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Research Article

Normal Families of Zero-Free Meromorphic Functions

Yuntong Li

Department of Basic Courses, Shaanxi Railway Institute, Weinan 714000, Shaanxi, China

Correspondence should be addressed to Yuntong Li, liyuntong2005@sohu.com

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Let $a(\neq 0),\ b \in \mathbb{C}$, and n and k be two positive integers such that $n \geq 2$. Let $\mathfrak{F}$ be a family of zero-free meromorphic functions defined in a domain $\mathfrak{D}$ such that for each $f \in \mathfrak{F}$, $f + a(f^{(k)})^n - b$ has at most nk zeros, ignoring multiplicity. Then $\mathfrak{F}$ is normal in $\mathfrak{D}$.

1. Introduction and Main Results

Let $\mathfrak{D}$ be a domain in $\mathbb{C}$, and let $\mathfrak{F}$ be a family of meromorphic functions defined in the domain $\mathfrak{D}$. $\mathfrak{F}$ is said to be normal in $\mathfrak{D}$, in the sense of Montel, if for every sequence $\{f_n\} \subseteq \mathfrak{F}$ contains a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges spherically uniformly on compact subsets of $\mathfrak{D}$ (see [1, Definition 3.1.1]).

$\mathfrak{F}$ is said to be normal at a point $z_0 \in \mathfrak{D}$ if there exists a neighborhood of z_0 in which $\mathfrak{F}$ is normal. It is well known that $\mathfrak{F}$ is normal in a domain $\mathfrak{D}$ if and only if it is normal at each of its points (see [1, Theorem 3.3.2]).

Let f be a meromorphic function in the complex plane. We use the standard nota-tions and results of value distribution theory as presented in [2–4]. In particular, $T(r, f)$ is Nevanlinna's characteristic function and $S(r, f)$ denotes a function with the property $S(r, f) = o(T(r, f))$ as $r \to \infty$ (outside an exceptional set of finite linear measure).

In 1959, Hayman [5] proved the following well-known result.

Theorem A. *Let f be a transcendental meromorphic function on the complex plane C, let a be a nonzero finite complex number, and let n be a positive integer. If $n \geq 5$, then $f' + af^n$ assumes each value $b \in C$ infinitely often.*

There are some examples constructed by Mues [6] which show that Theorem A is not true when $n = 3, 4$. Corresponding to Theorem A, Ye [7, Theorem 2.1] proved the following interesting result.

Theorem B. *Let f be a transcendental meromorphic function. If $a \neq 0$ is a finite complex number and $n \geq 3$ is an positive integer, then $f + af'^n$ assumes all finite complex number infinitely often.*

In [7, Theorem 2.2], Ye also obtained the following result, which may be considered as a normal family analogue of Theorem B.

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $\mathfrak{D}$, $f \neq b$ and $f + af'^n \neq b$ for every $f \in \mathcal{F}$, where $n \geq 2$ is an integer and $a \neq 0$, b are two finite complex numbers. Then, $\mathcal{F}$ is normal.*

Ye [7] asked whether Theorem B remains valid for $n = 2$. Recently, Fang and Zalcman showed that Theorem B holds for $n = 2$. In [8], the condition in Theorem C that $f \neq b$ can be relaxed to that all zeros of each function in $\mathcal{F}$ are of multiplicity at least 2. Actually. they obtained the following results.

Theorem D. *Let f be a transcendental meromorphic function. If $a \neq 0$ is a finite complex number and $n \geq 2$ is an positive integer, then $f + af'^n$ assumes all finite complex number infinitely often.*

Theorem E. *Let $\mathcal{F}$ be a family of meromorphic functions on the plane domain $\mathfrak{D}$, let $n \geq 2$ be a positive integer, and let $a \neq 0$, b be complex numbers. If, for each $f \in \mathcal{F}$, all zeros of f are multiple and $f + af'^n \neq b$ on D, then $\mathcal{F}$ is normal on D.*

A natural problem arises: what can we say if f' in Theorems E is replaced by the kth derivative $f^{(k)}$? In [9], Xu et al. proved the following result.

