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# SIMPLE AREAS

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**Let $\lambda \geqq 1, E = E^N$ and $g$ be continuous on $E \times E \times E$ with $g(a, \cdot, \cdot)$ convex, $g(a, kb, kc) = k^2 g(a, b, c)$ for all real $k$ and $(b^2 + c^2)/\lambda \leqq g(a, b, c) \leqq \lambda(b^2 + c^2)$ for all $a, b, c \in E$ where $b^2 = \| b \|^2$. If $f(a, d \wedge e) = \min_{b \wedge c = d \wedge e} g(a, b, c)$ then $f$ is a permissible integrand for the two-dimensional parametric variational problem.**

**Let $\gamma$ be a simple closed curve in $E, B$ be the closed unit circle in the plane, $C$ be the collection of functions $x$ continuous on $B$ into $E$ for which $x \mid \partial B \in \gamma$ and $D = \{x \in C \mid x$ is a $D$-map$\}$. Suppose that $D$ is not empty. It was shown in 'A problem of least area', [7], that the problem of minimizing $I(f)$ over $D$ is equivalent to minimizing $I(g)$ over $D$ where $I(f, x) = \iint f(x, p \wedge q), I(g, x) = \iint g(x, p, q)$, $p = x_u$, $q = x_v$ and both integrals are taken over $B$. The minimizing solution of $I(g)$ is known to have differentiability properties corresponding to $g$, and this solution also minimizes $I(f)$.**

**The function $f$ is simple, that is, for each $a \in E$, each supporting linear functional to $f(a, \cdot)$ is simple. If $N = 3$, then, of course, each parametric integrand is simple. In this paper we show that for each simple parametric integrand $F$ there exists $G$, satisfying the conditions imposed upon $g$, such that $F$ is obtained from $G$ as $f$ was obtained from $g$.**

In [7] we showed that the two-dimensional parametric problem in the calculus of variations considered by [1, 2, 4, 5, 6] could be reduced to a nonparametric problem provided the parametric integrand $f$ was properly related to a suitable nonparametric integrand $g, f = Ag$. When this occured, not only the existence of the minimizing solution $x$ was given by the nonparametric theory [3] but also its smoothness, if $g$ was smooth. Furthermore, we saw that $Ag$ was simple for each $g$, that is, each supporting linear functional of $Ag$ was simple. We shall show here that whenever $f$ is simple then there exists $g$ such that $f = Ag$.

Let $E = E^N$. If $a \in E$ or $a \in E^*$ let $a^2 = \| a \|^2$. Let $T_1 = E \wedge E$ with norm $N_1$, thus $N_1(a \wedge b)$ is the area of the parallelogram spanned by $a$ and $b$, and let $T_2 = E \times E$. We define $N_2$ on $T_2$ by $N_2(a, b) = (a^2 + b^2)/2$. Let $T^*$ be the set of all simple linear functionals over $T_1$ which have norm one. Hence, if $\zeta \in T^*$, there exist $\xi$ and $\eta$ in $E^*$ such that $\zeta = \xi \wedge \eta$ with $\xi^2 = \eta^2 = 1$ and $\xi \cdot \eta = 0$. We frequently

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write $\xi a$ for $\xi(a)$.

If $\varphi$ is defined on $P \times Q$ then $\varphi_p$ is defined on $Q$ by $\varphi_p(q) = \varphi(p, q)$ for all $p \in P$ and $q \in Q$.

Let $\mathscr{A}$ be the set of all continuous real-valued functions $f$ on $E \times T_1$ for which there exists $\lambda = \lambda(f) \geqq 1$ with $N_1/\lambda \leqq f_a \leqq \lambda N_1$ and such that $f_a$ is convex and positively homogeneous of degree one for each $a \in E$. Let $\mathscr{D}_0$ be the set of all continuous real-valued functions $g$ on $E \times T_2$ for which there exists $\lambda \geqq 1$ with $N_2/\lambda \leqq g_a \leqq \lambda N_2$ and such that $g_a$ is convex and homogeneous of degree two for each $a \in E$. For our purposes, $\mathscr{D}_0$ gives nothing more than $\mathscr{D} = \{h \in \mathscr{D}_0 \mid$ there exists $g \in \mathscr{D}_0$ such that $h(a, b, c) = \max_\theta g(a, b\cos\theta - c\sin\theta, b\sin\theta + c\cos\theta)\}$.

