin{aligned}
&\mathbb{D}_{\mathbf{A}}\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)\\
&\quad = \sum_{\alpha\geq 0}\left(\sum_{j\in I(\alpha)} \mathbf{A}_{j,\alpha+1}\left(V_0, V_1, (U_h)_{h\in I(\alpha+1)}\right)\right.\\
&\quad\quad \left.\times\left(\partial_{U_j}\Psi_\alpha\right)\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right)\right)\frac{W^\alpha}{\alpha!},\\
&\mathbb{D}_{\mathbf{B}}\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)\\
&\quad = \sum_{\alpha\geq 0}\left(\sum_{j\in I(\alpha)} \mathbf{B}_{j,\alpha+1}\left(V_0, V_1, (U_h)_{h\in I(\alpha+1)}\right)\right.\\
&\quad\quad \left.\times\left(\partial_{U_j}\Psi_\alpha\right)\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right)\right)\frac{W^\alpha}{\alpha!}
\end{aligned} \tag{49}$$

for all $\Psi \in \mathbb{G}[[V_0, V_1, (U_h)_{h\geq 0}, W]]$. We stress the fact that although these operators act on $\mathbb{G}[[V_0, V_1, (U_h)_{h\geq 0}, W]]$ their image does not have to belong to this space.

Proposition 5. *A formal series*

$$\begin{aligned}
&\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)\\
&\quad = \sum_{\alpha\geq 0}\left(\sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha)} \psi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}} \frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!}\Pi_{h\in I(\alpha)}\frac{U_h^{l_h}}{l_h!}\right)\\
&\quad\quad \times \frac{W^\alpha}{\alpha!}
\end{aligned} \tag{50}$$

satisfies the following functional equation:

$$
\begin{aligned}
&\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)\\
&= \sum_{k\in\mathcal{S}} \Big(B_{1,k}\left(V_0, V_1, U_0, W\right)\partial_W^{-S+k}\partial_{V_0}\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)\\
&\quad + B_{1,k}\left(V_0, V_1, U_0, W\right)\partial_W^{-S+k}\\
&\quad \times \mathbb{D}_{\mathbf{A}}\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)\Big)\\
&\quad + \sum_{k\in\mathcal{S}} \Big(B_{2,k}\left(V_0, V_1, U_0, W\right)\partial_W^{-S+k}\partial_{V_1}\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)\\
&\quad + B_{2,k}\left(V_0, V_1, U_0, W\right)\partial_W^{-S+k}\\
&\quad \times \mathbb{D}_{\mathbf{B}}\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)\Big)\\
&\quad + \sum_{k\in\mathcal{S}} B_{3,k}\left(V_0, V_1, U_0, W\right)\partial_W^{-S+k}\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)\\
&\quad + \widetilde{\Omega}\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)
\end{aligned}
\tag{51}
$$

if and only if its coefficients $\psi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}}$ satisfy the following recursion:

$$
\begin{aligned}
\frac{\psi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}}}{\alpha!} &= \sum_{k\in\mathcal{S}} \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\geq S-k}} \sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1\\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{1,1}!n_{1,2}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!}\\
&\quad \times \frac{b_{1,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1!}\Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}} \times \frac{\psi_{\alpha_2+k-S,n_{0,2}+1,n_{1,2},(l_{h,2})_{h\in I(\alpha_2+k-S)}}}{\alpha_2!}\Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,2}}\\
&\quad + \sum_{j\in I(\alpha_2+k-S)} \sum_{\substack{n_{0,1}+n_{0,2}+n_{0,3}=n_0, n_{1,1}+n_{1,2}+n_{1,3}=n_1\\ l_{h,1}+l_{h,2}+l_{h,3}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{0,3}!n_{1,1}!n_{1,2}!n_{1,3}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!l_{h,3}!}\\
&\quad \times \frac{b_{1,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1!}\Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}} \times A_{j,\alpha_2+k-S+1,n_{0,2},n_{1,2},(l_{h,2})_{h\in I(\alpha_2+k-S+1)}}\\
&\quad \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S+1)}\delta_{0,l_{h,2}} \times \frac{\psi_{\alpha_2+k-S,n_{0,3},n_{1,3},(l_{h,3})_{h\in I(\alpha_2+k-S),h\neq j},l_{j,3}+1}}{\alpha_2!} \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,3}}\\
&\quad + \sum_{k\in\mathcal{S}} \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\geq S-k}} \sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1\\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{1,1}!n_{1,2}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!}\\
&\quad \times \frac{b_{2,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1!}\Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}} \times \frac{\psi_{\alpha_2+k-S,n_{0,2},n_{1,2}+1,(l_{h,2})_{h\in I(\alpha_2+k-S)}}}{\alpha_2!}\Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,2}}\\
&\quad + \sum_{j\in I(\alpha_2+k-S)} \sum_{\substack{n_{0,1}+n_{0,2}+n_{0,3}=n_0, n_{1,1}+n_{1,2}+n_{1,3}=n_1\\ l_{h,1}+l_{h,2}+l_{h,3}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{0,3}!n_{1,1}!n_{1,2}!n_{1,3}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!l_{h,3}!}\\
&\quad \times \frac{b_{2,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1!}\Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}} \times B_{j,\alpha_2+k-S+1,n_{0,2},n_{1,2},(l_{h,2})_{h\in I(\alpha_2+k-S+1)}}\\
&\quad \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S+1)}\delta_{0,l_{h,2}} \times \frac{\psi_{\alpha_2+k-S,n_{0,3},n_{1,3},(l_{h,3})_{h\in I(\alpha_2+k-S),h\neq j},l_{j,3}+1}}{\alpha_2!} \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,3}}\\
&\quad + \sum_{k\in\mathcal{S}} \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\geq S-k}} \sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1\\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \frac{n_0!n_1!\Pi_{h\in I(\alpha)}l_h!}{n_{0,1}!n_{0,2}!n_{1,1}!n_{1,2}!\Pi_{h\in I(\alpha)}l_{h,1}!l_{h,2}!}
\end{aligned}
$$

$$\times \frac{b_{3,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1!} \Pi_{h\in I(\alpha)\setminus\{0\}} \delta_{0,l_{h,1}} \times \frac{\psi_{\alpha_2+k-S,n_{0,2},n_{1,2},(l_{h,2})_{h\in I(\alpha_2+k-S)}}}{\alpha_2!} \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)} \delta_{0,l_{h,2}}$$

$$+ \frac{\widetilde{\omega}_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}}}{\alpha!} \tag{52}$$

for all $\alpha \geq 0$, all $n_0, n_1, l_h \geq 0$ with $h \in I(\alpha)$.

Proof. We proceed by identification of the coefficients in the Taylor expansion with respect to the variables $V_0, V_1, (U_h)_{h\in I(\alpha)}$, and W for all $\alpha \geq 0$. By definition, we have that

$$B_{1,k}(V_0, V_1, U_0, W)\, \partial_W^{-S+k} \partial_{V_0} \Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right) = \sum_{\alpha\geq 0} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\geq S-k}} \mathscr{C}^1_{\alpha_1,\alpha_2} W^\alpha, \tag{53}$$

where the coefficients $\mathscr{C}^1_{\alpha_1,\alpha_2}$ can be rewritten, using the Kronecker symbol $\delta_{0,m}$, in the form

$$\begin{aligned}\mathscr{C}^1_{\alpha_1,\alpha_2} &= \left(\sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} \frac{b_{1,k,\alpha_1,n_0,n_1,l_0}}{\alpha_1!} \times \Pi_{h\in I(\alpha)\setminus\{0\}} \delta_{0,l_h} \frac{V_0^{n_0}}{n_0!} \frac{V_1^{n_1}}{n_1!} \Pi_{h\in I(\alpha)} \frac{U_h^{l_h}}{l_h!} \right) \\ &\quad \times \left(\sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} \frac{\psi_{\alpha_2+k-S,n_0+1,n_1,(l_h)_{h\in I(\alpha_2+k-S)}}}{\alpha_2!} \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)} \delta_{0,l_h} \frac{V_0^{n_0}}{n_0!} \frac{V_1^{n_1}}{n_1!} \Pi_{h\in I(\alpha)} \frac{U_h^{l_h}}{l_h!} \right).\end{aligned} \tag{54}$$

Hence,

$$\begin{aligned}\mathscr{C}^1_{\alpha_1,\alpha_2} &= \sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} \Bigg(\sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1 \\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \frac{b_{1,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1! n_{0,1}! n_{1,1}! \Pi_{h\in I(\alpha)} l_{h,1}!} \\ &\quad \times \Pi_{h\in I(\alpha)\setminus\{0\}} \delta_{0,l_{h,1}} \times \frac{\psi_{\alpha_2+k-S,n_{0,2}+1,n_{1,2},(l_{h,2})_{h\in I(\alpha_2+k-S)}}}{\alpha_2! n_{0,2}! n_{1,2}! \Pi_{h\in I(\alpha)} l_{h,2}!} \\ &\quad \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)} \delta_{0,l_{h,2}} \Bigg) \times V_0^{n_0} V_1^{n_1} \Pi_{h\in I(\alpha)} U_h^{l_h}.\end{aligned} \tag{55}$$

We also have that

$$B_{1,k}(V_0, V_1, U_0, W)\, \partial_W^{-S+k} \mathbb{D}_{\mathbf{A}} \Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right) = \sum_{\alpha\geq 0} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\geq S-k}} \mathscr{F}^1_{\alpha_1,\alpha_2} W^\alpha, \tag{56}$$

where the coefficients $\mathscr{F}^1_{\alpha_1,\alpha_2}$ can be rewritten in the form

$$\begin{aligned}\mathscr{F}^1_{\alpha_1,\alpha_2} &= \sum_{j\in I(\alpha_2-S+k)} \left(\sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} \frac{b_{1,k,\alpha_1,n_0,n_1,l_0}}{\alpha_1!} \times \Pi_{h\in I(\alpha)\setminus\{0\}} \delta_{0,l_h} \times \frac{V_0^{n_0}}{n_0!} \frac{V_1^{n_1}}{n_1!} \Pi_{h\in I(\alpha)} \frac{U_h^{l_h}}{l_h!} \right) \\ &\quad \times \left(\sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} A_{j,\alpha_2-S+k+1,n_0,n_1,(l_h)_{h\in I(\alpha_2-S+k+1)}} \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2-S+k+1)} \delta_{0,l_h} \times \frac{V_0^{n_0}}{n_0!} \frac{V_1^{n_1}}{n_1!} \Pi_{h\in I(\alpha)} \frac{U_h^{l_h}}{l_h!} \right) \\ &\quad \times \left(\sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} \frac{\psi_{\alpha_2-S+k,n_0,n_1,(l_h)_{h\in I(\alpha_2-S+k), h\neq j}, l_j+1}}{\alpha_2!} \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2-S+k)} \delta_{0,l_h} \times \frac{V_0^{n_0}}{n_0!} \frac{V_1^{n_1}}{n_1!} \Pi_{h\in I(\alpha)} \frac{U_h^{l_h}}{l_h!} \right).\end{aligned} \tag{57}$$

Therefore,

$$\mathcal{F}^{1}_{\alpha_1,\alpha_2} = \sum_{j\in I(\alpha_2-S+k)} \left(\sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} \left(\sum_{\substack{n_{0,1}+n_{0,2}+n_{0,3}=n_0, n_{1,1}+n_{1,2}+n_{1,3}=n_1 \\ l_{h,1}+l_{h,2}+l_{h,3}=l_h, h\in I(\alpha)}} \frac{b_{1,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1! n_{0,1}! n_{1,1}! \Pi_{h\in I(\alpha)} l_{h,1}!} \Pi_{h\in I(\alpha)\setminus\{0\}} \delta_{0,l_{h,1}} \right.\right.$$
$$\times \frac{A_{j,\alpha_2-S+k+1,n_{0,2},n_{1,2},(l_{h,2})_{h\in I(\alpha_2-S+k+1)}}}{n_{0,2}! n_{1,2}! \Pi_{h\in I(\alpha)} l_{h,2}!} \Pi_{h\in I(\alpha)\setminus I(\alpha_2-S+k+1)} \delta_{0,l_{h,2}}$$
$$\left. \times \frac{\psi_{\alpha_2-S+k,n_{0,3},n_{1,3},(l_{h,3})_{h\in I(\alpha_2-S+k),h\neq j},l_{j,3}+1}}{\alpha_2! n_{0,3}! n_{1,3}! \Pi_{h\in I(\alpha)} l_{h,3}!} \Pi_{h\in I(\alpha)\setminus I(\alpha_2-S+k)} \delta_{0,l_{h,3}} \right) \tag{58}$$
$$\left. \times V_0^{n_0} V_1^{n_1} \Pi_{h\in I(\alpha)} U_h^{l_h} \right).$$

On the other hand, using similar computations we get

$$B_{2,k}(V_0,V_1,U_0,W)\,\partial_W^{-S+k}\partial_{V_1}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right) = \sum_{\alpha\geq 0} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\geq S-k}} \mathcal{C}^{2}_{\alpha_1,\alpha_2} W^{\alpha}, \tag{59}$$

where

$$\mathcal{C}^{2}_{\alpha_1,\alpha_2} = \sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha)} \left(\sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1 \\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \frac{b_{2,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1! n_{0,1}! n_{1,1}! \Pi_{h\in I(\alpha)} l_{h,1}!} \times \Pi_{h\in I(\alpha)\setminus\{0\}} \delta_{0,l_{h,1}} \right.$$
$$\times \frac{\psi_{\alpha_2+k-S,n_{0,2},n_{1,2}+1,(l_{h,2})_{h\in I(\alpha_2+k-S)}}}{\alpha_2! n_{0,2}! n_{1,2}! \Pi_{h\in I(\alpha)} l_{h,2}!}$$
$$\left. \times \Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)} \delta_{0,l_{h,2}} \right)$$
$$\times V_0^{n_0} V_1^{n_1} \Pi_{h\in I(\alpha)} U_h^{l_h}. \tag{60}$$

