A CONGRUENCE THEOREM FOR TREES

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Let A and B be two trees with vertex sets $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ respectively. The trees are congruent, are isomorphic, or "are the same type", $(A \cong B)$, if there exists a one-to-one correspondence between their vertices which preserves the join-relationship between pairs of vertices. Let $c(a_i)$ denote the $(n-1)$-point subgraph of A obtained by deleting a_i and all joins (arcs, segments) at a_i from A. It is the purpose here to show that if there is a one-to-one correspondence in type, and frequency of type, between the sub-graphs of order $n-1$ in A and B, that is, if there exists a labeling such that $c(a_i) \cong c(b_i)$, $i=1, 2, \cdots, n$, then $A \cong B$. It is assumed throughout, therefore, that there is a labeling of the two trees A and B such that $c(a_i) \cong c(b_i)$, $i=1, 2, \cdots, n$, where $n \geqq 3$.

Some lemmas to the main theorem are established first. Let T denote a certain type of graph of order j, where $2 \leqq j < n$, which occurs as a subgraph α times in A and β times in B. If α_i is the number of T-type subgraphs which have a_i as a vertex, then,

$$\alpha = \left(\sum_1^n \alpha_i \right) \Big/ j .$$

Similarly,

$$\beta = \left(\sum_1^n \beta_i \right) \Big/ j ,$$

where b_i is the number of T-type subgraphs having b_i as a vertex. Because $c(a_i) \cong c(b_i)$, the number of T-type subgraphs which do not have a_i as a vertex is the same as the number which do not have b_i as a vertex. Thus

$$\alpha - \alpha_i = \beta - \beta_i , \qquad i=1, 2, \cdots, n.$$

Therefore

$$\sum_1^n (\alpha - \beta) = \sum_1^n (\alpha_i - \beta_i) ,$$

so $n(\alpha-\beta)=j(\alpha-\beta)$, which implies $\alpha=\beta$. This, in turn, implies $\alpha_i=\beta_i$, $i=1, 2, \cdots, n$, and the lemma is established.

LEMMA 1. *Every type of proper subgraph which occurs in A or B*

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occurs the same number of times in both, and a_i and b_i are vertices in the same number of these subgraphs, $i=1, 2, \cdots, n$.

The case $j=2$ gives a special result.

LEMMA 2. *The vertices a_i and b_i have the same degree, $i=1, 2, \cdots, n$.*

Next it is clear that if either A or B consists of just a path between two end points then the other is also a path of the same length. If neither is just a path, then their maximal-length paths are proper subgraphs and have the same length because of Lemma 1.

This proves the third lemma.

LEMMA 3. *The trees A and B have the same radius r and both trees are central or both are bicentral.*

A correspondence between $c(a_i)$ and $c(b_i)$, under which $c(a_i) \cong c(b_i)$, will be called an a_i-mapping (or b_i-mapping), and the main theorem is obtained by using these submappings to define a congruence of A and B. Because such a congruence is more easily obtained when the trees are central, the proof will be carried through for bicentral trees only, with the simpler proof implied by analogy. It is supposed therefore that A has bicenters $\bar{a}_1$ and $\bar{a}_2$ and that B has bicenters $\overline{b_1}$ and $\overline{b_2}$ (where $\bar{a}_1$ is not necessarily a_1).

Let F be a component in the graph obtained by deleting from A the bicenters and all joins to them. There is a point of F joined in A to one bicenter, say $\bar{a}_1$, and no point of F is joined in A to $\bar{a}_2$. By $(\bar{a} \cup F)$ is meant the graph, which has for its vertices $\bar{a}$, and the vertices of F, and whose joins are the same as they are in A. The graph $(\bar{a}_1 \cup F)$ is a *limb at* $\bar{a}_1$. It is a *radial* or *nonradial* limb according as it does not possess an *r-point*, that is, a point whose distance in A from the nearest bicenter is r. An easy consequence of Lemma 1 is that a_i is an r-point if and only if b_i is an r-point.