Theorem F. *Let $a(\neq 0)$, $b \in \mathbb{C}$ and n and k be two positive integers such that $n \geq k + 1$. Let $\mathcal{F}$ be a family of meromorphic functions defined on a domain $\mathfrak{D}$. If, for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least $k + 1$, and $f + a(f^{(k)})^n \neq b$ in D, then $\mathcal{F}$ is normal.*

Xu et al. [9] asked whether Theorem F remains valid for $n = 2$. We partially answer this question. If $f \neq 0$, we generalize Theorem F by allowing $f + a(f^{(k)})^n - b$ to have zeros but restricting their numbers.

Theorem 1.1. *Let $a(\neq 0), b \in \mathbb{C}$, and n and k be two positive integers such that $n \geq 2$. Let $\mathcal{F}$ be a family of zero-free meromorphic functions defined in a domain $\mathfrak{D}$ such that for each $f \in \mathcal{F}$, $f + a(f^{(k)})^n - b$ has at most nk zeros, ignoring multiplicity. Then, $\mathcal{F}$ is normal in $\mathfrak{D}$.*

Remark 1.2. Here, $f \neq 0$ can be replaced by $f \neq c$, where c is any finite complex numbers.

Example 1.3. Let $\mathfrak{D} = \{z : |z| < 1\}$. Let $\mathcal{F} = \{f_m\}$, where $f_m := e^{mz}$. Then, $f_m + af'_m = (1 + am)e^{mz} \neq 0$ in $\mathfrak{D}$ for every function $f \in \mathcal{F}$. However, it is easily obtained that $\mathcal{F}$ is not normal at the point $z = 0$.

Example 1.4. Let $\mathfrak{D} = \{z : |z| < 1\}$. Let $\mathcal{F} = \{f_m\}$, where $f_m := 1/mz$. Then, $f_m + a(f'_m)^2 = (mz^3 + 1)/m^2z^4$ has 3 zeros in $\mathfrak{D}$ for every function $f \in \mathcal{F}$. However, it is easily obtained that $\mathcal{F}$ is not normal at the point $z = 0$.

Example 1.5. Let $\mathfrak{D} = \{z : |z| < 1\}$. Let $\mathcal{F} = \{f_m\}$, where $f_m := mz$. It follows that $f_m + a(f_m')^2 = mz + m^2$ has no zero in $\mathfrak{D}$ for every function $f \in \mathcal{F}$. However, it is easily obtained that $\mathcal{F}$ is not normal at the point $z = 0$.

Examples 1.3 and 1.4 show that the conditions that $n \geq 2$ and $f + a(f^{(k)})^n - b$ have at most nk distinct zeros in Theorem 1.1 are shape. Example 1.5 shows the condition that $f \neq 0$ cannot be omitted.

2. Some Lemmas

To prove our results, we need some preliminary results.

Lemma 2.1 ([9], Lemma 2.2). *Let $n \geq 2$, k be positive integers, let a be a nonzero constant and let $P(z)$ be a polynomial. Then, the solution of the differential equation $a(W^{(k)}(z))^n + W(z) = P(z)$ must be polynomial.*

Lemma 2.2. *Let f be a nonzero transcendental meromorphic function. If a be a nonzero finite complex number and let $n \geq 2$ and k be two positive integers. Then, $f + a(f^{(k)})^n$ assumes each value $b \in \mathbb{C}$ infinitely often.*

Proof. Set

$$F = f + a\left(f^{(k)}\right)^n - b, \tag{2.1}$$

$$\phi = \frac{F'}{F} = \frac{f' + an\left(f^{(k)}\right)^{n-1} f^{(k+1)}}{f + a\left(f^{(k)}\right)^n - b}, \tag{2.2}$$

$$\psi = n\frac{f^{(k+1)}}{f^{(k)}} - \frac{F'}{F} = \frac{nf^{(k+1)}f - bnf^{(k)} - f'f^{(k)}}{f^{(k)}\left(f + a\left(f^{(k)}\right)^n - b\right)}. \tag{2.3}$$

We claim that $\phi\psi \not\equiv 0$. If $\phi \equiv 0$, then $F \equiv 0$. We can deduce that $F \equiv c$, where c is a finite complex number. We conclude from (2.1) and Lemma 2.1 that, f must be a polynomial, which is a contradiction.