We also have that

$$B_{2,k}(V_0,V_1,U_0,W)\,\partial_W^{-S+k}\mathbb{D}_{\mathbf{B}}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right) = \sum_{\alpha\geq 0} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\geq S-k}} \mathcal{F}^{2}_{\alpha_1,\alpha_2} W^{\alpha}, \tag{61}$$

where

$$\mathcal{F}^{2}_{\alpha_1,\alpha_2} = \sum_{j\in I(\alpha_2-S+k)} \left(\sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} \left(\sum_{\substack{n_{0,1}+n_{0,2}+n_{0,3}=n_0, n_{1,1}+n_{1,2}+n_{1,3}=n_1 \\ l_{h,1}+l_{h,2}+l_{h,3}=l_h, h\in I(\alpha)}} \frac{b_{2,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1! n_{0,1}! n_{1,1}! \Pi_{h\in I(\alpha)} l_{h,1}!} \Pi_{h\in I(\alpha)\setminus\{0\}} \delta_{0,l_{h,1}} \right.\right.$$
$$\times \frac{B_{j,\alpha_2-S+k+1,n_{0,2},n_{1,2},(l_{h,2})_{h\in I(\alpha_2-S+k+1)}}}{n_{0,2}! n_{1,2}! \Pi_{h\in I(\alpha)} l_{h,2}!} \Pi_{h\in I(\alpha)\setminus I(\alpha_2-S+k+1)} \delta_{0,l_{h,2}}$$

$$\left.\left.\times\frac{\psi_{\alpha_2-S+k,n_{0,3},n_{1,3},(l_{h,3})_{h\in I(\alpha_2-S+k),h\neq j},l_{j,3}+1}}{\alpha_2!n_{0,3}!n_{1,3}!\Pi_{h\in I(\alpha)}l_{h,3}!}\Pi_{h\in I(\alpha)\setminus I(\alpha_2-S+k)}\delta_{0,l_{h,3}}\right)\times V_0^{n_0}V_1^{n_1}\Pi_{h\in I(\alpha)}U_h^{l_h}\right), \tag{62}$$

$$B_{3,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)=\sum_{\alpha\geq 0}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\geq S-k}}\mathscr{C}^3_{\alpha_1,\alpha_2}W^{\alpha}, \tag{63}$$

where

$$\mathscr{C}^3_{\alpha_1,\alpha_2}=\sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha)}\left(\sum_{\substack{n_{0,1}+n_{0,2}=n_0,n_{1,1}+n_{1,2}=n_1\\ l_{h,1}+l_{h,2}=l_h,h\in I(\alpha)}}\frac{b_{3,k,\alpha_1,n_{0,1},n_{1,1},l_{0,1}}}{\alpha_1!n_{0,1}!n_{1,1}!\Pi_{h\in I(\alpha)}l_{h,1}!}\times\Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_{h,1}}\times\frac{\psi_{\alpha_2+k-S,n_{0,2},n_{1,2},(l_{h,2})_{h\in I(\alpha_2+k-S)}}}{\alpha_2!n_{0,2}!n_{1,2}!\Pi_{h\in I(\alpha)}l_{h,2}!}\times\Pi_{h\in I(\alpha)\setminus I(\alpha_2+k-S)}\delta_{0,l_{h,2}}\right)\times V_0^{n_0}V_1^{n_1}\Pi_{h\in I(\alpha)}U_h^{l_h}. \tag{64}$$

Finally, gathering the expansions (55), (58), (60), and (62) with (64) yields the result. □

Proposition 6. *The sequences $\varphi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}}$ and $\psi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}}$ satisfy the following inequalities:*

$$\varphi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}}\leq\psi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}} \tag{65}$$

for all $\alpha\geq 0$, all $n_0,n_1\geq 0$, all $l_h\geq 0$, $h\in I(\alpha)$.

Proof. For $\alpha=0$, using the recursions (16) and (52), we get that

$$\varphi_{0,n_0,n_1,(l_h)_{h\in I(0)}}=\widetilde{w}_{0,n_0,n_1,(l_h)_{h\in I(0)}}=\psi_{0,n_0,n_1,(l_h)_{h\in I(0)}} \tag{66}$$

for all $n_0,n_1,l_0\geq 0$. By induction on α and using the inequalities (30) together with the equalities (52), one gets the result. □

3. Convergent Series Solutions for a Functional Equation with Infinitely Many Variables

3.1. Banach Spaces of Formal Series. Let $\rho>1$ and $\sigma,\overline{V}_0,\overline{V}_1,\overline{W},\overline{\delta}>0$ be real numbers. For any given real number $b>1$, we define the sequences $r_b(\alpha)=\sum_{n=0}^{\alpha}1/(n+1)^b$ for all $\alpha\geq 0$ and $\overline{U}_h=\overline{\delta}/(h^b+1)$ for all $h\geq 0$.

Definition 7. Let $\alpha\geq 0$ be an integer. One denotes by $E_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}}$ the vector space of formal series

$$\Psi\left(V_0,V_1,(U_h)_{h\in I(\alpha)}\right)=\sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha)}\psi_{n_0,n_1,(l_h)_{h\in I(\alpha)}}\frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!}\times\Pi_{h\in I(\alpha)}\frac{U_h^{l_h}}{l_h!} \tag{67}$$

that belong to $\mathbb{C}[[V_0,V_1,(U_h)_{h\in I(\alpha)}]]$ such that the series

$$\left\|\Psi\left(V_0,V_1,(U_h)_{h\in I(\alpha)}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}}=\sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha)}\frac{\left|\psi_{n_0,n_1,(l_h)_{h\in I(\alpha)}}\right|}{\exp\left(\sigma r_b(\alpha)\rho\right)}\times\frac{\overline{V}_0^{n_0}\overline{V}_1^{n_1}\Pi_{h\in I(\alpha)}\overline{U}_h^{l_h}}{\left(n_0+n_1+\sum_{h\in I(\alpha)}l_h+\alpha\right)!} \tag{68}$$

is convergent. One denotes also by $G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$ the vector space of formal series

$$\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right) = \sum_{\alpha\geq 0} \Psi_\alpha\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right)\frac{W^\alpha}{\alpha!}, \tag{69}$$

where $\Psi_\alpha(V_0, V_1, (U_h)_{h\in I(\alpha)})$ belong to $E_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}}$ for all $\alpha \geq 0$, such that the series

$$\left\|\Psi\left(V_0, V_1, (U_h)_{h\geq 0}, W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} = \sum_{\alpha\geq 0} \left\|\Psi_\alpha\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}} \overline{W}^\alpha \tag{70}$$

is convergent. One checks that the space $G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$ equipped with the norm $\|\cdot\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$ is a Banach space.

In the next two propositions, we study norm estimates for linear operators acting on the Banach spaces $E_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}}$ constructed above.

Proposition 8. *Consider a formal series*

$$b\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right) = \sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} b_{n_0,n_1,(l_h)_{h\in I(\alpha)}} \frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!} \times \Pi_{h\in I(\alpha)} \frac{U_h^{l_h}}{l_h!} \tag{71}$$

which is absolutely convergent on the polydisc $D(0,\overline{V}_0) \times D(0,\overline{V}_1) \times_{h\in I(\alpha)} D(0,\overline{U}_h)$. One uses the notation

$$|b|\left(\overline{V}_0, \overline{V}_1, \left(\overline{U}_h\right)_{h\in I(\alpha)}\right) = \sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} \left|b_{n_0,n_1,(l_h)_{h\in I(\alpha)}}\right| \frac{\overline{V}_0^{n_0}}{n_0!}\frac{\overline{V}_1^{n_1}}{n_1!} \times \Pi_{h\in I(\alpha)} \frac{\overline{U}_h^{l_h}}{l_h!}. \tag{72}$$

Let $\Psi(V_0, V_1, (U_h)_{h\in I(\alpha)})$ belong to $E_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}}$. Then, the following inequality:

$$\begin{aligned}&\left\|b\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right) \times \Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}} \\ &\quad\leq |b|\left(\overline{V}_0, \overline{V}_1, \left(\overline{U}_h\right)_{h\in I(\alpha)}\right) \times \left\|\Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}}\end{aligned} \tag{73}$$

holds.

Proof. Let

$$\Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right) = \sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} \psi_{n_0,n_1,(l_h)_{h\in I(\alpha)}} \frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!} \times \Pi_{h\in I(\alpha)} \frac{U_h^{l_h}}{l_h!} \tag{74}$$

which belongs to $E_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}}$. By definition, we have that

$$\begin{aligned}&\left\|b\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right) \times \Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}} \\ &= \sum_{n_0,n_1,l_h\geq 0, h\in I(\alpha)} \left| \sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1 \\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \left(\left(n_0! n_1! \Pi_{h\in I(\alpha)} l_h!\right) \times \left(n_{0,1}! n_{0,2}! n_{1,1}! n_{1,2}! \times \Pi_{h\in I(\alpha)} l_{h,1}! l_{h,2}!\right)^{-1}\right) \times b_{n_{0,1},n_{1,1},(l_{h,1})_{h\in I(\alpha)}} \times \psi_{n_{0,2},n_{1,2},(l_{h,2})_{h\in I(\alpha)}} \right| \\ &\quad \times \frac{1}{\exp\left(\sigma r_b(\alpha)\rho\right)} \times \frac{\overline{V}_0^{n_0}\overline{V}_1^{n_1}\Pi_{h\in I(\alpha)}\overline{U}_h^{l_h}}{\left(n_0 + n_1 + \sum_{h\in I(\alpha)} l_h + \alpha\right)!}.\end{aligned} \tag{75}$$

We can give upper bounds for this latter expression

$$\begin{aligned}
&\left\| b\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right) \Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha)}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}} \\
&\quad\le \sum_{n_0,n_1,l_h\ge 0, h\in I(\alpha)} \sum_{\substack{n_{0,1}+n_{0,2}=n_0, n_{1,1}+n_{1,2}=n_1 \\ l_{h,1}+l_{h,2}=l_h, h\in I(\alpha)}} \left(\frac{n_0! n_1! \Pi_{h\in I(\alpha)} l_h!}{n_{0,2}! n_{1,2}! \Pi_{h\in I(\alpha)} l_{h,2}!} \times \frac{\left(n_{0,2}+n_{1,2}+\sum_{h\in I(\alpha)} l_{h,2}+\alpha\right)!}{\left(n_0+n_1+\sum_{h\in I(\alpha)} l_h+\alpha\right)!}\right) \\
&\qquad \times \frac{\left|b_{n_{0,1},n_{1,1},(l_{h,1})_{h\in I(\alpha)}}\right|}{n_{0,1}! n_{1,1}! \Pi_{h\in I(\alpha)} l_{h,1}!} \overline{V}_0^{n_{0,1}} \overline{V}_1^{n_{1,1}} \Pi_{h\in I(\alpha)} \overline{U}_h^{l_{h,1}} \times \left|\psi_{n_{0,2},n_{1,2},(l_{h,2})_{h\in I(\alpha)}}\right| \frac{1}{\exp\left(\sigma r_b(\alpha)\rho\right)} \frac{\overline{V}_0^{n_{0,2}} \overline{V}_1^{n_{1,2}} \Pi_{h\in I(\alpha)} \overline{U}_h^{l_{h,2}}}{\left(n_{0,2}+n_{1,2}+\sum_{h\in I(\alpha)} l_{h,2}+\alpha\right)!}.
\end{aligned} \tag{76}$$

Lemma 9. *For all integers $\alpha, n_0, n_1 \ge 0$, all $l_h \ge 0$, all $0 \le n_{0,2} \le n_0$, all $0 \le n_{1,2} \le n_1$, and all $0 \le l_{h,2} \le l_h$ for $h \in I(\alpha)$, one has that*

$$\frac{n_0! n_1! \Pi_{h\in I(\alpha)} l_h!}{n_{0,2}! n_{1,2}! \Pi_{h\in I(\alpha)} l_{h,2}!} \frac{\left(n_{0,2}+n_{1,2}+\sum_{h\in I(\alpha)} l_{h,2}+\alpha\right)!}{\left(n_0+n_1+\sum_{h\in I(\alpha)} l_h+\alpha\right)!} \le 1. \tag{77}$$

Proof. For any integers $a \le b$ and $\alpha \ge 0$, one has

$$\frac{(a+\alpha)!}{(b+\alpha)!} \le \frac{a!}{b!} \tag{78}$$

by using the factorization $(a+\alpha)! = (a+\alpha)(a+\alpha-1)\cdots(a+1)a!$. Therefore, one gets the inequality

$$\begin{aligned}
&\frac{n_0! n_1! \Pi_{h\in I(\alpha)} l_h!}{n_{0,2}! n_{1,2}! \Pi_{h\in I(\alpha)} l_{h,2}!} \frac{\left(n_{0,2}+n_{1,2}+\sum_{h\in I(\alpha)} l_{h,2}+\alpha\right)!}{\left(n_0+n_1+\sum_{h\in I(\alpha)} l_h+\alpha\right)!} \\
&\quad \le \frac{n_0! n_1! \Pi_{h\in I(\alpha)} l_h!}{n_{0,2}! n_{1,2}! \Pi_{h\in I(\alpha)} l_{h,2}!} \frac{\left(n_{0,2}+n_{1,2}+\sum_{h\in I(\alpha)} l_{h,2}\right)!}{\left(n_0+n_1+\sum_{h\in I(\alpha)} l_h\right)!}.
\end{aligned} \tag{79}$$

Now, from the identity $(A+B)^{n_0+n_1+\sum_{h\in I(\alpha)} l_h} = (A+B)^{n_0}(A+B)^{n_1} \times \Pi_{h\in I(\alpha)}(A+B)^{l_h}$ and the binomial formula, we deduce that

$$\begin{aligned}
&\frac{n_0! n_1! \Pi_{h\in I(\alpha)} l_h!}{n_{0,1}! n_{0,2}! n_{1,1}! n_{1,2}! \Pi_{h\in I(\alpha)} l_{h,1}! l_{h,2}!} \\
&\quad \le \frac{\left(n_0+n_1+\sum_{h\in I(\alpha)} l_h\right)!}{\left(n_{0,1}+n_{1,1}+\sum_{h\in I(\alpha)} l_{h,1}\right)! \left(n_{0,2}+n_{1,2}+\sum_{h\in I(\alpha)} l_{h,2}\right)!}
\end{aligned} \tag{80}$$

for all $n_{0,1}+n_{0,2}=n_0$, $n_{1,1}+n_{1,2}=n_1$, $l_{h,1}+l_{h,2}=l_h$. Therefore, we deduce that

$$\begin{aligned}
&\frac{n_0! n_1! \Pi_{h\in I(\alpha)} l_h!}{n_{0,2}! n_{1,2}! \Pi_{h\in I(\alpha)} l_{h,2}!} \frac{\left(n_{0,2}+n_{1,2}+\sum_{h\in I(\alpha)} l_{h,2}+\alpha\right)!}{\left(n_0+n_1+\sum_{h\in I(\alpha)} l_h+\alpha\right)!} \\
&\quad \le \frac{n_{0,1}! n_{1,1}! \Pi_{h\in I(\alpha)} l_{h,1}!}{\left(n_{0,1}+n_{1,1}+\sum_{h\in I(\alpha)} l_{h,1}\right)!} \le 1,
\end{aligned} \tag{81}$$

and the lemma follows from the inequalities (79) and (81).