Some special subgraphs of A and B are now defined. At $\bar{a}_i$ the radial limbs are

$$A_{i1},\ A_{i2},\ \cdots,\ A_{im_i}\ ,$$

and the non-radial limbs are

$$C_{i1},\ C_{i2},\ \cdots,\ C_{is_i}\ ,$$

while at $\overline{b_i}$ the radial limbs are

$$B_{i1},\ B_{i2},\ \cdots,\ B_{in_i}$$

and the non-radial limbs are

$$D_{i1}, D_{i2}, \cdots, D_{it_i}, \qquad i=1, 2 .$$

Next,

$$A_i=(A_{i1} \cup A_{i2} \cup \cdots \cup A_{im_i}), \quad B_i=(B_{i1} \cup B_{i2} \cup \cdots \cup B_{im_i}),$$
$$C_i=(C_{i1} \cup C_{i2} \cup \cdots \cup C_{is_i}),$$

and

$$D_i=(D_{i1} \cup D_{i2} \cup \cdots \cup D_{it_i}), \qquad i=1, 2.$$

Finally,

$$A_r=(A_1 \cup A_2),\ B_r=(B_1 \cup B_2),\ C=(C_1 \cup C_2),$$

and

$$D=(D_1 \cup D_2) .$$

In obtaining congruences for these special subgraphs, an important role is played by center preserving mappings, that is, those which pair $\bar{a}_1$ and $\bar{a}_2$ in some order with $\bar{b}_1$ and $\bar{b}_2$. It is useful, therefore, to define a vertex a_i to be a *nonessential point*, (n. e. point), if it is of degree one (is an end point) such that $c(a_i)$ is a bicentral tree of radius r. Every end point, which is not an r-point, is an n. e. point. An r-point is nonessential if it belongs to a limb with more than one r-point, or if the bicenter to which its limb belongs has more than one radial limb. If a_i is an n. e. point then b_i is an n. e. point and every a_i-mapping is center preserving. The following fact is also useful.

LEMMA 4. *If a_i is an* n. e. *point of A in A_r then b_i is an* n. e. *point of B in B_r.*

Proof. Assume $b_i \notin B_r$, that is, $b_i \in D$. Any a_i-mapping must pair the remainder of A_r (without a_i) with all of B_r, so the order of A_r is one greater than that of B_r. If A had a nonradial limb it would have an n. e. point in C, say a_j, and an a_j-mapping would have to pair A_r with all or part of B_r, which is impossible. Therefore A has no nonradial limb and b_i is the only point of B not in B_r. The sum of the degrees of $\bar{a}_1$ and $\bar{a}_2$ is therefore smaller than the sum of the degrees of $\bar{b}_1$ and $\bar{b}_2$. If B had an n. e. point b_l, distinct from b_i, the sum of the degrees of $\bar{b}_1$ and $\bar{b}_2$ would be the same in $c(b_l)$ as in B, and therefore a b_l-mapping could not be center preserving. From this it follows that a_i and b_i are the only n. e. points in A and B respectively. Thus

A consists of a $(2r+1)$-path and one extra point a_i joined to a point a_k, which is not a center, while B consists of a $(2r+1)$-path and one extra point b_i joined to a center. This center is b_k since a_k is the only point in A of degree three. But now it is clear that $c(a_k)$ has a component which is a path of greater length than any in $c(b_k)$, which contradicts $c(a_k) \cong c(b_k)$. The assumption that b_i is in D is therefore false and Lemma 4 is established.

THEOREM. *If A and B are trees with vertices $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$, $n \geqq 3$, respectively, and $c(a_i) \cong c(b_i)$, $i=1, 2, \cdots, n$ then $A \cong B$.*

Proof. As previously indicated, the details will be given only for the case where A and B are bicentral.

Case 1. One of the trees, say A, has a nonradial limb. Then A has an n.e. point a_k in C and, from Lemma 4, b_k is in D. An a_k-mapping, therefore, pairs A_r with B_r, so

$$(1) \qquad A_r \cong B_r\,.$$

Next,

(2) There is a congruence of C and D which pairs $\bar{a}_1$, $\bar{a}_2$ in some order with $\bar{b}_1$ and $\bar{b}_2$.