If $\psi \equiv 0$, from (2.3), we can obtain

$$c\left(f^{(k)}\right)^n = f + a\left(f^{(k)}\right)^n - b, \tag{2.4}$$

where c is a finite complex number, that is,

$$(a - c)\left(f^{(k)}\right)^n + f = b. \tag{2.5}$$

If $a - c = 0$, we can get that $f \equiv b$, which is a contradiction.

If $a - c \neq 0$, we conclude from (2.5) and Lemma 2.1 that f must be a polynomial, which is a contradiction.

By elementary Nevanlinna theory and (2.1), we have $T(r,F) = O(T(r,f))$. Thus, from (2.2) and (2.3), we have

$$m(r,\phi) = S(r,f), \qquad m(r,\psi) = S(r,f). \tag{2.6}$$

It follows from (2.2), (2.3) and Nevanlinna's First Fundamental Theorem that

$$\begin{aligned} N\left(r,\frac{1}{\phi}\right) &\leqslant m(r,\phi) + N(r,\phi) - m\left(r,\frac{1}{\phi}\right) + O(1) \\ &\leqslant N(r,\phi) + S(r,f) \leqslant \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f), \end{aligned} \tag{2.7}$$

$$\begin{aligned} N\left(r,\frac{1}{\psi}\right) &\leqslant m(r,\psi) + N(r,\psi) - m\left(r,\frac{1}{\psi}\right) + O(1) \\ &\leqslant N(r,\psi) + S(r,f) \leqslant \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + N\left(r,\frac{1}{F}\right) + S(r,f). \end{aligned} \tag{2.8}$$

By (2.2) and (2.3), we get

$$\phi(f-b) - f' = a\left(f^{(k)}\right)^n \psi. \tag{2.9}$$

We have by (2.6)-(2.7)

$$\begin{aligned} T(r,\phi(f-b) - f') &= T\left(r,(f-b)\left(\phi - \frac{f'}{f-b}\right)\right) \\ &\leqslant T(r,f-b) + T\left(r,\phi - \frac{f'}{f-b}\right) + S(r,f) \\ &\leqslant m(r,f-b) + N(r,f-b) + m\left(r,\phi - \frac{f'}{f-b}\right) + N\left(r,\phi - \frac{f'}{f-b}\right) + S(r,f) \\ &\leqslant m(r,f) + N(r,f) + m(r,\phi) + m\left(r,\frac{f'}{f-b}\right) + N\left(r,\phi - \frac{f'}{f-b}\right) + S(r,f) \\ &\leqslant T(r,f) + \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f). \end{aligned} \tag{2.10}$$

It follows from (2.6)–(2.10) that

$$\begin{aligned}
nT\left(r, f^{(k)}\right) &\leqslant T(r,\varphi) + T(r,\phi(f-b) - f') + S(r,f) \\
&\leqslant m(r,\varphi) + N(r,\varphi) + T(r,f) + \overline{N}(r,f) + N\left(r,\frac{1}{F}\right) + S(r,f) \\
&\leqslant \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + N\left(r,\frac{1}{F}\right) + m\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f}\right) + \overline{N}(r,f) \\
&\quad + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f) \\
&\leqslant \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + 2N\left(r,\frac{1}{F}\right) + m\left(r,\frac{f^{(k)}}{f}\right) + m\left(r,\frac{1}{f^{(k)}}\right) + N\left(r,\frac{1}{f}\right) \\
&\quad + \overline{N}(r,f) + S(r,f) \\
&\leqslant T\left(r,\frac{1}{f^{(k)}}\right) + 2N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + S(r,f) \\
&\leqslant T\left(r,f^{(k)}\right) + 2N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + S(r,f).
\end{aligned} \tag{2.11}$$