Finally, the inequality (73) follows from (76) and (77). □

Proposition 10. *Let α, α' be integers such that $\alpha' \ge 0$ and $\alpha'+1 < \alpha$. Let $j \in I(\alpha')$ and $k \in \{0,1\}$. One has that*

$$\begin{aligned}
&\left\|\partial_{U_j} \Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha')}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}} \\
&\quad \le \frac{\exp\left(-\sigma\rho\left(\left(\alpha-\alpha'\right)/(\alpha+1)^b\right)\right)}{\overline{U}_j \Pi_{l=1}^{\alpha-\alpha'-1}(\alpha-l+1)} \\
&\qquad \times \left\|\Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha')}\right)\right\|_{\rho,\alpha',\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha')}},
\end{aligned} \tag{82}$$

$$\begin{aligned}
&\left\|\partial_{V_k} \Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha')}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}} \\
&\quad \le \frac{\exp\left(-\sigma\rho\left(\left(\alpha-\alpha'\right)/(\alpha+1)^b\right)\right)}{\overline{V}_k \Pi_{l=1}^{\alpha-\alpha'-1}(\alpha-l+1)} \\
&\qquad \times \left\|\Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha')}\right)\right\|_{\rho,\alpha',\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha')}},
\end{aligned} \tag{83}$$

$$\begin{aligned}
&\left\|\Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha')}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}} \\
&\quad \le \frac{\exp\left(-\sigma\rho\left(\left(\alpha-\alpha'\right)/(\alpha+1)^b\right)\right)}{\Pi_{l=1}^{\alpha-\alpha'}(\alpha-l+1)} \\
&\qquad \times \left\|\Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha')}\right)\right\|_{\rho,\alpha',\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha')}}
\end{aligned} \tag{84}$$

for all $\Psi(V_0, V_1, (U_h)_{h\in I(\alpha')}) \in E_{\rho,\alpha',\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha')}}$.

Proof. Let $\Psi(V_0, V_1, (U_h)_{h\in I(\alpha')}) \in E_{\rho,\alpha',\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha')}}$ that we write in the form

$$\begin{aligned}
&\Psi\left(V_0, V_1, (U_h)_{h\in I(\alpha')}\right) \\
&\quad = \sum_{n_0,n_1,l_h\ge 0, h\in I(\alpha)} \psi_{n_0,n_1,(l_h)_{h\in I(\alpha')}} \\
&\qquad \times \Pi_{h\in I(\alpha)\setminus I(\alpha')} \delta_{0,l_h} \frac{V_0^{n_0}}{n_0!} \frac{V_1^{n_1}}{n_1!} \Pi_{h\in I(\alpha)} \frac{U_h^{l_h}}{l_h!}.
\end{aligned} \tag{85}$$

By definition, we get that

$$\left\|\partial_{U_j}\Psi\left(V_0,V_1,(U_h)_{h\in I(\alpha')}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}} = \sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha)} \frac{\left|\psi_{n_0,n_1,(l_h)_{h\in I(\alpha'),h\neq j},l_j+1}\prod_{h\in I(\alpha)\setminus I(\alpha')}\delta_{0,l_h}\right|}{\exp\left(\sigma r_b(\alpha)\rho\right)} \times \frac{\overline{V}_0^{n_0}\overline{V}_1^{n_1}\prod_{h\in I(\alpha)}\overline{U}_h^{l_h}}{\left(n_0+n_1+\sum_{h\in I(\alpha)}l_h+\alpha\right)!}. \quad (86)$$

We give upper bounds for this latter expression

$$\begin{aligned}&\left\|\partial_{U_j}\Psi\left(V_0,V_1,(U_h)_{h\in I(\alpha')}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}}\\ &= \sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha')}\left(\frac{\left(n_0+n_1+\sum_{h\in I(\alpha'),h\neq j}l_h+l_j+1+\alpha'\right)!}{\left(n_0+n_1+\sum_{h\in I(\alpha')}l_h+\alpha\right)!} \times \frac{1}{\overline{U}_j\exp\left(\sigma\rho\left(r_b(\alpha)-r_b(\alpha')\right)\right)}\right)\\ &\quad\times\frac{\left|\psi_{n_0,n_1,(l_h)_{h\in I(\alpha'),h\neq j},l_j+1}\right|}{\exp\left(\sigma r_b(\alpha')\rho\right)} \times \frac{\overline{V}_0^{n_0}\overline{V}_1^{n_1}\prod_{h\in I(\alpha'),h\neq j}\overline{U}_h^{l_h}\overline{U}_j^{l_j+1}}{\left(n_0+n_1+\sum_{h\in I(\alpha'),h\neq j}l_h+l_j+1+\alpha'\right)!}.\end{aligned} \quad (87)$$

Lemma 11. *One has*

$$\frac{\left(n_0+n_1+\sum_{h\in I(\alpha'),h\neq j}l_h+l_j+1+\alpha'\right)!}{\left(n_0+n_1+\sum_{h\in I(\alpha')}l_h+\alpha\right)!} \times \frac{1}{\exp\left(\sigma\rho\left(r_b(\alpha)-r_b(\alpha')\right)\right)} \leq \frac{\exp\left(-\sigma\rho\left((\alpha-\alpha')/(\alpha+1)^b\right)\right)}{\prod_{l=1}^{\alpha-\alpha'-1}(\alpha-l+1)}. \quad (88)$$

Proof. We notice that

$$r_b(\alpha)-r_b(\alpha') = \sum_{n=\alpha'+1}^{\alpha}\frac{1}{(n+1)^b} \geq \frac{\alpha-\alpha'}{(\alpha+1)^b} \quad (89)$$

and, with the help of (78), that for all integers $a\geq 0$,

$$\frac{(a+1+\alpha')!}{(a+\alpha)!} \leq \frac{1}{\prod_{l=1}^{\alpha-\alpha'-1}(\alpha-l+1)}. \quad (90)$$

The lemma follows.

We get that the inequality (82) follows from (87) together with (88). Finally, using similar arguments, one gets also the inequalities (83) and (84). □

In the next two propositions, we study norm estimates for linear operators acting on the Banach space $G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$.

Proposition 12. *Let a formal series $b(V_0,V_1,U_0,W)\in\mathbb{C}[[V_0,V_1,U_0,W]]$ be absolutely convergent on the polydisc $D(0,\overline{V}_0)\times D(0,\overline{V}_1)\times D(0,\overline{U}_0)\times D(0,\overline{W})$. Let $\Psi(V_0,V_1,(U_h)_{h\geq 0},W)$ belong to $G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$. Then, the product $b(V_0,V_1,U_0,W)\Psi(V_0,V_1,(U_h)_{h\geq 0},W)$ belongs to $G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$ and the following inequality:*

$$\begin{aligned}&\left\|b(V_0,V_1,U_0,W)\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}\\ &\leq |b|\left(\overline{V}_0,\overline{V}_1,\overline{U}_0,\overline{W}\right) \times \left\|\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}\end{aligned} \quad (91)$$

holds.

Proof. Let

$$\begin{aligned}b(V_0,V_1,U_0,W) &= \sum_{\alpha\geq 0}b_\alpha(V_0,V_1,U_0)\frac{W^\alpha}{\alpha!},\\ \Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right) &= \sum_{\alpha\geq 0}\Psi_\alpha\left(V_0,V_1,(U_h)_{h\in I(\alpha)}\right)\frac{W^\alpha}{\alpha!}.\end{aligned} \quad (92)$$

By definition, we get

$$\begin{aligned}&\left\|b(V_0,V_1,U_0,W)\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}\\ &= \sum_{\alpha\geq 0}\left\|\sum_{\alpha_1+\alpha_2=\alpha}\alpha!\frac{b_{\alpha_1}(V_0,V_1,U_0)}{\alpha_1!} \times \frac{\Psi_{\alpha_2}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2)}\right)}{\alpha_2!}\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}}\overline{W}^\alpha.\end{aligned} \quad (93)$$

Lemma 13. *One has*

$$\begin{aligned}&\left\|b_{\alpha_1}(V_0,V_1,U_0)\Psi_{\alpha_2}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2)}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}}\\ &\leq \frac{\alpha_2!}{\alpha!}\left|b_{\alpha_1}\right|\left(\overline{V}_0,\overline{V}_1,\overline{U}_0\right) \times \left\|\Psi_{\alpha_2}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2)}\right)\right\|_{\rho,\alpha_2,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2)}}.\end{aligned} \quad (94)$$

Proof. We can write

$$b_{\alpha_1}(V_0,V_1,U_0) = \sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha)} b_{\alpha_1,n_0,n_1,l_0} \times \Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_h}\frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!}\Pi_{h\in I(\alpha)}\frac{U_h^{l_h}}{l_h!},$$
$$\Psi_{\alpha_2}(V_0,V_1,(U_h)_{h\in I(\alpha_2)}) = \sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha)} \psi_{\alpha_2,n_0,n_1,(l_h)_{h\in I(\alpha_2)}}\Pi_{h\in I(\alpha)\setminus I(\alpha_2)} \times \delta_{0,l_h}\frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!}\Pi_{h\in I(\alpha)}\frac{U_h^{l_h}}{l_h!}. \quad (95)$$

By remembering (73) of Proposition 8, we deduce that

$$\left\|b_{\alpha_1}(V_0,V_1,U_0)\Psi_{\alpha_2}(V_0,V_1,(U_h)_{h\in I(\alpha_2)})\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}} \leq |b_{\alpha_1}|(\overline{V}_0,\overline{V}_1,\overline{U}_0) \times\left(\sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha_2)}\frac{\left|\psi_{\alpha_2,n_0,n_1,(l_h)_{h\in I(\alpha_2)}}\right|}{\exp(\sigma r_b(\alpha)\rho)} \times\frac{\overline{V}_0^{n_0}\overline{V}_1^{n_1}\Pi_{h\in I(\alpha_2)}\overline{U}_h^{l_h}}{\left(n_0+n_1+\sum_{h\in I(\alpha_2)}l_h+\alpha\right)!}\right). \quad (96)$$

Lemma 14. *One has*

$$\frac{1}{\left(n_0+n_1+\sum_{h\in I(\alpha_2)}l_h+\alpha\right)!} \leq \frac{\alpha_2!}{\alpha!}\frac{1}{\left(n_0+n_1+\sum_{h\in I(\alpha_2)}l_h+\alpha_2\right)!}. \quad (97)$$

Proof. We write

$$\frac{1}{\left(n_0+n_1+\sum_{h\in I(\alpha_2)}l_h+\alpha\right)!} = \frac{\left(n_0+n_1+\sum_{h\in I(\alpha_2)}l_h+\alpha_2\right)!}{\left(n_0+n_1+\sum_{h\in I(\alpha_2)}l_h+\alpha\right)!}\frac{1}{\left(n_0+n_1+\sum_{h\in I(\alpha_2)}l_h+\alpha_2\right)!} \quad (98)$$

and we use the inequality

$$\frac{(a+\alpha_2)!}{(a+\alpha)!}\leq\frac{\alpha_2!}{\alpha!} \quad (99)$$

for all $\alpha=\alpha_1+\alpha_2$ and all $a\in\mathbb{N}$ which follows from (78). This yields the lemma.

Using the fact that $\exp(\sigma r_b(\alpha)\rho) \geq \exp(\sigma r_b(\alpha_2)\rho)$ and gathering the inequalities (96) and (97) yield (94).

Finally, using (93) with (94), one gets

$$\left\|b(V_0,V_1,U_0,W) \times\Psi(V_0,V_1,(U_h)_{h\geq 0},W)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \leq \sum_{\alpha\geq 0}\left(\sum_{\alpha_1+\alpha_2=\alpha}\frac{|b_{\alpha_1}|(\overline{V}_0,\overline{V}_1,\overline{U}_0)}{\alpha_1!} \times\left\|\Psi_{\alpha_2}(V_0,V_1,(U_h)_{h\in I(\alpha_2)})\right\|_{\rho,\alpha_2,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2)}}\right) \times\overline{W}^{\alpha} \quad (100)$$

from which the inequality (91) follows. □

Proposition 15. (1) *Let* $S,k\geq 0$ *be integers such that*

$$S\geq k+1+\max\left(b(d_{1,k}+2)+3,d+1+b(d+d_{1,k}+1)\right). \quad (101)$$

Then, there exists a constant $C_{8.1}>0$ *(which is independent of* $\rho>1$*) such that*

$$\left\|B_{1,k}(V_0,V_1,U_0,W)\partial_W^{-S+k} \times\mathbb{D}_{\mathbf{A}}\Psi(V_0,V_1,(U_h)_{h\geq 0},W)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \leq C_{8.1}\overline{W}^{S-k} \times\left\|\Psi(V_0,V_1,(U_h)_{h\geq 0},W)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \quad (102)$$

for all $\Psi(V_0,V_1,(U_h)_{h\geq 0},W)\in G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$.

(2) *Let* $S,k\geq 0$ *be integers such that*

$$S\geq k+3+b(2+d_{2,k}). \quad (103)$$

Then, there exists a constant $C_{8.2}>0$ *(which is independent of* $\rho>1$*) such that*

$$\left\|B_{2,k}(V_0,V_1,U_0,W)\partial_W^{-S+k} \times\mathbb{D}_{\mathbf{B}}\Psi(V_0,V_1,(U_h)_{h\geq 0},W)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \leq C_{8.2}\overline{W}^{S-k} \times\left\|\Psi(V_0,V_1,(U_h)_{h\geq 0},W)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \quad (104)$$

for all $\Psi(V_0,V_1,(U_h)_{h\geq 0},W)\in G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$.

