RANGES AND INVERSES OF PERTURBED LINEAR OPERATORS

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1. Introduction. Let X and Y denote normed linear spaces and let $T \neq 0$ be a linear operator with domain $D(T) \subset X$ and range $R(T) \supset Y$. In this paper, $D(T)$ is not required to be dense in X and T need not be continuous. Furthermore, X and Y shall be assumed complete only when necessary. Under these general conditions, we investigate some invariant properties of the range and inverse of T when T is perturbed by a bounded linear operator A. For example, it is shown that if the range of T is not dense in Y and T has a bounded inverse, then $T + A$ has the same properties provided that $D(A) \supset D(T)$ and the norm of A is sufficiently small. In addition, a theorem of Yood ([5], Th. 2.1) is generalized with some of the proofs simplified.

DEFINITION. Let $X_1 = \overline{D(T)} \subset X$. When X_1 is considered as a normed linear space, the conjugate transformation T' is defined as follows: Its domain $D(T')$ consists of the set of all y' in the conjugate space Y' for which $y'T$ is continuous on $D(T)$; for such a y' we define $T'y' = x'$ where x' is the unique bounded linear extension of $y'T$ to X_1; that is, x' is in the conjugate space X_1' of X_1.

The above notations shall be retained throughout the discussion.

2. Ranges and inverses of $T + A$.

LEMMA 1. *If T has a bounded inverse, then so does $T + A$ whenever $\|A\| < \|T^{-1}\|^{-1}$.*

Proof. $\|(T + A)x\| > (\|T^{-1}\|^{-1} - \|A\|)\|x\|$.

THEOREM 1. *If $\overline{R(T)} = Y$ and T has a bounded inverse, then $\overline{R(T + A)} = Y$ and $T + A$ has a bounded inverse whenever $\|A\| < \|T^{-1}\|^{-1}$ and $D(T) \subset D(A)$.*

Proof. By [4] Th. 1.4, $(T')^{-1} = (T^{-1})'$ exists and is continuous on X_1'. Hence from the lemma we conclude that $(T + A)' = T' + A'$ has a bounded inverse since $\|A'\| = \|A\| < \|T^{-1}\|^{-1} = \|(T')^{-1}\|^{-1}$. The theorem now follows from [4] Th. 1.2.

If for $X = Y$, the resolvent of a linear operator T is defined as the set of scalars λ such that $(T - \lambda I)^{-1}$ exists and is continuous on a

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domain dense in X, then the following corollary is an immediate result of the theorem.

COROLLARY. *The resolvent of a linear operator is open.*

DEFINITION. For each $z \neq 0$ in Y, let

$$m_z(T) = \sup \{k / \|z - Tx\| \geqq k \|Tx\|, x \in D(T)\} .$$

We define $m(T) = \sup_{0 \neq z \in Y} m_z(T)$.

REMARK. $m(T) \leqq 1$; This follows from the fact that for $Tx \neq 0$ and for each $z \in Y$, $\|z - T\alpha x\| / \|T\alpha x\| \leqq 1 + \|z\| / \|T\alpha x\| \rightarrow 1$ as $|\alpha| \rightarrow \infty$.

LEMMA 2. *Let Y be complete. Then $\overline{R(T)} = Y$ if and only if $m(T) = 0$.*

Proof. If $\overline{R(T)} = Y$, it is easy to see that $m(T) = 0$. Suppose there exists an element $y_0 \in Y$ which is not in $\overline{R(T)}$. The 1-dimensional linear manifold $[y_0]$ spanned by y_0 and the linear manifold $[y_0] + \overline{R(T)}$ are closed in Y; moreover, $[y_0] \cap \overline{R(T)} = (0)$. Hence by [2] Th. 2.1, there exists a $k > 0$ such that $\|y_0 - y\| \geqq k\|y\|$ for all $y \in R(T)$; that is, $m(T) > 0$.

THEOREM 2. *If $\overline{R(T)} \neq Y$ and T has a bounded inverse, then $\overline{R(T + A)} \neq Y$ and $T + A$ has a bounded inverse whenever $\|A\| < m(T)/\|T^{-1}\|$, and $D(T) \subset D(A)$.*

Proof. Clearly there is no loss of generality if the theorem is proved for the completion $\tilde{Y}$ of Y. Thus it may be assumed that Y is complete. We now simplify and apply an argument given by Yood [5, p. 489]. From Lemma 1, $T + A$ has a bounded inverse. By Lemma 2, there exists, for each $\varepsilon > 0$, an element $y_0 \in Y$ but not in $\overline{R(T)}$ such that

$$\|y_0 - Tx\| \geqq (m(T) - \varepsilon)\|Tx\| \text{ for all } x \in D(T) . \tag{1}$$

Suppose that the theorem is not true. Then $y_0 \in \overline{R(T + A)} = Y$ and thus we may choose an element $x \in D(T)$ so that

$$\|(T + A)x - y_0\| < \min(\varepsilon d, \|y_0\|) ,$$

where d is the distance between y_0 and $\overline{R(T)}$. In particular,

(2) $\quad ||(T+A)x - y_0|| < \varepsilon d \leqq \varepsilon||y_0 - Tx||$ and $x \neq 0$.