Consider the n. e. points of A in A_r. First, suppose the limb to which one of these points a_i belongs is still of length r after a_i is deleted from it. Then an $a_i$mapping cannot take this sub-limb into D, and so must pair the remainder of A_r with the remainder of B_r. It therefore pairs C with D as stated in (2). Next, suppose every n.e. point which belongs to A_r or B_r is the end of an r-path limb. If a_i and b_i are such points, then deleting them from their limbs produces two $(r-1)$-path limbs. Since these sub-limbs are congruent, an a_i-mapping either pairs C and D as stated in (2) or else can be redefined to do so. The only remaining possibility is that no n.e. point occurs in either A_r or B_r, so each is a $(2r+1)$-path. Let a_i be the r-point in A_i and b_i be the r-point in B_i, $i=1, 2$. Since $C(a_1)$ is a tree of radius r and center $\bar{a}_2$ and $c(b_1)$ is a congruent tree with center $\bar{b}_2$, an a_1-mapping pairs $\bar{a}_2$ and $\bar{b}_2$. It must also pair the nonradial limbs of A at $\bar{a}_2$ with the nonradial limbs of B at $\bar{b}_2$, and hence $C_2 \cong D_2$. By the same reasoning, an a_2-mapping establishes a congruence of C_1 and D_1 which pairs $\bar{a}_1$ with $\bar{b}_1$, so there is clearly a congruence of C and D satisfying (2).

If a congruence of C and D, satisfying (2), and a congruence of

A_r and B_r both pair the bicenters in the same order, then clearly $A \cong B$. Assume, on the contrary, that every congruence of A_r with B_r pairs the bicenters in one order, say $\bar{a}_1$ with $\bar{b}_1$ and $\bar{a}_2$ with $\bar{b}_2$, while every congruence of C and D, satisfying (2), pairs the bicenters in the opposite order, namely $\bar{a}_1$ with $\bar{b}_2$ and $\bar{a}_2$ with $\bar{b}_1$. It will be shown that this leads to a contradiction and hence that $A \cong B$ under Case 1. First, from the assumption about the congruence of A_r and B_r, it follows that each is not just a $(2r+1)$-path. Therefore A has an n. e. point a_i in A_r, and it may be supposed that $a_i \in A_1$. Since an a_i-mapping implies a congruence of C and D satisfying (2) it must pair $\bar{a}_1$ and $\bar{a}_2$ with $\bar{b}_2$ and $\bar{b}_1$ in that order. By assumption, $A_1 \cong B_1$ and $A_2 \cong B_2$, so $b_i \in B_2$ would imply that an a_i-mapping pairs A_2 with B_1. But then $A_1 \cong A_2 \cong B_1 \cong B_2$ would contradict the unique mapping of bicenters in any congruence of A_r with B_r. Therefore $b_i \in B_1$. Let f_1 be the order of A_1 and B_1 and f_2 be that of A_2 and B_2. An a_i-mapping shows $f_2 = f_1 - 1$. Suppose A has an n. e. point a_j in A_2. Then an a_j-mapping pairs $\bar{a}_1$ and $\bar{b}_2$, so it pairs A_1 with all or part of B_2. But this is impossible because $f_1 > f_2$. Therefore there is no n. e. point of A in A_2, and, by the same reasoning, there is no n. e. point of B in B_2. Thus A_2 and B_2 are paths of length r, and A and B each have just two end points in A_1 and B_1 respectively.

Now consider nonradial limbs. At least one exists so, from Lemma 4, at least one each exists in each tree. Suppose there is a nonradial limb at a_1, and let a_j be an end point of A in this limb. Then $b_j \in D$. Because an a_j-mapping includes a congruence of A_r and B_r, it pairs a_1 with $\bar{b}_1$ and $\bar{a}_2$ with $\bar{b}_2$. If b_j were in D_1 such a mapping would imply $C_2 \cong D_2$, and this, with $C_1 \cong D_2$ and $C_2 \cong D_1$, would yield $C_1 \cong C_2 \cong D_1 \cong D_2$, contradicting the unique center pairings in a congruence of C and D. Therefore $b_j \in D_2$. Let f_3 be the order of C_1 and D_2 and f_4 by the order of C_2 and D_1. An a_j-mapping shows that $f_3 - 1 = f_4$. Therefore there is no n. e. point in C_2 and none in D_1. For if a_k in C_2 were an n. e. point, an a_k-mapping would pair $\bar{a}_1$ with $\bar{b}_1$ and therefore would pair C_1 with all or part of D_1. This is impossible because $f_3 > f_4$. There are, therefore, no nonradial limbs at $\bar{a}_2$ or $\bar{b}_1$, and there is just one nonradial limb at $\bar{a}_1$ and at b_2, each of length one. The center $\bar{a}_1$ and $\bar{b}_2$ are of degree three and $\bar{a}_2$ and $\bar{b}_1$ have degree two. Let a_1 be the end point of A in A_2. The tree $c(a_1)$ has only one center, namely $\bar{a}_1$ of degree three. If the r-point b_1 were in B_2, the center of $c(b_1)$ would have degree two, contradicting $c(a_1) \cong c(b_1)$. So $b_1 \in B_1$. Also b_1 is the only r-point in B_1, for otherwise $c(b_1)$ would be a bicentral tree. If a_2 and b_2 denote the other r-points of A and B respectively, it follows that $c(a_2)$ and $c(b_2)$ are central trees and that $\bar{a}_2$ and $\bar{b}_1$ are their respective centers. But in $c(a_2)$ the radial arm to a_1 is a path in $c(b_2)$ both radial arms branch, and this contradicts $c(a_2) \cong c(b_2)$. The supposition that a nonradial limb exists at $\bar{a}_2$ rather than $\bar{a}_1$ leads to the same kind of contradiction, hence $A \cong B$ under all the possibilities of Case 1.