Proof. (1) We show the first inequality (102). We expand

$$B_{1,k}(V_0,V_1,U_0,W)=\sum_{\alpha\geq 0}B_{1,k,\alpha}(V_0,V_1,U_0)\frac{W^{\alpha}}{\alpha!}. \quad (105)$$

By definition, we have

$$\left\| B_{1,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\mathbb{D}_{\mathbf{A}}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$$
$$= \sum_{\alpha\geq 0}\left\| \sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k} \alpha!\frac{B_{1,k,\alpha_1}\left(V_0,V_1,U_0\right)}{\alpha_1!} \times \left(\sum_{j\in I(\alpha_2-S+k)} \frac{\mathbf{A}_{j,\alpha_2-S+k+1}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k+1)}\right)}{\alpha_2!} \times \left(\partial_{U_j}\Psi_{\alpha_2-S+k}\right)\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}} \overline{W}^{\alpha}. \tag{106}$$

Now, using Lemma 13, we deduce that

$$\left\| B_{1,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\mathbb{D}_{\mathbf{A}}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$$
$$\leq \sum_{\alpha\geq 0}\left(\sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k} \frac{\left|B_{1,k,\alpha_1}\right|\left(\overline{V}_0,\overline{V}_1,\overline{U}_0\right)}{\alpha_1!} \times \left\| \sum_{j\in I(\alpha_2-S+k)} \mathbf{A}_{j,\alpha_2-S+k+1}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k+1)}\right) \times \left(\partial_{U_j}\Psi_{\alpha_2-S+k}\right)\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2)}}\right)\overline{W}^{\alpha}. \tag{107}$$

In the next lemma, we give estimates for the coefficients of the series $\mathbf{A}_{j,\alpha}$ and $|B_{1,k,\alpha}|$.

Lemma 16. (1) *The coefficients of the Taylor series of* $\mathbf{A}_{j,\alpha_2-S+k+1}(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k+1)})$

$$\mathbf{A}_{j,\alpha_2-S+k+1}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k+1)}\right) = \sum_{n_0,n_1,l_h\geq 0,h\in I(\alpha_2-S+k+1)} A_{j,\alpha_2-S+k+1,n_0,n_1,(l_h)_{h\in I(\alpha_2-S+k+1)}} \times \frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!}\Pi_{h\in I(\alpha_2-S+k+1)}\frac{U_h^{l_h}}{l_h!} \tag{108}$$

satisfy the next estimates. There exist constants $a,\delta>0$, *with* $\delta>\overline{\delta}$, $a_p>0$, $0\leq p\leq d$ *such that*

$$\frac{A_{j,\alpha_2-S+k+1,n_0,n_1,(l_h)_{h\in I(\alpha_2-S+k+1)}}}{n_0!n_1!\Pi_{h\in I(\alpha_2-S+k+1)}l_h!} \leq \left(a\nu(\alpha_2-S+k+1)^2(\rho+\delta) + (d+1)\max_{0\leq p\leq d}a_p(\rho+\delta)^d\mathcal{P}_d\left(\alpha_2-S+k\right)\right) \times\left(\delta^{n_0+n_1+\sum_{h\in I(\alpha_2-S+k+1)}l_h}\right)^{-1} \tag{109}$$

for all $\alpha_2\geq S-k$, *all* $j\in I(\alpha_2-S+k)$, *all* $n_0,n_1,l_h\geq 0$, $h\in I(\alpha_2-S+k+1)$ *where* $\mathcal{P}_d$ *is defined in* (115).

(2) *The coefficients of the Taylor series of* $|B_{1,k,\alpha_1}|(\overline{V}_0,\overline{V}_1,\overline{U}_0)$

$$\left|B_{1,k,\alpha_1}\right|\left(\overline{V}_0,\overline{V}_1,\overline{U}_0\right) = \sum_{n_0,n_1,l_0\geq 0} b_{1,k,\alpha_1,n_0,n_1,l_0}\frac{\overline{V}_0^{n_0}}{n_0!}\frac{\overline{V}_1^{n_1}}{n_1!}\frac{\overline{U}_0^{l_0}}{l_0!} \tag{110}$$

satisfy the following inequalities. There exist constants $\delta>\overline{\delta}$, $D_{1,k},\widehat{D}_{1,k}>0$ *with*

$$\frac{b_{1,k,\alpha_1,n_0,n_1,l_0}}{n_0!n_1!l_0!} \leq \frac{D_{1,k}(\rho+\delta)^{d_{1,k}}\alpha_1!\widehat{D}_{1,k}^{\alpha_1}}{\delta^{n_0+n_1+l_0}} \tag{111}$$

for all $\alpha_1\geq 0$, *all* $n_0,n_1,l_0\geq 0$.

Proof. We first treat the estimates for $\mathbf{A}_{j,\alpha}$. From the Cauchy formula in several variables, one can write

$$\begin{aligned}&\left(\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha_2-S+k+1)}\partial_{u_h}^{l_h}A_j\left(v_0,v_1,(u_h)_{h\in I(\alpha_2-S+k+1)}\right)\right)\\&\quad\times\left(n_0!n_1!\Pi_{h\in I(\alpha_2-S+k+1)}l_h!\right)^{-1}\\&\quad=\left(\frac{1}{2i\pi}\right)^{\alpha_2-S+k+4}\\&\qquad\times\int_{C(v_0,\delta)}\int_{C(v_1,\delta)}\Pi_{h\in I(\alpha_2-S+k+1)}\\&\qquad\times\int_{C(u_h,\delta)}A_j\left(\chi_0,\chi_1,(\xi_h)_{h\in I(\alpha_2-S+k+1)}\right)\\&\qquad\qquad\times\left(\left(d\chi_0 d\chi_1\Pi_{h\in I(\alpha_2-S+k+1)}d\xi_h\right)\right.\\&\qquad\qquad\quad\times\left((\chi_0-v_0)^{n_0+1}(\chi_1-v_1)^{n_1+1}\right.\\&\qquad\qquad\qquad\left.\left.\times\Pi_{h\in I(\alpha_2-S+k+1)}(\xi_h-u_h)^{l_h+1}\right)^{-1}\right)\end{aligned}\tag{112}$$

for all $|v_0|<R$, $|v_1|<R$, $|u_h|<\rho$, $h\in I(\alpha_2-S+k+1)$ and $j\in I(\alpha_2-S+k)$ where R is introduced in Section 2.2. The integration is made along positively oriented circles with radius $\delta>0$, $C(v_0,\delta)$,$C(v_1,\delta)$ and $C(u_h,\delta)$ for $h\in I(\alpha_2-S+k+1)$. We choose the real number $\delta>\overline{\delta}$ in such a way that $R+\delta<R'$ where R' is defined in Section 2.1 and $\overline{\delta}$ at the beginning of Section 3.1. Now, since the functions $a(\chi_0,\chi_1)$ and $a_p(\chi_0,\chi_1)$ are holomorphic on $D(0,R')^2$, the number $\nu>0$ (see (10)) can be chosen large enough such that there exist real numbers $a,a_p>0$, for $0\le p\le d$, with

$$\begin{aligned}&\sup_{|\chi_0|<R+\delta,|\chi_1|<R+\delta}\left|\frac{\partial_{\chi_1}^{l_1}a(\chi_0,\chi_1)}{l_1!\nu^{l_1}}\right|\le a,\\&\sup_{|\chi_0|<R+\delta,|\chi_1|<R+\delta}\left|\frac{\partial_{\chi_1}^{l_0}a_p(\chi_0,\chi_1)}{l_0!\nu^{l_0}}\right|\le a_p\end{aligned}\tag{113}$$

for all $l_0,l_1\ge 0$. We recall also that for any integers $k,n\ge 1$, the number of tuples $(b_1,\ldots,b_k)\in\mathbb{N}^k$ such that $b_1+\cdots+b_k=n$ is $(n+k-1)!/((k-1)!n!)$. From these latter statements and the definition of A_j given by (27), we deduce that

$$\begin{aligned}&\left|A_j\left(\chi_0,\chi_1,(\xi_h)_{h\in I(\alpha_2-S+k+1)}\right)\right|\\&\quad\le a\nu(j+1)^2(\rho+\delta)\\&\qquad+(d+1)\max_{0\le p\le d}a_p(\rho+\delta)^d\mathscr{P}_d(j)\end{aligned}\tag{114}$$

(since $\rho>1$), where

$$\mathscr{P}_d(j)=\frac{(j+d)!}{j!}=\Pi_{l=1}^{d}(j+l)\tag{115}$$

is a polynomial of degree d in j with positive coefficients, for all $|\chi_0|<R+\delta$, $|\chi_1|<R+\delta$, $|\xi_h|<\rho+\delta$, $h\in I(\alpha_2-S+k+1)$ and $j\in I(\alpha_2-S+k)$. Gathering (112) and (114) yields (109).

Again, from the Cauchy formula in several variables, one can write

$$\begin{aligned}&\frac{\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\partial_{u_0}^{l_0}b_{1,k,\alpha_1}(v_0,v_1,u_0)}{n_0!n_1!l_0!}\\&\quad=\left(\frac{1}{2i\pi}\right)^{3}\\&\qquad\times\int_{C(v_0,\delta)}\int_{C(v_1,\delta)}\int_{C(u_0,\delta)}b_{1,k,\alpha_1}(\chi_0,\chi_1,\xi_0)\\&\qquad\qquad\times\left((d\chi_0 d\chi_1 d\xi_0)\right.\\&\qquad\qquad\quad\times\left((\chi_0-v_0)^{n_0+1}(\chi_1-v_1)^{n_1+1}\right.\\&\qquad\qquad\qquad\left.\left.\times(\xi_0-u_0)^{l_0+1}\right)^{-1}\right)\end{aligned}\tag{116}$$

for all $|v_0|<R$, $|v_1|<R$, $|u_0|<\rho$. Again, one chooses the real number $\delta>\overline{\delta}$ in such a way that $R+\delta<R'$. By construction of b_{1,k,α_1} in Section 2.2, we know that there exist two constants $D_{1,k},\widehat{D}_{1,k}>0$ such that

$$\left|b_{1,k,\alpha_1}(\chi_0,\chi_1,\xi_0)\right|\le D_{1,k}(\rho+\delta)^{d_{1,k}}\alpha_1!\widehat{D}_{1,k}^{\alpha_1}\tag{117}$$

for all $\alpha_1\ge 0$, all $|\chi_0|<R+\delta$, $|\chi_1|<R+\delta$, $|\xi_0|<\rho+\delta$. Gathering (116) and (117) yields (111).

From (111), we deduce that

$$\begin{aligned}&\frac{\left|B_{1,k,\alpha_1}\right|\left(\overline{V}_0,\overline{V}_1,\overline{U}_0\right)}{\alpha_1!}\\&\quad\le\frac{D_{1,k}(\rho+\delta)^{d_{1,k}}\widehat{D}_{1,k}^{\alpha_1}}{\left(1-\left(\overline{V}_0/\delta\right)\right)\left(1-\left(\overline{V}_1/\delta\right)\right)\left(1-\left(\overline{U}_0/\delta\right)\right)}.\end{aligned}\tag{118}$$

On the other hand, from Proposition 8, we deduce that

$$\begin{aligned}&\left\|\mathbf{A}_{j,\alpha_2-S+k+1}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k+1)}\right)\right.\\&\quad\times\left(\partial_{U_j}\Psi_{\alpha_2-S+k}\right)\\&\quad\left.\times\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2)}}\\&\le\left|\mathbf{A}_{j,\alpha_2-S+k+1}\right|\left(\overline{V}_0,\overline{V}_1,\left(\overline{U}_h\right)_{h\in I(\alpha_2-S+k+1)}\right)\\&\quad\times\left\|\left(\partial_{U_j}\Psi_{\alpha_2-S+k}\right)\right.\\&\qquad\left.\times\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2)}}.\end{aligned}\tag{119}$$

From (109), we deduce that

$$\begin{aligned}
&\left|\mathbf{A}_{j,\alpha_2-S+k+1}\right|\left(\overline{V}_0,\overline{V}_1,\left(\overline{U}_h\right)_{h\in I(\alpha_2-S+k+1)}\right)\\
&\quad\le\left(a\nu(\alpha_2-S+k+1)^2(\rho+\delta)+(d+1)\right.\\
&\qquad\left.\times\max_{0\le p\le d}a_p(\rho+\delta)^d\mathcal{P}_d(\alpha_2-S+k)\right)\\
&\qquad\times\left(\left(1-\frac{\overline{V}_0}{\delta}\right)\left(1-\frac{\overline{V}_1}{\delta}\right)\right.\\
&\qquad\left.\times\Pi_{h\in I(\alpha_2-S+k+1)}\left(1-\frac{\overline{U}_h}{\delta}\right)\right)^{-1}
\end{aligned}\tag{120}$$

for all $j\in I(\alpha_2-S+k)$. Now, from the definition of $\overline{U}_h=\overline{\delta}/(h^b+1)$, where $b>1$, we know that there exists $\kappa>0$ such that

$$\Pi_{h\in I(\alpha)}\left(1-\frac{\overline{U}_h}{\delta}\right)\ge\kappa\tag{121}$$

for all $\alpha\ge0$. From Proposition 10, we have that

$$\begin{aligned}
&\left\|\left(\partial_{U_j}\Psi_{\alpha_2-S+k}\right)\left(V_0,V_1,\left(U_h\right)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2)}}\\
&\quad\le\frac{\exp\left(-\sigma\rho\left((S-k)/(\alpha_2+1)^b\right)\right)}{\overline{U}_j\Pi_{l=1}^{S-k-1}(\alpha_2-l+1)}\\
&\qquad\times\left\|\Psi_{\alpha_2-S+k}\right.\\
&\qquad\quad\left.\times\left(V_0,V_1,\left(U_h\right)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2-S+k,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2-S+k)}}.
\end{aligned}\tag{122}$$

Collecting the estimates (120), (121), and (122), we get from (119) that

$$\begin{aligned}
&\left\|\sum_{j\in I(\alpha_2-S+k)}\mathbf{A}_{j,\alpha_2-S+k+1}\left(V_0,V_1,\left(U_h\right)_{h\in I(\alpha_2-S+k+1)}\right)\right.\\
&\qquad\times\left(\partial_{U_j}\Psi_{\alpha_2-S+k}\right)\\
&\qquad\left.\times\left(V_0,V_1,\left(U_h\right)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2)}}\\
&\quad\le\mathcal{A}_{\rho,\alpha_2}\\
&\qquad\times\left\|\Psi_{\alpha_2-S+k}\right.\\
&\qquad\quad\left.\times\left(V_0,V_1,\left(U_h\right)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2-S+k,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2-S+k)}},
\end{aligned}\tag{123}$$

where

$$\begin{aligned}
&\mathcal{A}_{\rho,\alpha_2}\\
&\quad=\left(\left(a\nu(\alpha_2-S+k+1)^2(\rho+\delta)+(d+1)\right.\right.\\
&\qquad\left.\times\max_{0\le p\le d}a_p(\rho+\delta)^d\mathcal{P}_d(\alpha_2-S+k)\right)\\
&\qquad\left.\times\left(\left(1-\frac{\overline{V}_0}{\delta}\right)\left(1-\frac{\overline{V}_1}{\delta}\right)\kappa\right)^{-1}\right)\\
&\qquad\times\frac{\exp\left(-\sigma\rho\left((S-k)/(\alpha_2+1)^b\right)\right)}{\Pi_{l=1}^{S-k-1}(\alpha_2-l+1)}\frac{1}{\overline{\delta}}\\
&\qquad\times(\alpha_2-S+k+1)\\
&\qquad\times\left((\alpha_2-S+k)^b+1\right).
\end{aligned}\tag{124}$$