From (1) and (2),

$$||A||||x|| \geqq ||Ax|| \geqq ||Tx - y_0|| - ||y_0 - (T+A)x|| > (1-\varepsilon)||y_0 - Tx||$$
$$\geqq (1-\varepsilon)(m(T)-\varepsilon)||Tx|| \geqq ||T^{-1}||^{-1}(1-\varepsilon)(m(T)-\varepsilon)||x|| .$$

Since $\varepsilon > 0$ was arbitrary, $||A|| \geqq ||T^{-1}||^{-1}(m(T))$ which is impossible.

LEMMA 3. *Suppose X and Y are complete. If T is a closed linear operator, then $R(T) = Y$ and T^{-1} does not exist if and only if $\overline{R(T')} \neq X_1'$ and T' has a bounded inverse.*

Proof. This follows from the "state diagram" for closed operators [1].

THEOREM 3. *Suppose X and Y are complete. If T is closed, $R(T) = Y$ and T^{-1} does not exist, then $R(T+A) = Y$ and $(T+A)^{-1}$ does not exist whenever $D(T) \subset D(A)$ and $A < m(T')/||(T')^{-1}||$.*

Proof. By Lemma 3, $\overline{R(T')} \neq X_1'$ and T' has a bounded inverse. Futhermore, $D(A') = Y' \supset D(T')$ and $T' \neq 0$ since $D(T')$ is total ([4] Th. 1.1). From Theorem 2, it is clear that $\overline{R(T'+A')} \neq X_1'$ and $T'+A'$ has a bounded inverse. Since $T'+A' = (T+A)'$ and $T+A$ is closed, the theorem follows from Lemma 3.

3. A generalization of a theorem. In ([5] Th. 2.1), Yood proves a theorem about the range of a bounded linear transformation T and its conjugate T', where T maps Banach Space X into Banach space Y. We now generalize the theorem by requiring instead that T be a closed linear operator on $D(T)$. The results are stated in a different but more precise form than in [5].

DEFINITION. If T has a bounded inverse, let $K(T) = ||T^{-1}||$, otherwise let $K(T) = 0$. We now define a number $\alpha(T)$ as follows:

$$\alpha(T) = \min\left((m(T), \frac{m(T)}{K(T)}\right) \text{ if } m(T) > 0$$
$$= \infty \text{ if } m(T) = 0 .$$

$\alpha(T')$ shall be defined in a similar manner.

THEOREM 4. *Suppose X and Y are complete. Let T be a closed linear transformation and let A represent a bounded linear transform-*

ation such that $D(A) \supset D(T)$. *Then the following statements concerning* T *are equivalent.*

(1) *Either* T *has bounded inverse or* $R(T) = Y$.
(2) $\overline{R(T' + A')} \subset R(T')$ *if* $\|A\| < \alpha(T')$.
(3) $R(T' + A') \subset R(T')$ *if* $\|A\| < \alpha(T')$.
(4) $R(T')$ *is not a proper dense subset of* X_1' *and* $\|A\| < \alpha(T')$ *implies that* $\overline{R(T' + A')} \subset R(T')$.
(5) $R(T')$ *is not a proper dense of* X_1' *and* $\|A\| < \alpha(T')$ *implies that* $R(T' + A') \subset R(T')$.
(6) $\overline{R(T + A)} \subset R(T)$ *if* $\|A\| < \alpha(T)$.
(7) $R(T + A) \subset R(T)$ *if* $\|A\| < \alpha(T)$.
(8) $R(T)$ *is not a proper dense subset of* Y *and* $\|A\| < \alpha(T)$ *implies that* $\overline{R(T + A)} \subset R(T)$.
(9) $R(T)$ *is not a proper dense subset of* Y *and* $\|A\| < \alpha(T)$ *implies that* $R(T + A) \subset R(T)$.

Proof. (1) implies (2): (T need not be closed): If T has a bounded inverse, then by [1] $R(T') = X_1' \supset R(T' + A')$ for all A. If T has no bounded inverse, then $R(T) = Y$ so that $R(T') \neq X_1'$ and T' has a bounded inverse by [1]. Since T' is closed, it follows that $R(T')$ is closed; i.e. $m(T') > 0$. If (2) is false, there exists an $x_0' \in \overline{R(T' + A')}$ but at a positive distance d from $R(T')$. By the argument as in Theorem 2, $\|A\| = \|A'\| \geqq \dfrac{m(T')}{K(T')} \geqq \alpha(T') > \|A\|$ which is impossible.