So, we have

$$(n-1)T\left(r,f^{(k)}\right) \leqslant 2N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + S(r,f). \tag{2.12}$$

We have

$$(n-1)T\left(r,f^{(k)}\right) \geq (n-1)N\left(r,f^{(k)}\right) \geq (n-1)N(r,f) + (n-1)\overline{N}(r,f). \tag{2.13}$$

Since $f \neq 0$, if $f + a(f^{(k)})^n$ assumes the value b only finitely often, we by (2.12) can get

$$N(r,f) = S(r,f). \tag{2.14}$$

Hence,

$$(n-1)T\left(r,f^{(k)}\right) \leqslant 2N\left(r,\frac{1}{F}\right) + S(r,f). \tag{2.15}$$

So $f + a(f^{(k)})^n$ assumes each value $b \in \mathbb{C}$ infinitely often.

We complete the proof of Lemma 2.2. □

Using the method of Chang [10, Lemma 4], we obtain the following lemma.

Lemma 2.3. *Let f be a nonconstant zero-free rational function, $n \geq 2$, let k be two positive integers, and $a \neq 0$, b be two complex constants. Then, the function $f + a(f^{(k)})^n - b$ has at least $nk+1$ distinct zeros in $\mathbb{C}$.*

Proof. Since $f(z)$ is a nonconstant zero-free rational function, $f(z)$ is not a polynomial, and hence it has at least one finite pole. Thus, we can write

$$f(z) = \frac{C_1}{\prod_{i=1}^{m}(z+z_i)^{p_i}}, \tag{2.16}$$

where C_1 is a nonzero constant, m and p_i are positive integers, the z_i (when $1 \leq i \leq m$) are distinct complex numbers, and denote $p = \sum_{i=1}^{m} p_i$.

By induction, we deduce from (2.16) that

$$f^{(k)}(z) = \frac{P_{(m-1)k}}{\prod_{i=1}^{m}(z+z_i)^{p_i+k}}, \tag{2.17}$$

where $P_{(m-1)k}$ is polynomial of degree $(m-1)k$.

So the degree of numerator of the function $f + a(f^{(k)})^n$ is equal to $\sum_{i=1}^{m}(n-1)p_i + nk$. By calculation, $f + a(f^{(k)})^n - b$ has at least one zero in $\mathbb{C}$. Thus, we can write

$$f + a\left(f^{(k)}\right)^n - b = \frac{C_2\prod_{i=1}^{s}(z+\alpha_i)^{l_i}}{\prod_{i=1}^{m}(z+z_i)^{n(p_i+k)}}, \tag{2.18}$$

where C_2 is a nonzero constant, l_i are positive integers, α_i (when $1 \leq i \leq s$), and z_i (when $1 \leq i \leq m$) are distinct complex numbers. Thus, by (2.16), (2.17), and (2.18), we get

$$C_1\prod_{i=1}^{m}(z+z_i)^{(n-1)p_i+nk} + a\left(P_{(m-1)k}\right)^n = b\prod_{i=1}^{m}(z+z_i)^{n(p_i+k)} + C_2\prod_{i=1}^{s}(z+\alpha_i)^{l_i}. \tag{2.19}$$