Now, we recall the following classical estimates. Let $\delta,m_1,m_2>0$ be positive real numbers, and then

$$\sup_{x\ge0}(x+\delta)^{m_1}\exp(-m_2x)\le\left(\frac{m_1}{m_2}\right)^{m_1}\exp(-m_1)\exp(\delta m_2)\tag{125}$$

holds. Hence,

$$\begin{aligned}
&(\rho+\delta)^{d_{1,k}}\mathcal{A}_{\rho,\alpha_2}\\
&\quad\le\left(\left(a\nu(\alpha_2-S+k+1)^2(\alpha_2+1)^{b(1+d_{1,k})}\right.\right.\\
&\qquad\left.\times\left(\frac{\exp(-1)(1+d_{1,k})}{\sigma(S-k)}\right)^{1+d_{1,k}}\exp(\delta\sigma(S-k))\right)\\
&\qquad\times\left(\left(1-\frac{\overline{V}_0}{\delta}\right)\left(1-\frac{\overline{V}_1}{\delta}\right)\kappa\right)^{-1}\\
&\qquad+\left((d+1)\max_{0\le p\le d}a_p\mathcal{P}_d(\alpha_2-S+k)(\alpha_2+1)^{b(d+d_{1,k})}\right.\\
&\qquad\left.\times\left(\frac{(d+d_{1,k})\exp(-1)}{\sigma(S-k)}\right)^{d+d_{1,k}}\exp(\delta\sigma(S-k))\right)\\
&\qquad\left.\times\left(\left(1-\frac{\overline{V}_0}{\delta}\right)\left(1-\frac{\overline{V}_1}{\delta}\right)\kappa\right)^{-1}\right)\\
&\qquad\times\frac{(\alpha_2-S+k+1)\left((\alpha_2-S+k)^b+1\right)}{\overline{\delta}\Pi_{l=1}^{S-k-1}(\alpha_2-l+1)}.
\end{aligned}\tag{126}$$

Under the assumptions (101), one gets a constant $\widetilde{C}_{8.1}>0$ (depending on $a,\max_{0\le p\le d}a_p,\delta,\overline{\delta},b,d,\ d_{1,k},\sigma,\nu,S,k,\kappa,$

$\overline{V}_0, \overline{V}_1$) such that

$$(\rho+\delta)^{d_{1,k}} \mathscr{A}_{\rho,\alpha_2} \le \widetilde{C}_{8.1} \tag{127}$$

for all $\rho \ge 0$, all $\alpha_2 \ge S-k$. Finally, gathering (107), (118), (123), and (127), one gets that

$$\begin{aligned}
&\left\| B_{1,k}(V_0,V_1,U_0,W)\,\partial_W^{-S+k}\mathbb{D}_{\mathbf{A}}\Psi\left(V_0,V_1,(U_h)_{h\ge 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\ge0},\overline{W})} \\
&\le \sum_{\alpha\ge 0}\Bigg(\sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\ge S-k}\frac{D_{1,k}}{\left(1-(\overline{V}_0/\delta)\right)\left(1-(\overline{V}_1/\delta)\right)\left(1-(\overline{U}_0/\delta)\right)}\widehat{D}_{1,k}^{\alpha_1} \\
&\qquad\qquad \times \widetilde{C}_{8.1}\left\|\Psi_{\alpha_2-S+k}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2-S+k,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2-S+k)}}\Bigg) \\
&\quad \times \overline{W}^{\alpha_1+\alpha_2-S+k}\,\overline{W}^{S-k} \\
&= \frac{\widetilde{C}_{8.1}D_{1,k}}{\left(1-(\overline{V}_0/\delta)\right)\left(1-(\overline{V}_1/\delta)\right)\left(1-(\overline{U}_0/\delta)\right)\left(1-\widehat{D}_{1,k}\overline{W}\right)} \times \overline{W}^{S-k}\left\|\Psi\left(V_0,V_1,(U_h)_{h\ge0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\ge0},\overline{W})}
\end{aligned} \tag{128}$$

provided that $\overline{V}_0 < \delta$, $\overline{V}_1 < \delta$, $\overline{U}_0 < \delta$, and $\overline{W} < 1/\widehat{D}_{1,k}$, which yields (102).

(2) Now, we turn towards the estimates (104) which will follow from the same arguments as in (1). Using Lemma 13, we get that

$$\begin{aligned}
&\Big\| B_{2,k}(V_0,V_1,U_0,W)\,\partial_W^{-S+k} \\
&\quad\times \mathbb{D}_{\mathbf{B}}\Psi\left(V_0,V_1,(U_h)_{h\ge0},W\right)\Big\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\ge0},\overline{W})} \\
&\le \sum_{\alpha\ge0}\Bigg(\sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\ge S-k}\frac{\left|B_{2,k,\alpha_1}\right|\left(\overline{V}_0,\overline{V}_1,\overline{U}_0\right)}{\alpha_1!} \\
&\qquad \times \Bigg\|\sum_{j\in I(\alpha_2-S+k)}\mathbf{B}_{j,\alpha_2-S+k+1}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k+1)}\right) \\
&\qquad\quad \times\left(\partial_{U_j}\Psi_{\alpha_2-S+k}\right) \\
&\qquad\quad \times\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\Bigg\|_{\rho,\alpha_2,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2)}}\Bigg) \\
&\quad \times \overline{W}^{\alpha}.
\end{aligned} \tag{129}$$

In the next lemma, we give estimates for the coefficients of the series $\mathbf{B}_{j,\alpha}$ and $|B_{2,k,\alpha}|$.

Lemma 17. (1) *The coefficients of the Taylor series of* $\mathbf{B}_{j,\alpha_2-S+k+1}(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k+1)})$

$$\begin{aligned}
&\mathbf{B}_{j,\alpha_2-S+k+1}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k+1)}\right) \\
&\quad = \sum_{n_0,n_1,l_h\ge0,h\in I(\alpha_2-S+k+1)} B_{j,\alpha_2-S+k+1,n_0,n_1,(l_h)_{h\in I(\alpha_2-S+k+1)}} \\
&\qquad \times \frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!}\Pi_{h\in I(\alpha_2-S+k+1)}\frac{U_h^{l_h}}{l_h!}
\end{aligned} \tag{130}$$

satisfy the next estimates. There exist a constant $\delta > 0$, with $\delta > \overline{\delta}$ such that

$$\begin{aligned}
&\frac{B_{j,\alpha_2-S+k+1,n_0,n_1,(l_h)_{h\in I(\alpha_2-S+k+1)}}}{n_0!n_1!\Pi_{h\in I(\alpha_2-S+k+1)}l_h!} \\
&\quad \le \frac{\nu(\alpha_2-S+k+1)(\rho+\delta)}{\delta^{n_0+n_1+\sum_{h\in I(\alpha_2-S+k+1)}l_h}}
\end{aligned} \tag{131}$$

for all $\alpha_2 \ge S-k$, all $j \in I(\alpha_2-S+k)$, all $n_0,n_1,l_h \ge 0$, $h \in I(\alpha_2-S+k+1)$.

(2) *The coefficients of the Taylor series of $|B_{2,k,\alpha_1}|(\overline{V}_0,\overline{V}_1,\overline{U}_0)$*

$$\begin{aligned}
&\left|B_{2,k,\alpha_1}\right|\left(\overline{V}_0,\overline{V}_1,\overline{U}_0\right) \\
&\quad = \sum_{n_0,n_1,l_0\ge0} b_{2,k,\alpha_1,n_0,n_1,l_0}\frac{\overline{V}_0^{n_0}}{n_0!}\frac{\overline{V}_1^{n_1}}{n_1!}\frac{\overline{U}_0^{l_0}}{l_0!}
\end{aligned} \tag{132}$$

satisfy the following inequalities. There exist constants $\delta > \overline{\delta}$, $D_{2,k}, \widehat{D}_{2,k} > 0$ with

$$\frac{b_{2,k,\alpha_1,n_0,n_1,l_0}}{n_0!n_1!l_0!} \le \frac{D_{2,k}(\rho+\delta)^{d_{2,k}}\alpha_1!\widehat{D}_{2,k}^{\alpha_1}}{\delta^{n_0+n_1+l_0}} \tag{133}$$

for all $\alpha_1 \ge 0$, all $n_0,n_1,l_0 \ge 0$.

Proof. (1) From the Cauchy formula in several variables, one can check that

$$\begin{aligned}
&\Big(\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\Pi_{h\in I(\alpha_2-S+k+1)}\partial_{u_h}^{l_h} \\
&\quad \times B_j\left(v_0, v_1, (u_h)_{h\in I(\alpha_2-S+k+1)}\right)\Big) \\
&\quad \times \left(n_0! n_1! \Pi_{h\in I(\alpha_2-S+k+1)} l_h!\right)^{-1} \\
&= \left(\frac{1}{2i\pi}\right)^{\alpha_2-S+k+4} \\
&\quad \times \int_{C(v_0,\delta)}\int_{C(v_1,\delta)} \Pi_{h\in I(\alpha_2-S+k+1)} \\
&\quad \times \int_{C(u_h,\delta)} B_j\left(\chi_0, \chi_1, (\xi_h)_{h\in I(\alpha_2-S+k+1)}\right) \\
&\qquad \times \Big(\left(d\chi_0 d\chi_1 \Pi_{h\in I(\alpha_2-S+k+1)} d\xi_h\right) \\
&\qquad \times \Big((\chi_0 - v_0)^{n_0+1}(\chi_1 - v_1)^{n_1+1} \\
&\qquad \times \Pi_{h\in I(\alpha_2-S+k+1)}(\xi_h - u_h)^{l_h+1}\Big)^{-1}\Big)
\end{aligned} \tag{134}$$

for all $|v_0| < R$, $|v_1| < R$, $|u_h| < \rho$, $h \in I(\alpha_2 - S + k + 1)$, and $j \in I(\alpha_2 - S + k)$. We choose the real number $\delta > \overline{\delta}$ in such a way that $R + \delta < R'$. From the definition given in (28), we get that

$$\left|B_j\left(\chi_0, \chi_1, (\xi_h)_{h\in I(\alpha_2-S+k+1)}\right)\right| \le \nu\left(j+1\right)\left(\rho+\delta\right) \tag{135}$$

for all $|\chi_0| < R+\delta$, $|\chi_1| < R+\delta$, $|\xi_h| < \rho+\delta$, $h \in I(\alpha_2-S+k+1)$, and $j \in I(\alpha_2 - S + k)$. Gathering (134) and (135) yields (131).

(2) The proof is exactly the same as (2) in Lemma 16.

From (133), we deduce that

$$\begin{aligned}
&\frac{\left|B_{2,k,\alpha_1}\right|\left(\overline{V}_0, \overline{V}_1, \overline{U}_0\right)}{\alpha_1!} \\
&\quad \le \frac{D_{2,k}(\rho+\delta)^{d_{2,k}}\widehat{D}_{2,k}^{\alpha_1}}{\left(1-\left(\overline{V}_0/\delta\right)\right)\left(1-\left(\overline{V}_1/\delta\right)\right)\left(1-\left(\overline{U}_0/\delta\right)\right)}.
\end{aligned} \tag{136}$$

Using Propositions 8 and 10, we deduce that

$$\begin{aligned}
&\Bigg\|\sum_{j\in I(\alpha_2-S+k)} \mathbf{B}_{j,\alpha_2-S+k+1}\left(V_0, V_1, (U_h)_{h\in I(\alpha_2-S+k+1)}\right) \\
&\qquad \times \left(\partial_{U_j}\Psi_{\alpha_2-S+k}\right) \\
&\qquad \times \left(V_0, V_1, (U_h)_{h\in I(\alpha_2-S+k)}\right)\Bigg\|_{\rho,\alpha_2,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2)}} \\
&\quad \le \mathscr{B}_{\rho,\alpha_2} \\
&\quad \times \Big\|\Psi_{\alpha_2-S+k} \\
&\qquad \times \left(V_0, V_1, (U_h)_{h\in I(\alpha_2-S+k)}\right)\Big\|_{\rho,\alpha_2-S+k,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2-S+k)}},
\end{aligned} \tag{137}$$

where

$$\begin{aligned}
\mathscr{B}_{\rho,\alpha_2} &= \frac{\nu\left(\alpha_2 - S + k + 1\right)\left(\rho+\delta\right)}{\left(1-\left(\overline{V}_0/\delta\right)\right)\left(1-\left(\overline{V}_1/\delta\right)\right)\kappa} \\
&\quad \times \frac{\exp\left(-\sigma\rho\left((S-k)/(\alpha_2+1)^b\right)\right)}{\Pi_{l=1}^{S-k-1}\left(\alpha_2 - l + 1\right)} \\
&\quad \times \frac{1}{\delta}\left(\alpha_2 - S + k + 1\right)\left(\left(\alpha_2 - S + k\right)^b + 1\right)
\end{aligned} \tag{138}$$

and where κ is introduced in (121). Using the estimates (125), we get

$$\begin{aligned}
&(\rho+\delta)^{d_{2,k}}\mathscr{B}_{\rho,\alpha_2} \\
&\le \Bigg(\Bigg(\nu\left(\alpha_2 - S + k + 1\right)\left(\alpha_2+1\right)^{b(1+d_{2,k})} \\
&\qquad \times \left(\frac{\exp(-1)\left(1+d_{2,k}\right)}{\sigma\left(S-k\right)}\right)^{1+d_{2,k}} \exp\left(\delta\sigma\left(S-k\right)\right)\Bigg) \\
&\quad \times \left(\left(1-\frac{\overline{V}_0}{\delta}\right)\left(1-\frac{\overline{V}_1}{\delta}\right)\kappa\right)^{-1}\Bigg) \\
&\quad \times \frac{\left(\alpha_2 - S + k + 1\right)\left(\left(\alpha_2 - S + k\right)^b + 1\right)}{\overline{\delta}\Pi_{l=1}^{S-k-1}\left(\alpha_2 - l + 1\right)}.
\end{aligned} \tag{139}$$

Under the assumptions (103), one gets a constant $\widetilde{C}_{8.2} > 0$ (depending on $\delta, \overline{\delta}, b, d_{2,k}, \sigma, \nu, S, k, \kappa, \overline{V}_0, \overline{V}_1$) such that

$$(\rho+\delta)^{d_{2,k}}\mathscr{B}_{\rho,\alpha_2} \le \widetilde{C}_{8.2} \tag{140}$$

for all $\rho \ge 0$, all $\alpha_2 \ge S - k$. Finally, gathering (129), (136), (137), and (140), we get (104). □

Proposition 18. (1) *Let $S, k \ge 0$ be integers such that*

$$S \ge k + 1 + b\max\left(d_{1,k}, d_{2,k}\right). \tag{141}$$

Then, for $m \in \{0, 1\}$, there exists a constant $C_9 > 0$ (which is independent of $\rho > 1$) such that

$$\begin{aligned}
&\Big\|B_{m+1,k}\left(V_0, V_1, U_0, W\right)\partial_W^{-S+k}\partial_{V_m} \\
&\quad \times \Psi\left(V_0, V_1, (U_h)_{h\ge 0}, W\right)\Big\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\ge 0},\overline{W})} \\
&\le C_9\overline{W}^{S-k} \\
&\quad \times \left\|\Psi\left(V_0, V_1, (U_h)_{h\ge 0}, W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\ge 0},\overline{W})}
\end{aligned} \tag{142}$$

for all $\Psi(V_0, V_1, (U_h)_{h\ge 0}, W) \in G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\ge 0},\overline{W})}$.