Case 2. There are no nonradial limbs but one tree has at least three radial limbs. Suppose there are at least two radial limbs at $\bar{a}_2$, and now let $\bar{a}_1 = a_1$ and $\bar{b}_2 = b_2$. One and only one component of $c(a_1)$ is a central tree of radius r. Its center is $\bar{a}_2$ and all of its limbs are radial. Let b_1' be the center of the corresponding, congruent tree in $c(b_1)$. If b_1 is neither $\bar{b}_1$ or $\bar{b}_2$, then b_1' is a non-end point of B in some limb of B, say a limb at $\bar{b}_1$. There is a path P from $\bar{b}_1$, to b_1' and there also exists a path P' starting at b_1' and having no join in common with P. Since the length of P' must be less than r, all the limbs of the tree centered at b_1' cannot be radial. The supposition that b_1 is neither $\bar{b}_1$ or $\bar{b}_2$ is therefore false, and b_1 may be taken to be $\bar{b}_1$. Then $A_2 \cong B_2$ is implied by an a_1-mapping.

If there are at least two radial arms at either a_1 or b_1, the same reasoning shows that $A_1 \cong B_1$ and this, with $A_2 \cong B_2$, implies $A \cong B$. Suppose, then, that A_{11} and B_{11} are the only limbs at a_1 and b_1 respectively, and let the order of A_{11} be at least as great as that of B_{11}. There is an r-point a_j in A_{21} and it is an n. e. point. An a_j-mapping must pair a_1 with b_1 because these are of degree two while $\bar{a}_2$ and $\bar{b}_2$ are of degree at least three. The mapping therefore pairs A_{11} with all or part of B_{11}, and since the latter case is excluded by the orders of A_{11} and B_{11}, it follows that $A_{11} \cong B_{11}$. This, with $A_2 \cong B_2$, implies $A \cong B$ and completes Case 2.

Case 3. Each tree has exactly two limbs. Let n_i be the order of A_i and n_i' be the order of B_i, $i=1, 2$. Assume that the pair n_1, n_2 is not the pair n_1', n_2' in either order. Then, because $n_1+n_2=n_1'+n_2'$, one of the four numbers is a strict maximum. Suppose $n_2 > \max(n_1, n_1', n_2')$. Then A_2 is not congruent to B_1 or B_2 or any of their subgraphs, and therefore A_1 has no n. e. points. It is therefore a path with one r-point of A, say a_3. Then vertex b_3 is an r-point and is the only r-point of its limb because a_3 is not an n. e. point. The tree $c(a_3)$ is central, has radius r, and $\bar{a}_2$ is its center, so its two radial limbs have orders n_1 and n_2. The center of $c(b_3)$ is either $\bar{b}_1$ or $\bar{b}_2$, but in either case the two limbs have orders n_1' and n_2', so a congruence of $c(a_3)$ and $c(b_3)$ is impossible. From this contradiction it follows that n_1 and n_2 are in some order the numbers n_1' and n_2' and it may be supposed that $n_1=n_1'$

and $n_2=n_2'$.

Now consider the n. e. points. If none exist, then both trees are $(2r+1)$-paths and hence are congruent. If, on the other hand, a_i is an n. e. point of A, then b_i is an n. e. point of B and the following applies:

(3) If a_i and b_i are n. e. points, with a_i in A_1 and b_i in B_2 (or a_i in A_2 and b_i in B_1), then $A \cong B$.