Case 1. If $b = 0$, it follows that $\sum_{i=1}^{m}[(n-1)p_i + nk] = \sum_{i=1}^{s} l_i$ and $C_1 = C_2$. Thus, it follows from (2.19) that

$$\prod_{i=1}^{m}(1+z_it)^{(n-1)p_i+nk} - \prod_{i=1}^{s}(1+\alpha_it)^{l_i} = t^{(n-1)p+nk}Q(t), \tag{2.20}$$

where $Q(t) = (-a/C_1)t^{(m-1)nk}(P_{(m-1)k}(1/t))^n$ is a polynomial. Then, $Q(t)$ is a polynomial of degree less than $(m-1)nk$, and it follows that

$$\frac{\prod_{i=1}^{m}(1+z_it)^{(n-1)p_i+nk}}{\prod_{i=1}^{s}(1+\alpha_it)^{l_i}} = 1 + \frac{t^{(n-1)p+nk}Q(t)}{\prod_{i=1}^{s}(1+\alpha_it)^{l_i}} = 1 + O\left(t^{(n-1)p+nk}\right) \tag{2.21}$$

as $t \to 0$.

Logarithmic differentiation of both sides of (2.21) shows that

$$\sum_{i=1}^{m}\frac{\left((n-1)p_i+nk\right)z_i}{1+z_it}-\sum_{i=1}^{s}\frac{l_i\alpha_i}{1+\alpha_it}=O\left(t^{(n-1)p+nk-1}\right) \tag{2.22}$$

as $t \to 0$.

Comparing the coefficient of (2.22) for t^j, $j=0,1,\dots,(n-1)p+nk-2$, we have

$$\sum_{i=1}^{m}\left((n-1)p_i+nk\right)z_i^j-\sum_{i=1}^{s}l_i\alpha_i^j=0 \tag{2.23}$$

for $j=1,\dots,(n-1)p+nk-1$.

Set $z_{m+i}=-\alpha_i$ when $1\le i\le s$. Noting that $\sum_{i=1}^{m}[(n-1)p_i+nk]=\sum_{i=1}^{s}l_i$, then it follows from (2.23) that the system of linear equations,

$$\sum_{i=1}^{m+s}z_i^jx_i=0, \tag{2.24}$$

where $0\le j\le(n-1)p+nk-1$, has a nonzero solution

$$(x_1,\dots,x_m,x_{m+1},\dots,x_{m+s})=\left((n-1)p_1+nk,\dots,(n-1)p_m+nk,l_1,\dots,l_s\right). \tag{2.25}$$

If $(n-1)p+nk\ge m+s$, then the determinant $\det(z_i^j)_{(m+s)\times(m+s)}$ of the coefficients of the system of (2.24), where $0\le j\le(n-1)p+nk-1$, is equal to zero, by Cramer's rule (see, e.g., [11]). However, the z_i are distinct complex numbers when $1\le i\le m+s$, and the determinant is a Vandermonde determinant, so it cannot be 0 (see [11]), which is a contradiction.

Hence, we conclude that $(n-1)p+nk<m+s$. Noting that $n\ge 2$, it follows from this and $p=\sum_{i=1}^{m}p_i\ge m$ that $s\ge nk+1$. □

Case 2. If $b\neq 0$, set

$$b\prod_{i=1}^{m}(z+z_i)^{n(p_i+k)}-C_1\prod_{i=1}^{m}(z+z_i)^{(n-1)p_i+nk}=b\prod_{i=1}^{m}(z+z_i)^{(n-1)p_i+nk}\prod_{i=1}^{q}(z+\beta_i)^{t_i}, \tag{2.26}$$

where t_i are positive integers. It follows that β_i (when $1\le i\le q$) and z_i (when $1\le i\le m$) are distinct complex numbers, and $\sum_{i=1}^{q}t_i=p$.