(2) *Let* $S, k \geq 0$ *be integers such that*

$$S \geq k + bd_{3,k}. \tag{143}$$

Then, there exists a constant $C_{9.1} > 0$ *(which is independent of* $\rho > 1$*) such that*

$$\begin{aligned} &\left\| B_{3,k}\left(V_0, V_1, U_0, W\right) \partial_W^{-S+k} \right. \\ &\quad \left. \times \Psi\left(V_0, V_1, \left(U_h\right)_{h\geq 0}, W\right) \right\|_{(\rho, \overline{V}_0, \overline{V}_1, (\overline{U}_h)_{h\geq 0}, \overline{W})} \\ &\leq C_{9.1} \overline{W}^{S-k} \\ &\quad \times \left\| \Psi\left(V_0, V_1, \left(U_h\right)_{h\geq 0}, W\right) \right\|_{(\rho, \overline{V}_0, \overline{V}_1, (\overline{U}_h)_{h\geq 0}, \overline{W})} \end{aligned} \tag{144}$$

for all $\Psi(V_0, V_1, (U_h)_{h\geq 0}, W) \in G_{(\rho, \overline{V}_0, \overline{V}_1, (\overline{U}_h)_{h\geq 0}, \overline{W})}$.

Proof. (1) We expand

$$B_{m+1,k}\left(V_0, V_1, U_0, W\right) = \sum_{\alpha \geq 0} B_{m+1,k,\alpha}\left(V_0, V_1, U_0\right) \frac{W^\alpha}{\alpha!}. \tag{145}$$

By definition, we have

$$\begin{aligned} &\left\| B_{m+1,k}\left(V_0, V_1, U_0, W\right) \partial_W^{-S+k} \partial_{V_m} \right. \\ &\quad \left. \times \Psi\left(V_0, V_1, \left(U_h\right)_{h\geq 0}, W\right) \right\|_{(\rho, \overline{V}_0, \overline{V}_1, (\overline{U}_h)_{h\geq 0}, \overline{W})} \\ &= \sum_{\alpha \geq 0} \left\| \sum_{\alpha_1 + \alpha_2 = \alpha, \alpha_2 \geq S-k} \alpha! \frac{B_{m+1,k,\alpha_1}\left(V_0, V_1, U_0\right)}{\alpha_1!} \right. \\ &\qquad \times \left(\left(\partial_{V_m} \Psi_{\alpha_2 - S+k} \right. \right. \\ &\qquad \quad \left. \times \left(V_0, V_1, \left(U_h\right)_{h \in I(\alpha_2 - S+k)} \right) \right) \\ &\qquad \quad \left. \left. \times \left(\alpha_2!\right)^{-1} \right) \right\|_{\rho, \alpha, \overline{V}_0, \overline{V}_1, (\overline{U}_h)_{h\in I(\alpha)}} \\ &\quad \times \overline{W}^\alpha. \end{aligned} \tag{146}$$

Now, using Lemma 13, we deduce that

$$\begin{aligned} &\left\| B_{m+1,k}\left(V_0, V_1, U_0, W\right) \partial_W^{-S+k} \partial_{V_m} \right. \\ &\quad \left. \times \Psi\left(V_0, V_1, \left(U_h\right)_{h\geq 0}, W\right) \right\|_{(\rho, \overline{V}_0, \overline{V}_1, (\overline{U}_h)_{h\geq 0}, \overline{W})} \\ &\leq \sum_{\alpha \geq 0} \left(\sum_{\alpha_1 + \alpha_2 = \alpha, \alpha_2 \geq S-k} \frac{\left| B_{m+1,k,\alpha_1} \right| \left(\overline{V}_0, \overline{V}_1, \overline{U}_0 \right)}{\alpha_1!} \right. \\ &\qquad \times \left\| \left(\partial_{V_m} \Psi_{\alpha_2 - S+k} \right) \right. \\ &\qquad \quad \left. \left. \times \left(V_0, V_1, \left(U_h\right)_{h \in I(\alpha_2 - S+k)} \right) \right\|_{\rho, \alpha_2, \overline{V}_0, \overline{V}_1, (\overline{U}_h)_{h\in I(\alpha_2)}} \right) \\ &\quad \times \overline{W}^\alpha. \end{aligned} \tag{147}$$

From Proposition 10, we know that

$$\begin{aligned} &\left\| \left(\partial_{V_m} \Psi_{\alpha_2 - S+k} \right) \right. \\ &\quad \left. \times \left(V_0, V_1, \left(U_h\right)_{h \in I(\alpha_2 - S+k)} \right) \right\|_{\rho, \alpha_2, \overline{V}_0, \overline{V}_1, (\overline{U}_h)_{h\in I(\alpha_2)}} \\ &\quad \leq \frac{\exp\left(-\sigma\rho\left((S-k)/(\alpha_2+1)^b\right)\right)}{\overline{V}_m \Pi_{l=1}^{S-k-1}\left(\alpha_2 - l + 1\right)} \\ &\qquad \times \left\| \Psi_{\alpha_2 - S+k} \right. \\ &\qquad \quad \left. \times \left(V_0, V_1, \left(U_h\right)_{h \in I(\alpha_2 - S+k)} \right) \right\|_{\rho, \alpha_2 - S+k, \overline{V}_0, \overline{V}_1, (\overline{U}_h)_{h\in I(\alpha_2 - S+k)}}. \end{aligned} \tag{148}$$

From (118), (136), (147), and (148), we get that

$$\begin{aligned} &\left\| B_{m+1,k}\left(V_0, V_1, U_0, W\right) \partial_W^{-S+k} \partial_{V_m} \right. \\ &\quad \left. \times \Psi\left(V_0, V_1, \left(U_h\right)_{h\geq 0}, W\right) \right\|_{(\rho, \overline{V}_0, \overline{V}_1, (\overline{U}_h)_{h\geq 0}, \overline{W})} \\ &\leq \sum_{\alpha \geq 0} \left(\sum_{\alpha_1 + \alpha_2 = \alpha, \alpha_2 \geq S-k} \widehat{D}_{m+1,k}^{\alpha_1} \times \mathscr{C}_{\rho, \alpha_2} \right. \\ &\qquad \times \left\| \Psi_{\alpha_2 - S+k} \right. \\ &\qquad \quad \left. \left. \times \left(V_0, V_1, \left(U_h\right)_{h \in I(\alpha_2 - S+k)} \right) \right\|_{\rho, \alpha_2 - S+k, \overline{V}_0, \overline{V}_1, (\overline{U}_h)_{h\in I(\alpha_2 - S+k)}} \right) \\ &\quad \times \overline{W}^\alpha, \end{aligned} \tag{149}$$

where

$$\begin{aligned} \mathscr{C}_{\rho, \alpha_2} &= \frac{D_{m+1,k}(\rho+\delta)^{d_{m+1,k}}}{\left(1 - \left(\overline{V}_0/\delta\right)\right)\left(1 - \left(\overline{V}_1/\delta\right)\right)\left(1 - \left(\overline{U}_0/\delta\right)\right)} \\ &\quad \times \frac{\exp\left(-\sigma\rho\left((S-k)/(\alpha_2+1)^b\right)\right)}{\overline{V}_m \Pi_{l=1}^{S-k-1}\left(\alpha_2 - l + 1\right)}. \end{aligned} \tag{150}$$

Using the estimates (125), we deduce that

$$\begin{aligned} &\mathscr{C}_{\rho, \alpha_2} \\ &\quad \leq \frac{D_{m+1,k}\left(d_{m+1,k}\exp(-1)/\sigma(S-k)\right)^{d_{m+1,k}} \exp(\delta\sigma(S-k))}{\left(1 - \left(\overline{V}_0/\delta\right)\right)\left(1 - \left(\overline{V}_1/\delta\right)\right)\left(1 - \left(\overline{U}_0/\delta\right)\right)} \\ &\qquad \times \frac{\left(\alpha_2+1\right)^{bd_{m+1,k}}}{\overline{V}_m \Pi_{l=1}^{S-k-1}\left(\alpha_2 - l + 1\right)}. \end{aligned} \tag{151}$$

Under the assumption (141), we get a constant $\widetilde{C}_9 > 0$ (depending on $D_{m+1,k}, d_{m+1,k}, S, k, \delta, \sigma, \overline{V}_0, \overline{V}_1, \overline{U}_0, b$) such that

$$\mathscr{C}_{\rho, \alpha_2} \leq \widetilde{C}_9 \tag{152}$$

for all $\rho > 1$, all $\alpha_2 \geq S - k$. Finally, collecting (149) and (152), we get

$$\begin{aligned}
&\left\|B_{m+1,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\partial_{V_m}\Psi(V_0,V_1,(U_h)_{h\geq 0},W)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}\\
&\leq \sum_{\alpha\geq 0}\left(\sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k}\widehat{D}_{m+1,k}^{\alpha_1}\times\widetilde{C}_9\left\|\Psi_{\alpha_2-S+k}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2-S+k,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2-S+k)}}\right)\overline{W}^{\alpha_1+\alpha_2-S+k}\overline{W}^{S-k}\\
&=\frac{\widetilde{C}_9}{1-\widehat{D}_{m+1,k}\overline{W}}\overline{W}^{S-k}\left\|\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}
\end{aligned} \tag{153}$$

which yields (142).

(2) We expand

$$B_{3,k}\left(V_0,V_1,U_0,W\right)=\sum_{\alpha\geq 0}B_{3,k,\alpha}\left(V_0,V_1,U_0\right)\frac{W^\alpha}{\alpha!}. \tag{154}$$

By definition, we have

$$\begin{aligned}
&\left\|B_{3,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\right.\\
&\quad\left.\times\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}\\
&=\sum_{\alpha\geq 0}\left\|\sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k}\alpha!\frac{B_{3,k,\alpha_1}\left(V_0,V_1,U_0\right)}{\alpha_1!}\right.\\
&\qquad\times\left(\left(\Psi_{\alpha_2-S+k}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\right)\right.\\
&\qquad\left.\left.\times\left(\alpha_2!\right)^{-1}\right)\right\|_{\rho,\alpha,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha)}}\\
&\quad\times\overline{W}^\alpha.
\end{aligned} \tag{155}$$

Now, using Lemma 13, we deduce that

$$\begin{aligned}
&\left\|B_{3,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\right.\\
&\quad\left.\times\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}\\
&=\sum_{\alpha\geq 0}\left(\sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k}\frac{\left|B_{3,k,\alpha_1}\right|\left(\overline{V}_0,\overline{V}_1,\overline{U}_0\right)}{\alpha_1!}\right.\\
&\qquad\times\left\|\Psi_{\alpha_2-S+k}\right.\\
&\qquad\quad\left.\left.\times\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2)}}\right)\\
&\quad\times\overline{W}^\alpha.
\end{aligned} \tag{156}$$

From Proposition 10, we know that

$$\begin{aligned}
&\left\|\Psi_{\alpha_2-S+k}(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)})\right\|_{\rho,\alpha_2,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2)}}\\
&\quad\leq\frac{\exp\left(-\sigma\rho\left(\left(S-k\right)/\left(\alpha_2+1\right)^b\right)\right)}{\Pi_{l=1}^{S-k}\left(\alpha_2-l+1\right)}\\
&\qquad\times\left\|\Psi_{\alpha_2-S+k}\right.\\
&\qquad\quad\left.\times\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2-S+k,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2-S+k)}}.
\end{aligned} \tag{157}$$

On the other hand, the coefficients of the Taylor series of $|B_{3,k,\alpha_1}|(\overline{V}_0,\overline{V}_1,\overline{U}_0)$

$$\begin{aligned}
&\left|B_{3,k,\alpha_1}\right|\left(\overline{V}_0,\overline{V}_1,\overline{U}_0\right)\\
&\quad=\sum_{n_0,n_1,l_0\geq 0}b_{3,k,\alpha_1,n_0,n_1,l_0}\frac{\overline{V}_0^{n_0}}{n_0!}\frac{\overline{V}_1^{n_1}}{n_1!}\frac{\overline{U}_0^{l_0}}{l_0!}
\end{aligned} \tag{158}$$

satisfy the following inequalities. There exist constants $\delta>\overline{\delta}$, $D_{3,k},\widehat{D}_{3,k}>0$ with

$$\frac{b_{3,k,\alpha_1,n_0,n_1,l_0}}{n_0!n_1!l_0!}\leq\frac{D_{3,k}(\rho+\delta)^{d_{3,k}}\alpha_1!\widehat{D}_{3,k}^{\alpha_1}}{\delta^{n_0+n_1+l_0}} \tag{159}$$

for all $\alpha_1\geq 0$, all $n_0,n_1,l_0\geq 0$. The proof copies (2) from Lemma 16. From (159), we deduce that

$$\begin{aligned}
&\frac{\left|B_{3,k,\alpha_1}\right|\left(\overline{V}_0,\overline{V}_1,\overline{U}_0\right)}{\alpha_1!}\\
&\quad\leq\frac{D_{3,k}(\rho+\delta)^{d_{3,k}}\widehat{D}_{3,k}^{\alpha_1}}{\left(1-\left(\overline{V}_0/\delta\right)\right)\left(1-\left(\overline{V}_1/\delta\right)\right)\left(1-\left(\overline{U}_0/\delta\right)\right)}.
\end{aligned} \tag{160}$$