(2) implies (3). Obvious

(3) implies (1): (cf. [5]): If $R(T) \neq Y$ and T has no bounded inverse, then we show that (3) fails to hold. By [1], $R(T') \neq X_1'$ and T' has no bounded inverse. Therefore, we may choose an element $x_0' \in X_1'$, $\|x_0'\| = 1$ and $x_0' \notin R(T')$. For each $\varepsilon > 0$, there exists an element $y_0' \in D(T')$ such that $\|y_0'\| = 1$, $\|T'y_0'\| < \varepsilon$ and an element y_0 such that $\|y_0\| = 1$, $y_0'y_0 = \beta$ is real and $1 \geqq \beta \geqq 1/2$. Let A be defined by $Ax = \varepsilon(x_0'x - (\varepsilon\beta)^{-1}T'y_0'x)\, y_0$ for $x \in D(T)$. Hence

$$A'y_0' = \varepsilon y_0'y_0(x_0' - (\varepsilon\beta)^{-1}T'y_0') = \varepsilon\beta x_0' - T'y_0' ,$$

so that

$$(T' + A')y_0' = \varepsilon\beta x_0' \notin R(T'). \quad \text{Moreover, } \|A\| \leqq \varepsilon\left(1 + \frac{1}{\beta}\right) \leqq 3\varepsilon .$$

Since $\varepsilon > 0$ was arbitrary, it follows that (3) does not hold.

(4) and (5) implies (1): Follows from the above argument.

(1) implies (4) and (5): (T need not be closed): This follows from the fact that

(1) implies that $R(T')$ is closed and also that (1) implies (2).

(1) implies (6): If $R(T) = Y$, then (6) is satisfied. Suppose $R(T) \neq Y$ but that T has a bounded inverse. Hence $R(T)$ is closed so that $m(T) > 0$. If (6) is false, there exists an element $y_0 \varepsilon Y = \overline{R(T + A)}$ but $y_0 \notin R(T)$. The remaining argument is now as in Theorem 2.

(6) implies (7): Obvious

(7) implies (1): If $R(T) \neq Y$ and T has no bounded inverse, then for $\varepsilon > 0$, there exists an element $x_0 \in D(T)$, $\|x_0\| = 1$ such that $\|Tx_0\| < \varepsilon$. An element $x_0' \in X_1'$ is chosen so that $\|x_0'\| = 1$ and $x_0'x_0 = 1$. Suppose that $y \notin R(T)$ and $\|y\| = 1$. We define A by the relation

$$Ax = \varepsilon x_0'x(y - \varepsilon^{-1}Tx_0),\ x \in D(T)\ .$$

Then $(T + A)x_0 = \varepsilon y \notin R(T)$. Moreover, $\|A\| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, (7) cannot hold. Thus the assertion is proved.

(8) and (9) are equivalent to (1): This is shown in the same way that (4) and (5) were shown equivalent to (1).

If there is no restriction put on the inverse but only on the range of T, we may still infer something about the range of $T + A$. In fact, A need not be continuous. The following theorem illustrates this.

THEOREM 5. *Suppose X and Y are complete. If T is a closed linear operator with a closed range, then there exists a $\rho > 0$ such that $T + A$ is also a closed linear operator with a closed range whenever A is a linear operator* (*not necessarily continuous*) *with $D(A) \supset D(T)$ and $\|Ax\| \leqq \rho(\|x\| + \|Tx\|)$ for every $x \in D(T)$.*

Proof. We introduce another norm $\|x\|_1$, on $D(T)$ by defining $\|x\|_1, = \|x\| + \|Tx\|$. D_1 shall denote $D(T)$ with this new norm. Since X and Y are complete and T is closed, it is easy to see that D_1 is a complete normed linear space. Moreover, T_1 as a transformation of D_1 into Y is bounded and has an inverse. Thus by the closed graph theorem, T^{-1} is bounded; i.e. there exists an $m > 0$ such that $\|Tx\| \geqq m(\|x\| + \|Tx\|)$ for $x \in D_1$. Choose $\rho > 0$ so that $1 > \rho$ and $m - \rho > 0$. Thus $\|(T + A)x\| \geqq (m - \rho)(\|x\| + \|Tx\|)$, whence $T + A$ has a bounded inverse from $R(T + A)$ onto D_1. Clearly $T + A$ is continuous on D_1. Since defining a new norm in $D(T)$ does not alter the situation in Y, it follows that $R(T + A)$ is closed. In [3], Nagy proves that $T + A$ is a closed operator from $D(T)$ into Y, which completes the proof of the theorem.

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