By (2.19), we have

$$b\prod_{i=1}^{m}(z+z_i)^{(n-1)p_i+nk}\prod_{i=1}^{q}(z+\beta_i)^{t_i}+C_2\prod_{i=1}^{s}(z+\alpha_i)^{l_i}=a\left(P_{(m-1)k}\right)^n. \tag{2.27}$$

It follows that

$$\sum_{i=1}^{m}\left[(n-1)p_i+nk\right]+\sum_{i=1}^{q}t_i=np+nmk=\sum_{i=1}^{s}l_i, \tag{2.28}$$

and $C_2=-b$. Thus, by (2.27),

$$\prod_{i=1}^{m}(1+z_it)^{(n-1)p_i+nk}\prod_{i=1}^{q}(1+\beta_it)^{t_i}-\prod_{i=1}^{s}(1+\alpha_it)^{l_i}=t^{n(p+k)}Q(t), \tag{2.29}$$

where $Q(t)=(a/b)t^{(m-1)nk}(P_{(m-1)k}(1/t))^n$ is a polynomial. Then, $Q(t)$ is a polynomial of degree less than $(m-1)nk$, and it follows that

$$\frac{\prod_{i=1}^{m}(1+z_it)^{(n-1)p_i+nk}\prod_{i=1}^{q}(1+\beta_it)^{t_i}}{\prod_{i=1}^{s}(1+\alpha_it)^{l_i}}=1+\frac{t^{n(p+k)}Q(t)}{\prod_{i=1}^{s}(1+\alpha_it)^{l_i}}=O\left(t^{n(p+k)}\right) \tag{2.30}$$

as $t\rightarrow 0$.

Thus, by taking logarithmic derivatives of both sides of (2.12), we get

$$\sum_{i=1}^{m}\frac{((n-1)p_i+nk)z_i}{1+z_it}+\sum_{i=1}^{q}\frac{t_i\beta_i}{1+\beta_it}-\sum_{i=1}^{s}\frac{l_i\alpha_i}{1+\alpha_it}=O\left(t^{n(p+k)-1}\right). \tag{2.31}$$

We consider two cases.

Subcase 2.1 ($\{\alpha_1,\dots,\alpha_s\}\cap\{\beta_1,\dots,\beta_q\}=\emptyset$). Applying the reasoning of Case 1 and noting that $p\geq q$, we deduce that $s\geq nk$.

Subcase 2.2 ($\{\alpha_1,\dots,\alpha_s\}\cap\{\beta_1,\dots,\beta_q\}\neq\emptyset$). Without loss of generality, we may assume that $\alpha_{q-i}=\beta_i$, for $(1\leq i\leq M)$. Denote

$$z_i=\begin{cases} z_i & \text{for } 1\leq i\leq m,\\ \beta_{i-m} & \text{for } m+1\leq i\leq m+q,\\ \alpha_{M+i-m-q} & \text{for } m+q+1\leq i\leq m+q+s-M,\end{cases}$$
$$N_i=\begin{cases} (n-1)p_i+nk & \text{for } 1\leq i\leq m,\\ t_{i-m} & \text{for } m+1\leq i\leq m+s-M,\\ t_{i-m}-l_{i-m-s+M} & \text{for } m+s-M+1\leq i\leq m+q,\\ l_{i-m-q+M} & \text{for } m+q+1\leq i\leq m+q+s-M.\end{cases} \tag{2.32}$$

The formula (2.31) can be rewritten:

$$\sum_{i=1}^{m+q+s-M}\frac{N_iz_i}{1+z_it}=O\left(t^{n(p+k)-1}\right). \tag{2.33}$$

Applying the reasoning of Case 1, and noting that $p \geq q$, we deduce that $s \geq nk + 1$. This completes the proof of Lemma 2.3.

Lemma 2.4 ([10], Lemma 4). *Let f be a nonconstant zero-free rational function, let $a \neq 0$ be a complex constant, and let k be a positive integer. Then $f^{(k)} - a$ has at least $k + 1$ distinct zeros in $\mathbb{C}$.*

Lemma 2.5 (see [12], Lemma 2, Zalcman's lemma). *Let $\mathcal{F}$ be a family of functions meromorphic on a domain $\mathfrak{D}$, all of whose zeros have multiplicity at least k. Suppose that there exists $A \geqslant 1$ such that $|f^{(k)}(z)| \leqslant A$ whenever $f(z) = 0$. Then, if $\mathcal{F}$ is not normal at $z_0 \in \mathfrak{D}$, there exist, for each $0 \leqslant \alpha \leqslant k$,*

(a) *points $z_n, z_n \to z_0$;*

(b) *functions $f_n \in \mathcal{F}$;*

(c) *positive numbers $\rho_n \to 0^+$;*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros of $g(\xi)$ are of multiplicity at least k, such that $g^{\#}(\xi) \leq g^{\#}(0) = kA + 1$.