From (160), (156), and (157), we get that

$$\left\|B_{3,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$$
$$\leq \sum_{\alpha\geq 0}\left(\sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k}\widehat{D}_{3,k}^{\alpha_1}\times\mathscr{D}_{\rho,\alpha_2}\left\|\Psi_{\alpha_2-S+k}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2-S+k,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2-S+k)}}\right)\overline{W}^{\alpha}, \tag{161}$$

where

$$\mathscr{D}_{\rho,\alpha_2}=\frac{D_{3,k}(\rho+\delta)^{d_{3,k}}}{\left(1-\left(\overline{V}_0/\delta\right)\right)\left(1-\left(\overline{V}_1/\delta\right)\right)\left(1-\left(\overline{U}_0/\delta\right)\right)}\times\frac{\exp\left(-\sigma\rho\left((S-k)/(\alpha_2+1)^b\right)\right)}{\Pi_{l=1}^{S-k}\left(\alpha_2-l+1\right)}. \tag{162}$$

Using the estimates (125), we deduce that

$$\mathscr{D}_{\rho,\alpha_2}\leq\frac{D_{3,k}(d_{3,k}\exp(-1)/\sigma(S-k))^{d_{3,k}}\exp(\delta\sigma(S-k))}{\left(1-\left(\overline{V}_0/\delta\right)\right)\left(1-\left(\overline{V}_1/\delta\right)\right)\left(1-\left(\overline{U}_0/\delta\right)\right)}\times\frac{(\alpha_2+1)^{bd_{3,k}}}{\Pi_{l=1}^{S-k}\left(\alpha_2-l+1\right)}. \tag{163}$$

Under the assumption (143), we get a constant $\widetilde{C}_{9.1}>0$ (depending on $D_{3,k},d_{3,k},S,k,\delta,\sigma,\overline{V}_0,\overline{V}_1,\overline{U}_0,b$) such that

$$\mathscr{D}_{\rho,\alpha_2}\leq\widetilde{C}_{9.1} \tag{164}$$

for all $\rho>1$, all $\alpha_2\geq S-k$. Finally, collecting (161) and (164), we get

$$\left\|B_{3,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$$
$$\leq \sum_{\alpha\geq 0}\left(\sum_{\alpha_1+\alpha_2=\alpha,\alpha_2\geq S-k}\widehat{D}_{3,k}^{\alpha_1}\times\widetilde{C}_{9.1}\left\|\Psi_{\alpha_2-S+k}\left(V_0,V_1,(U_h)_{h\in I(\alpha_2-S+k)}\right)\right\|_{\rho,\alpha_2-S+k,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\in I(\alpha_2-S+k)}}\overline{W}^{\alpha_1+\alpha_2-S+k}\overline{W}^{S-k}\right) \tag{165}$$
$$=\frac{\widetilde{C}_{9.1}}{1-\widehat{D}_{3,k}\overline{W}}\overline{W}^{S-k}\left\|\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$$

which yields (144). □

3.2. A Functional Partial Differential Equation in the Banach Spaces of Infinitely Many Variables $G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U})_{h\geq 0},\overline{W})}$. In the next proposition, we solve a functional fixed point equation within the Banach spaces of formal series introduced in the previous subsection.

Proposition 19. *One makes the following assumptions:*

$$S\geq k+1+\max\left(b\left(d_{1,k}+2\right)+3,d+1+b\left(d+d_{1,k}+1\right)\right),$$
$$S\geq k+3+b\left(2+d_{2,k}\right),$$
$$S\geq k+1+b\max\left(d_{1,k},d_{2,k}\right),\qquad S\geq k+bd_{3,k} \tag{166}$$

for all $k\in\mathscr{S}$. Then, for given $\overline{V}_0,\overline{V}_1,\overline{\delta}>0$, there exists $\overline{W}>0$ (independent of $\rho>1$) such that, for all $\widetilde{I}(V_0,V_1,(U_h)_{h\geq 0},W)\in G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$, the functional equation

$$\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)=\sum_{k\in\mathscr{S}}B_{1,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\partial_{V_0}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)$$
$$+B_{1,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\mathbb{D}_{\mathbf{A}}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)$$
$$+\sum_{k\in\mathscr{S}}B_{2,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\partial_{V_1}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)$$
$$+B_{2,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\mathbb{D}_{\mathbf{B}}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)$$
$$+\sum_{k\in\mathscr{S}}B_{3,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)$$
$$+\widetilde{I}\left(V_0,V_1,(U_h)_{h\geq 0},W\right) \tag{167}$$

has a unique solution $\Psi(V_0,V_1,(U_h)_{h\geq 0},W) \in G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$. *Moreover, one has that*

$$\left\|\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \leq 2\left\|\tilde{I}\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}. \tag{168}$$

Proof. We consider the map $\mathfrak{M}$ from the space $\mathbb{G}[[V_0,V_1,(U_h)_{h\geq 0},W]]$ of formal series (introduced in Definition 4) into itself defined as follows:

$$\begin{aligned}\mathfrak{M}&\left(\Delta\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right)\\ &=\sum_{k\in\mathcal{S}}B_{1,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\partial_{V_0}\Delta\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\\ &\quad+B_{1,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\mathbb{D}_{\mathbf{A}}\Delta\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\\ &\quad+\sum_{k\in\mathcal{S}}B_{2,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\partial_{V_1}\Delta\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\\ &\quad+B_{2,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\mathbb{D}_{\mathbf{B}}\Delta\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\\ &\quad+\sum_{k\in\mathcal{S}}B_{3,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\Delta\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\end{aligned} \tag{169}$$

for all $\Delta(V_0,V_1,(U_h)_{h\geq 0},W) \in \mathbb{G}[[V_0,V_1,(U_h)_{h\geq 0},W]]$.

In order to prove the proposition, we need the following lemma.

Lemma 20. *Let* id *be the identity map* $x \mapsto x$ *from* $\mathbb{G}[[V_0,V_1,(U_h)_{h\geq 0},W]]$ *into itself. Then, for a well-chosen* $\overline{W} > 0$, *the map* $\mathrm{id}-\mathfrak{M}$ *defines an invertible map such that* $(\mathrm{id}-\mathfrak{M})^{-1}$ *is defined from* $G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$ *into itself. Moreover, one has that*

$$\left\|(\mathrm{id}-\mathfrak{M})^{-1}\left(\Xi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \leq 2\left\|\Xi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \tag{170}$$

for all $\Xi(V_0,V_1,(U_h)_{h\geq 0},W) \in G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$.

Proof. Taking care of the constraints (166), we get from Propositions 15 and 18 a constant $C_{10} > 0$ (depending on the constants introduced above and also on the aforementioned propositions: $a, \max_{0\leq p\leq d}a_p$, $\delta,\overline{\delta}$, b,d, $\max_{k\in\mathcal{S}}d_{1,k}$, $\max_{k\in\mathcal{S}}D_{1,k}$, $\max_{k\in\mathcal{S}}d_{2,k}, \max_{k\in\mathcal{S}}D_{2,k}$, $\max_{k\in\mathcal{S}}d_{3,k}$, $\max_{k\in\mathcal{S}}D_{3,k}$, σ,ν, $S,\mathcal{S}$, and $\kappa,\overline{V}_0,\overline{V}_1$ but independent of $\rho > 1$) such that

$$\begin{aligned}\left\|\mathfrak{M}\left(\Delta\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} &\leq C_{10}\left(\sum_{k\in\mathcal{S}}\overline{W}^{S-k}\right)\\ &\quad\times\left\|\Delta\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}\end{aligned} \tag{171}$$

for all $\Delta(V_0,V_1,(U_h)_{h\geq 0},W) \in G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$ with $0 \leq \overline{W} \leq \min_{m\in\{0,1,2\},k\in\mathcal{S}}1/(2\widehat{D}_{m+1,k})$. Since $S > k$ for all $k \in \mathcal{S}$, we can choose $\overline{W} > 0$ such that

$$C_{10}\sum_{k\in\mathcal{S}}\overline{W}^{S-k} < \frac{1}{2} \tag{172}$$

together with $\overline{W} \leq \min_{m\in\{0,1,2\},k\in\mathcal{S}}1/(2\widehat{D}_{m+1,k})$. We deduce that

$$\left\|\mathfrak{M}\left(\Delta\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \leq \frac{1}{2}\left\|\Delta\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \tag{173}$$

for all $\Delta(V_0,V_1,(U_h)_{h\geq 0},W) \in G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$. This yields the estimates (170).

Finally, let $\tilde{I}(V_0,V_1,(U_h)_{h\geq 0},W) \in G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$ for $\overline{W} > 0$ chosen as in Lemma 20. We define

$$\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right) = (\mathrm{id}-\mathfrak{M})^{-1}\left(\tilde{I}\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right). \tag{174}$$

By construction, $\Psi(V_0,V_1,(U_h)_{h\geq 0},W)$ belongs to $G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$ and solves (167) with the estimates (168). □

4. Analytic Solutions with Growth Estimates of Linear Partial Differential Equations in $\mathbb{C}^3$

We are now in position to state the main result of our work.

Theorem 21. *Let* $b_{m,k}(t,z,u_0,w)$ *be the functions defined in* (15) *for* $m = 1,2,3$ *and* $k \in \mathcal{S}$. *Let one assume that there exists* $b > 1$ *such that*

$$\begin{aligned}S &\geq k+1+\max\left(b\left(d_{1,k}+2\right)+3, d+1+b\left(d+d_{1,k}+1\right)\right),\\ S &\geq k+3+b\left(2+d_{2,k}\right),\\ S &\geq k+1+b\max\left(d_{1,k},d_{2,k}\right), \qquad S \geq k+bd_{3,k}\end{aligned} \tag{175}$$

for all $k \in \mathcal{S}$. *For all* $0 \leq j \leq S-1$, *one considers functions* $\omega_j(t,z)$ *which are assumed to be holomorphic and bounded on the product* $D(0,R')^2$.

Then, there exist constants $\sigma,\overline{W},C_{12} > 0$ *such that the problem*

$$\begin{aligned}\partial_w^S Y(t,z,w) = \sum_{k\in\mathcal{S}}&\left(b_{1,k}(t,z,X(t,z),w)\partial_t\partial_w^k Y(t,z,w)\right.\\ &+b_{2,k}(t,z,X(t,z),w)\partial_z\partial_w^k Y(t,z,w)\\ &\left.+b_{3,k}(t,z,X(t,z),w)\partial_w^k Y(t,z,w)\right)\end{aligned} \tag{176}$$

with initial data

$$\left(\partial_w^j Y\right)(t,z,0) = \omega_j(t,z), \quad 0 \le j \le S-1, \tag{177}$$

has a solution $Y(t,z,w)$ *which is holomorphic on* $\mathrm{Int}\,(K) \times D(0,\overline{W}/2)$ *and which fulfills the following estimates:*

$$\sup_{(t,z)\in \mathrm{Int}\,(K), w\in D(0,\overline{W}/2)} |Y(t,z,w)| \le C_{12}\exp(\sigma\zeta(b)\rho) + \sum_{j=0}^{S-1}\sup_{(t,z)\in \mathrm{Int}\,(K)}\left|\omega_j(t,z)\right|\frac{\left(\overline{W}/2\right)^j}{j!}, \tag{178}$$

where $\zeta(b) = \sum_{n\ge 0} 1/(n+1)^b$, *for any compact set* $K \subset D(0,R)^2 \setminus \Theta$ *with nonempty interior* $\mathrm{Int}\,(K)$ *for some* $R < R'$ *and any* $\rho > 1$ *which satisfies* (10). *One stresses that the constants* $\sigma, \overline{W}, C_{12} > 0$ *do not depend neither on* K *nor on* $\rho > 1$.

Proof. By convention, we will put $\omega_j(t,z) \equiv 0$ for all $j \ge S$. On the other hand, we specialize the functions $\widetilde{\omega}_\alpha$ which were introduced in (12) in order that

$$\begin{aligned}\widetilde{\omega}_\alpha&\left(v_0,v_1,(u_h)_{h\in I(\alpha)}\right)\\ &= \widehat{\omega}_\alpha(v_0,v_1,u_0)\\ &= \sum_{k\in\mathcal{S}}\sum_{\alpha_1+\alpha_2=\alpha}\alpha!\left(\frac{b_{1,k,\alpha_1}(v_0,v_1,u_0)}{\alpha_1!}\frac{\partial_{v_0}\omega_{\alpha_2+k}(v_0,v_1)}{\alpha_2!}\right.\\ &\qquad + \frac{b_{2,k,\alpha_1}(v_0,v_1,u_0)}{\alpha_1!}\frac{\partial_{v_1}\omega_{\alpha_2+k}(v_0,v_1)}{\alpha_2!}\\ &\qquad \left.+\frac{b_{3,k,\alpha_1}(v_0,v_1,u_0)}{\alpha_1!}\frac{\omega_{\alpha_2+k}(v_0,v_1)}{\alpha_2!}\right).\end{aligned} \tag{179}$$

By construction and using the definition (26), we can write with the help of the Kronecker symbol,

$$\widetilde{\omega}_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}} = \widehat{\omega}_{\alpha,n_0,n_1,l_0} \times \Pi_{h\in I(\alpha)\setminus\{0\}}\delta_{0,l_h}, \tag{180}$$

where

$$\widehat{\omega}_{\alpha,n_0,n_1,l_0} = \sup_{|v_0|<R,|v_1|<R,|u_0|<\rho}\left|\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\partial_{u_0}^{l_0}\widehat{\omega}_\alpha(v_0,v_1,u_0)\right|. \tag{181}$$