For, suppose $a_i \in A_1$ and $b_i \in B_2$. Then because of the orders of the limbs, an a_i-mapping pairs A_2 with B_1, so $A_2 \cong B_1$ and $n_1=n_2$. If there is no n. e. point of A in A_2 then A_2 is an r-path and so is B_1 because it is congruent to A_2. But then, because $n_1=n_2$, both A_1 and B_2 are also r-paths, which is contradictory. Therefore, there exists an n. e. point a_j in A_2. Because $n_1=n_2$, an a_j-mapping pairs A_1 either with B_2 or with B_1. The first case, together with $A_2 \cong B_1$, implies $A \cong B$ directly. The second case implies $A_1 \cong B_1 \cong A_2$, and from this it follows that there is an n. e. point, say b_k, in B_1. Then a b_k-mapping pairs B_2 with either A_1 or A_2. Therefore all the limbs are the limbs are the same type and $A \cong B$.

Because of (3), it is now only necessary to consider the case $a_i \in A_1$ and $b_i \in B_1$. There are two sub-cases.

Case 3.1. There is no n. e. point in either A_2 or B_2. Then $A_2 \cong B_2$ since they are both r-paths. Let the end point of A_2 be a_3. Then $c(a_3)$ is a central tree, of radius r, whose center is $\bar{a}_1$. From $c(a_3) \cong c(b_3)$, it follows that b_3 is the only r-point of some limb in B. Assume $b_3 \in B_1$. Let b_4 be the r-point of B in B_2. Then a_4 is the only r-point of A in A_1. An a_4-mapping pairs $\bar{a}_2$ with $\overline{b_1}$ and also pairs the limb of $c(a_4)$ which is not a path with the limit of $c(b_4)$ which is not a path. It therefore pairs $\bar{a}_1$ with b_{11}, the first point in the limb B_1. Because $\bar{a}_1$ is of degree two, the point b_{11} is of degree two and so is joined to a well defined second point in B_1, say b_{12}. An a_3-mapping pairs $\overline{b_2}$ and $\bar{a}_1$, and, by the same reasoning as before, pairs $\overline{b_1}$ with the first point, say a_{11}, in A_1. Then a_{11} is of degree two and so is joined to a well defined second point a_{12} in A_1. An a_4-mapping must, then, pairs a_{11} with b_{12}, so b_{21} is of degree two and joins the third point in B_1. Alternating this way between the a_3 and a_4 mappings, it follows that all points of A_1 and B_1 are of degree two, which is absurd. The assumption that b_3 is in B_1 is therefore false, so $b_3 \in B_2$. Now an a_3-mapping must pair $\bar{a}_1$ with $\overline{b_1}$ and must also pair the branching and non-branching limbs at $\bar{a}_1$ and $\overline{b_1}$. Therefore $A_1 \cong B_1$, and this, with $A_2 \cong B_2$, implies $A \cong B$.

Case 3.2. There is an n. e. point in A_2 or else there is one in B_2. Suppose $a_j \in A_2$ is nonessential. If b_j is in B_1, then, from (3), $A \cong B$, so suppose $b_j \in B_2$. If an a_i-mapping pairs A_2 with B_2 and an a_j-mapping pairs A_1 with B_1 then clearly $A \cong B$. So suppose an a_i-mapping pairs A_2 with the remainder of B_1 (without b_i). Then $n_1 = n_2 + 1$ and because of this an a_j-mapping pairs A_1 with B_1, hence $A_1 \cong B_1$. Let a_k be the point of A_1 paired with b_i in an a_j-mapping. Then A_1 minus a_k, that is the graph obtained from A_1 by deleting a_k and all joins to a_k, is congruent to B_1 minus b_i. But an a_i-mapping pairs A_2 with B_1 minus b_i. Therefore $c(a_k)$ is a bicentral tree both of whose limbs are congruent to A_2. From Lemma 1 there is a subgraph of the same type in B and hence $B_2 \cong A_2$. This, with $A_1 \cong B_1$, implies $A \cong B$ and completes the proof.

It is natural to wonder if any two graphs must be isomorphic when they have the same composition in terms of $(n-1)$-point subgraphs. The author has considered the question for graphs having at most one join for any pair of points, with no point joined to itself. Actual inspection shows that the theorem is valid for all such graphs up to order seven. It also holds for any two such graphs of general, finite order if either is disconnected or its transpose is disconnected. (The transpose is obtained by reversing the join relationship between every pair of vertices.) However, the author was unable to prove or disprove the general case. As a final comment, it is not true that the same composition in terms of $(n-2)$-point subgraphs implies isomorphism.

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