Here, as usual, $g^{\#}(\xi) = |g'(\xi)|/(1 + |g(\xi)|^2)$ is the spherical derivative.

3. Proof of Theorem

Suppose that $\mathcal{F}$ is not normal in $\mathfrak{D}$. Then, there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point $z_0 \in \mathfrak{D}$. Without loss of generality, we assume that $z_0 = 0$. We consider two cases.

Case 1 ($b = 0$). By Zalcman's lemma, there exist:

(a) points $z_n, z_n \to z_0$;

(b) functions $f_n \in \mathcal{F}$;

(c) positive numbers $\rho_n \to 0^+$;

such that

$$g_j(\xi) = \rho_j^{-nk/(n-1)} f_j(z_j + \rho_j \xi) \longrightarrow g(\xi), \tag{3.1}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a nonconstant meromorphic function in $\mathbb{C}$. Since $f_j \neq 0$, by Hurwitz's theorem, it implies that $g(\xi) \neq 0$.

On every compact subset of $\mathbb{C}$ which contains no poles of g, from (3.1), we get

$$g_j(\xi) + a\left(g_j^k(\xi)\right)^n = \rho_j^{-nk/(n-1)}\left(f_j(z_j + \rho_j \xi) \; + a\left(f_j^k(z_j + \rho_j \xi)\right)^n\right) \longrightarrow g(\xi) + a\left(g^k(\xi)\right)^n, \tag{3.2}$$

also locally uniformly with respect to the spherical metric.

We claim that $g(\xi) + a(g^k(\xi))^n$ has at most nk distinct zeros.

Suppose that $g(\xi) + a(g^k(\xi))^n$ has $nk + 1$ distinct zeros ξ_i, $1 \le i \le nk + 1$, and choose $\delta(> 0)$ small enough such that $\bigcap_{i=1}^{nk+1} D(\xi_i, \delta) = \emptyset$, where $D(\xi_0, \delta) = \{\xi \mid |\xi - \xi_i| < \delta\}$.

From (3.2), by Hurwitz's theorem, there exist points $\xi_i^j \in D(\xi_i, \delta)$ $(1 \le i \le nk + 1)$ such that for sufficiently large j,

$$f_j\left(z_j + \rho_j \xi_i^j\right) + a\left(f_j^k\left(z_j + \rho_j \xi_i^j\right)\right)^n = 0, \tag{3.3}$$

for $1 \le i \le nk + 1$.

Since $z_j \to 0$ and $\rho_j \to 0^+$, we have $z_j + \rho_j \xi_i^j \in D(0, \sigma)$ (σ is a positive constant) for sufficiently large j, so $f_j(z) + a(f_j^k(z))^n$ has $nk + 1$ distinct zeros, which contradicts the fact that $f_j(z) + a(f_j^k(z))^n$ has at most nk zero.

However, by Lemmas 2.2 and 2.3, there do not exist nonconstant meromorphic functions that have the above properties. This contradiction shows that $\mathcal{F}$ is normal in $\mathfrak{D}$.

Case 2 ($b \neq 0$). By Zalcman's lemma, there exist:

(a) points $z_n, z_n \to z_0$;

(b) functions $f_n \in \mathcal{F}$;

(c) positive numbers $\rho_n \to 0^+$;

such that

$$g_j(\xi) = \rho_j^{-k} f_j(z_j + \rho_j \xi) \longrightarrow g(\xi) \tag{3.4}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a nonconstant meromorphic function in $\mathbb{C}$. Since $f_j \neq 0$, by Hurwitz's theorem, it implies that $g(\xi) \neq 0$.