Lemma 22. *There exist* $\widetilde{V}_0, \widetilde{V}_1, \widetilde{W} > 0$ *such that the formal series*

$$\begin{aligned}\widetilde{\Omega}&\left(V_0,V_1,(U_h)_{h\ge 0},W\right)\\ &= \sum_{\alpha\ge 0}\left(\sum_{n_0,n_1,l_h\ge 0,h\in I(\alpha)}\widetilde{\omega}_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}}\right.\\ &\qquad\left.\times\frac{V_0^{n_0}}{n_0!}\frac{V_1^{n_1}}{n_1!}\Pi_{h\in I(\alpha)}\frac{U_h^{l_h}}{l_h!}\right)\frac{W^\alpha}{\alpha!}\end{aligned} \tag{182}$$

belongs to $G_{(\rho,\widetilde{V}_0,\widetilde{V}_1,(\overline{U}_h)_{h\ge 0},\widetilde{W})}$. *Moreover, there exists a constant* $C_{11} > 0$ *(independent of* ρ*) such that*

$$\left\|\widetilde{\Omega}(V_0,V_1,(U_h)_{h\ge 0},W)\right\|_{(\rho,\widetilde{V}_0,\widetilde{V}_1,(\overline{U}_h)_{h\ge 0},\widetilde{W})} \le C_{11}. \tag{183}$$

Proof. Let $k \in \mathcal{S}$. Due to the estimates (14) for the functions $b_{m,k,\alpha}(t,z,u_0)$, we get couples of constants $D_{1,k}, \widehat{D}_{1,k} > 0$, $D_{2,k}, \widehat{D}_{2,k} > 0$, and $D_{3,k}, \widehat{D}_{3,k} > 0$ such that

$$\begin{aligned}\left|b_{1,k,\alpha_1}(\chi_0,\chi_1,\xi_0)\right| &\le D_{1,k}(\rho+\delta)^{d_{1,k}}\alpha_1!\left(\widehat{D}_{1,k}\right)^{\alpha_1},\\ \left|b_{2,k,\alpha_1}(\chi_0,\chi_1,\xi_0)\right| &\le D_{2,k}(\rho+\delta)^{d_{2,k}}\alpha_1!\left(\widehat{D}_{2,k}\right)^{\alpha_1},\\ \left|b_{3,k,\alpha_1}(\chi_0,\chi_1,\xi_0)\right| &\le D_{3,k}(\rho+\delta)^{d_{3,k}}\alpha_1!\left(\widehat{D}_{3,k}\right)^{\alpha_1}\end{aligned} \tag{184}$$

for all $\alpha_1 \ge 0$, all $|\chi_0| < R+\delta < R'$, $|\chi_1| < R+\delta < R'$, $|\xi_0| < \rho+\delta$. Moreover, we also get couples of constants $E_{1,k}, \widehat{E}_{1,k} > 0$, $E_{2,k}, \widehat{E}_{2,k} > 0$, and $E_{3,k}, \widehat{E}_{3,k} > 0$ such that

$$\begin{aligned}\left|\partial_{\chi_0}\omega_{\alpha_2+k}(\chi_0,\chi_1)\right| &\le E_{1,k}\alpha_2!(\widehat{E}_{1,k})^{\alpha_2},\\ \left|\partial_{\chi_1}\omega_{\alpha_2+k}(\chi_0,\chi_1)\right| &\le E_{2,k}\alpha_2!(\widehat{E}_{2,k})^{\alpha_2},\\ \left|\omega_{\alpha_2+k}(\chi_0,\chi_1)\right| &\le E_{3,k}\alpha_2!\left(\widehat{E}_{3,k}\right)^{\alpha_2}\end{aligned} \tag{185}$$

for all $\alpha_2 \ge 0$, all $|\chi_0| < R+\delta < R'$, $|\chi_1| < R+\delta < R'$. From (184) and (185) we deduce

$$\begin{aligned}&\left|\widehat{\omega}_\alpha(\chi_0,\chi_1,\xi_0)\right|\\ &\le \sum_{k\in\mathcal{S}}\sum_{\alpha_1+\alpha_2=\alpha}\alpha!\left(D_{1,k}E_{1,k}(\rho+\delta)^{d_{1,k}}\left(\widehat{D}_{1,k}\right)^{\alpha_1}\left(\widehat{E}_{1,k}\right)^{\alpha_2}\right.\\ &\qquad + D_{2,k}E_{2,k}(\rho+\delta)^{d_{2,k}}\left(\widehat{D}_{2,k}\right)^{\alpha_1}\left(\widehat{E}_{2,k}\right)^{\alpha_2}\\ &\qquad \left.+D_{3,k}E_{3,k}(\rho+\delta)^{d_{3,k}}\left(\widehat{D}_{3,k}\right)^{\alpha_1}\left(\widehat{E}_{3,k}\right)^{\alpha_2}\right)\end{aligned} \tag{186}$$

for all $\alpha \ge 0$, all $|\chi_0| < R+\delta < R'$, $|\chi_1| < R+\delta < R'$, $|\xi_0| < \rho+\delta$. From the Cauchy formula in several variables, one can write

$$\begin{aligned}&\frac{\partial_{v_0}^{n_0}\partial_{v_1}^{n_1}\partial_{u_0}^{l_0}\widehat{\omega}_\alpha(v_0,v_1,u_0)}{n_0!n_1!l_0!}\\ &= \left(\frac{1}{2i\pi}\right)^3\\ &\quad\times\int_{C(v_0,\delta)}\int_{C(v_1,\delta)}\int_{C(u_0,\delta)}\widehat{\omega}_\alpha(\chi_0,\chi_1,\xi_0)\\ &\qquad\times\left((d\chi_0 d\chi_1 d\xi_0)\right.\\ &\qquad\times\left((\chi_0-v_0)^{n_0+1}\right.\\ &\qquad\times(\chi_1-v_1)^{n_1+1}\\ &\qquad\left.\left.\times(\xi_0-u_0)^{l_0+1}\right)^{-1}\right)\end{aligned} \tag{187}$$

for all $|v_0| < R$, $|v_1| < R$, $|u_0| < \rho$. We deduce that

$$\frac{\widehat{\omega}_{\alpha,n_0,n_1,l_0}}{n_0!n_1!l_0!} \leq \frac{1}{\delta^{n_0+n_1+l_0}} \times \sum_{k\in\mathcal{S}} \sum_{\alpha_1+\alpha_2=\alpha} \alpha! \times \Big(D_{1,k}E_{1,k}(\rho+\delta)^{d_{1,k}}\big(\widehat{D}_{1,k}\big)^{\alpha_1}\big(\widehat{E}_{1,k}\big)^{\alpha_2} + D_{2,k}E_{2,k}(\rho+\delta)^{d_{2,k}}\big(\widehat{D}_{2,k}\big)^{\alpha_1}\big(\widehat{E}_{2,k}\big)^{\alpha_2} + D_{3,k}E_{3,k}(\rho+\delta)^{d_{3,k}}\big(\widehat{D}_{3,k}\big)^{\alpha_1}\big(\widehat{E}_{3,k}\big)^{\alpha_2}\Big) \tag{188}$$

for all $\alpha \geq 0$, all $n_0, n_1, l_0 \geq 0$. Using (180), we get that

$$\left\|\widetilde{\Omega}(V_0,V_1,(U_h)_{h\geq 0},W)\right\|_{(\rho,\widetilde{V}_0,\widetilde{V}_1,(\overline{U}_h)_{h\geq 0},\widetilde{W})} = \sum_{\alpha\geq 0}\left(\sum_{n_0,n_1,l_0\geq 0} \frac{\left|\widehat{\omega}_{\alpha,n_0,n_1,l_0}\right|}{\exp(\sigma r_b(\alpha)\rho)} \times \frac{\widetilde{V}_0^{n_0}\widetilde{V}_1^{n_1}\overline{U}_0^{l_0}}{(n_0+n_1+l_0+\alpha)!}\right)\widetilde{W}^{\alpha}. \tag{189}$$

From (188), (125), and with the help of the classical estimates

$$(n_0+n_1+l_0+\alpha)! \geq n_0!n_1!l_0!\alpha!, \tag{190}$$

for all $n_0, n_1, l_0, \alpha \geq 0$, we get a constant $C_{11,1} > 0$ (depending on $D_{1,k}, d_{1,k}, E_{1,k}, D_{2,k}, d_{2,k}, E_{2,k}, D_{3,k}, d_{3,k}, E_{3,k}$ for all $k \in \mathcal{S}$, σ, δ) such that

$$\left\|\widetilde{\Omega}\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\right\|_{(\rho,\widetilde{V}_0,\widetilde{V}_1,(\overline{U}_h)_{h\geq 0},\widetilde{W})} \leq \frac{C_{11,1}}{\left(1-\left(\widetilde{V}_0/\delta\right)\right)\left(1-\left(\widetilde{V}_1/\delta\right)\right)\left(1-\left(\overline{U}_0/\delta\right)\right)} \times \sum_{k\in\mathcal{S}} \frac{1}{\left(1-\widehat{D}_{1,k}\widetilde{W}\right)\left(1-\widehat{E}_{1,k}\widetilde{W}\right)} + \frac{1}{\left(1-\widehat{D}_{2,k}\widetilde{W}\right)\left(1-\widehat{E}_{2,k}\widetilde{W}\right)} + \frac{1}{\left(1-\widehat{D}_{3,k}\widetilde{W}\right)\left(1-\widehat{E}_{3,k}\widetilde{W}\right)}. \tag{191}$$

We choose

$$0 < \widetilde{W} < \min_{k\in\mathcal{S}}\left(\frac{1}{\left(2\widehat{D}_{1,k}\right)}, \frac{1}{\left(2\widehat{D}_{2,k}\right)}, \frac{1}{\left(2\widehat{D}_{3,k}\right)}, \frac{1}{\left(2\widehat{E}_{1,k}\right)}, \frac{1}{\left(2\widehat{E}_{2,k}\right)}, \frac{1}{\left(2\widehat{E}_{3,k}\right)}\right), \quad 0 < \widetilde{V}_0 < \frac{\delta}{2},\ 0 < \widetilde{V}_1 < \frac{\delta}{2},\ 0 < \overline{U}_0 < \frac{\delta}{2}. \tag{192}$$

From (191) we deduce the inequality (183).

Under the assumption (175), we get from Proposition 19 four constants $0 < \overline{V}_0 < \widetilde{V}_0$, $0 < \overline{V}_1 < \widetilde{V}_1$, $0 < \overline{U}_0$, and $0 < \overline{W} < \widetilde{W}$ (independent of ρ) such that the functional equation

$$\begin{aligned}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right) &= \sum_{k\in\mathcal{S}}\Big(B_{1,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\partial_{V_0}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right) \\ &\quad + B_{1,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\mathbb{D}_{\mathbf{A}}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\Big) \\ &\quad + \sum_{k\in\mathcal{S}}\Big(B_{2,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\partial_{V_1}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right) \\ &\quad + B_{2,k}(V_0,V_1,U_0,W)\,\partial_W^{-S+k}\mathbb{D}_{\mathbf{B}}\Psi(V_0,V_1,(U_h)_{h\geq 0},W)\Big) \\ &\quad + \sum_{k\in\mathcal{S}} B_{3,k}\left(V_0,V_1,U_0,W\right)\partial_W^{-S+k}\Psi\left(V_0,V_1,(U_h)_{h\geq 0},W\right) \\ &\quad + \widetilde{\Omega}\left(V_0,V_1,(U_h)_{h\geq 0},W\right)\end{aligned} \tag{193}$$

has a unique solution $\Psi(V_0,V_1,(U_h)_{h\geq 0},W)$ belonging to $G_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})}$ which satisfies moreover the estimates

$$\left\|\Psi(V_0,V_1,(U_h)_{h\geq 0},W)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \leq 2\left\|\widetilde{\Omega}(V_0,V_1,(U_h)_{h\geq 0},W)\right\|_{(\rho,\overline{V}_0,\overline{V}_1,(\overline{U}_h)_{h\geq 0},\overline{W})} \leq 2C_{11}. \tag{194}$$

Now, from Proposition 6, we know that the sequence $\varphi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}}$ introduced in (25) satisfies the inequality

$$\varphi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}} \leq \psi_{\alpha,n_0,n_1,(l_h)_{h\in I(\alpha)}} \tag{195}$$

for all $\alpha \geq 0$, all $n_0, n_1, l_h \geq 0$, for $h \in I(\alpha)$. Gathering (194) and (195) and from the definition of the Banach spaces in Section 3.1, we get, in particular, for $n_0 = n_1 = l_h = 0$, for all $h \in I(\alpha)$, all $\alpha \geq 0$, that

$$\begin{aligned}\sup_{|v_0|<R,|v_1|<R,|u_h|<\rho,h\in I(\alpha)} &\left|\phi_\alpha\left(v_0,v_1,(u_h)_{h\in I(\alpha)}\right)\right| \\ &\leq \psi_{\alpha,0,0,(0)_{h\in I(\alpha)}} \leq 2C_{11}\exp\left(\sigma r_b(\alpha)\rho\right)\left(\frac{1}{\overline{W}}\right)^{\alpha}\alpha! \\ &\leq 2C_{11}\exp\left(\sigma\zeta(b)\rho\right)\left(\frac{1}{\overline{W}}\right)^{\alpha}\alpha!\end{aligned} \tag{196}$$

for all $\alpha \geq 0$ and where $\zeta(b) = \sum_{n\geq 0} 1/(n+1)^b$. From (196), we get that the formal series $U(t,z,w)$ introduced in (11) actually defines a holomorphic function (denoted again by $U(t,z,w)$) on $\mathrm{Int}(K) \times D(0,\overline{W}/2)$ for which the estimates

$$\sup_{(t,z)\in \mathrm{Int}(K),w\in D(0,\overline{W}/2)} |U(t,z,w)| \leq 4C_{11}\exp\left(\sigma\zeta(b)\rho\right) \tag{197}$$

hold and which satisfies (17) on $\mathrm{Int}(K) \times D(0,\overline{W}/2)$.

Finally, we define the function

$$Y(t,z,w) = \partial_w^{-S}U(t,z,w) + \sum_{j=0}^{S-1}\omega_j(t,z)\frac{w^j}{j!}. \tag{198}$$

By construction, $Y(t,z,w)$ defines a holomorphic function on $\mathrm{Int}(K) \times D(0, \overline{W}/2)$ with bounds estimates

$$\begin{aligned} &\sup_{(t,z)\in \mathrm{Int}(K), w\in D(0,\overline{W}/2)} |Y(t,z,w)| \\ &\quad \leq 4\left(\frac{\overline{W}}{2}\right)^{S} C_{11} \exp(\sigma\zeta(b)\rho) \\ &\qquad + \sum_{j=0}^{S-1} \sup_{(t,z)\in \mathrm{Int}(K)} \left|\omega_j(t,z)\right| \frac{(\overline{W}/2)^j}{j!} \end{aligned} \tag{199}$$

and solves the problem (176), (177). This yields the result. □

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