On every compact subset of $\mathbb{C}$ which contains no poles of g, from (3.4), we get

$$\rho_j^k g_j(\xi) + a\left(g_j^k(\xi)\right)^n - b \longrightarrow a\left(g^k(\xi)\right)^n - b \tag{3.5}$$

also locally uniformly with respect to the spherical metric.

Noting that

$$\rho_j^k g_j(\xi) + a\left(g_j^k(\xi)\right)^n - b = f_j(z_j + \rho_j \xi) + a\left(f_j^k(z_j + \rho_j \xi)\right)^n - b, \tag{3.6}$$

we claim that $a(g^k(\xi))^n - b$ has at most nk distinct zeros.

Suppose that $g(\xi) + a(g^k(\xi))^n - b$ has $nk + 1$ distinct zeros ξ_i, $1 \le i \le nk + 1$, and choose $\delta(> 0)$ small enough such that $\bigcap_{i=1}^{nk+1} D(\xi_i, \delta) = \emptyset$, where $D(\xi_0, \delta) = \{\xi \mid |\xi - \xi_i| < \delta\}$.

From (3.2), by Hurwitz's theorem, there exist points $\xi_i^j \in D(\xi_i, \delta)$ $(1 \le i \le nk + 1)$ such that for sufficiently large j

$$f_j\left(z_j + \rho_j \xi_i^j\right) + a\left(f_j^k\left(z_j + \rho_j \xi_i^j\right)\right)^n - b = 0, \tag{3.7}$$

for $1 \le i \le nk + 1$.

Since $z_j \to 0$ and $\rho_j \to 0^+$, we have $z_j + \rho_j \xi_i^j \in D(0,\ \sigma)$ (σ is a positive constant) for sufficiently large j, so $f_j(z) + a(f_j^k(z))^n - b$ has $nk+1$ distinct zeros, which contradicts the fact that $f_j(z) + a(f_j^k(z))^n - b$ has at most nk zero.

Denote $c_1, c_2, \ldots, c_n$ by the different roots of $\omega^n = b/a$, then

$$a\left(g^k(\xi)\right)^n - b = a\prod_{i=1}^{n}\left(g^k(\xi) - c_i\right). \tag{3.8}$$

Subcase 2.1 (If $g(\xi)$ is a rational function). By Lemma 2.4 and (3.8), we can deduce that $a(g^k(\xi))^n - b$ has at least $nk + n$ distinct zeros. This contradicts the claim that $a(g^k(\xi))^n - b$ has at most nk distinct zeros.

Subcase 2.2 (If $g(\xi)$ is a transcendental meromorphic function). By Nevanlinnas second main theorem, we have

$$\begin{aligned}
T\left(r, g^{(k)}\right) &\le \overline{N}\left(r, g^{(k)}\right) + \sum_{i=1}^{n}\overline{N}\left(r, \frac{1}{g^{(k)} - c_i}\right) + S\left(r, g^{(k)}\right) \\
&= \overline{N}\left(r, g^{(k)}\right) + \overline{N}\left(r, \frac{1}{a\left(g^{(k)}\right)^n - b}\right) + S\left(r, g^{(k)}\right) \\
&\le \frac{1}{k+1}N\left(r, g^{(k)}\right) + S\left(r, g^{(k)}\right) \\
&\le \frac{1}{k+1}T\left(r, g^{(k)}\right) + S\left(r, g^{(k)}\right).
\end{aligned} \tag{3.9}$$

It follows that $T(r, g^{(k)}) \le S(r, g^{(k)})$, which is a contradiction. This contradiction shows that $\mathfrak{F}$ is normal in $\mathfrak{D}$.

Hence, Theorem 1.1 is proved.

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