If $g \in \mathscr{D}$ then let $Ag(a, b \wedge c) = \min_{d \wedge e = b \wedge c} g(a, b, c)$ and

$$Ag(a, \alpha) = \inf\left\{\sum_{i=1}^{k} Ag(a, b_i \wedge c_i) \,\middle|\, \sum_{i=1}^{k} b_i \wedge c_i = \alpha\right\}$$

for all $\alpha \in T_1$. We saw in [7] that $Ag \in \mathscr{A}$ and that $Ag$ is simple. Evidently $Ag(a, b \wedge c) = \min_{r \neq 0} g(a, rb, sb + r^{-1}c)$.

If $g \in \mathscr{D}$ then $2g_a^{1/2}$ is convex and positively homogeneous of degree one. Suppose that $\xi, \eta \in E^*$, and so $(\xi, \eta) \in T_2^*$. We say that $(\xi, \eta)$ *supports* $2g_a^{1/2}$ at $(b, c)$ if $\xi b + \eta c = 2[g(a, b, c)]^{1/2}$ and if $\xi d + \eta e \leqq 2[g(a, d, e)]^{1/2}$ for all $(d, e)$. Furthermore, $(\xi, \eta)$ supports $2g_a^{1/2}$ *properly* at $(b, c)$ if $(\xi, \eta)$ supports $2g_a^{1/2}$ at $(b, c)$ and if $\xi b = \eta c$, $\xi c = \eta b = 0$.

The following lemma appears in [7]

LEMMA 1. *If $(\xi, \eta)$ supports $2g_a^{1/2}$ properly at $(b, c)$ then $g(a, b, c) = Ag(a, b \wedge c) = [b \wedge c, \xi \wedge \eta]$ where $[d \wedge e, \rho \wedge \sigma] = \rho(d)\sigma(e) - \rho(e)\sigma(d)$.*

*Proof.* If $r \neq 0$ then $4g(a, rb, sb + r^{-1}c) \geqq (r\xi(b) + r^{-1}\eta(c))^2 = (r + r^{-1})^2(\xi b + \eta c)^2/4 \geqq (\xi b + \eta c)^2 = 4g(a, b, c)$ and $g(a, b, c) = [b \wedge c, \xi \wedge \eta]$.

Now suppose that $\xi, \eta \in E^*$, $\xi^2 = \eta^2 = 1$ and $\xi \cdot \eta = 0$. Let $H_{\xi,\eta}(b, c) = [(\xi b + \eta c)^2 + (\xi c - \eta b)^2]/4$. It is easy to see that $H_{\xi,\eta} = H_{\rho,\sigma}$ if $\xi \wedge \eta = \rho \wedge \sigma$, $\rho^2 = \sigma^2 = 1$ and $\rho \cdot \sigma = 0$. Hence we can define $h_{\xi \wedge \eta} = H_{\xi,\eta}$. It quickly follows that $h_\zeta(b\cos\theta - c\sin\theta, b\sin\theta + c\cos\theta) = h_\zeta(b, c)$ for all $\zeta \in T^*$ and all real $\theta$. As the sum of squares of linear functionals, $h$ is continuous, convex and homogeneous of degree two. An easy computation shows that $\rho \wedge \sigma = \zeta$ if $(\rho, \sigma)$ supports $2h_\zeta^{1/2}$ at $(b, c)$ where $h_\zeta(b, c) \neq 0$.

We define $Ah_\zeta(b \wedge c) = \inf_{d \wedge e = b \wedge c} h_\zeta(d, e)$.

If $\phi$ is a real number let $\phi^+ = \max\{\phi, 0\}$.

LEMMA 2. $Ah_\zeta(b \wedge c) = [b \wedge c, \zeta]^+$.

*Proof.* Suppose that $\zeta = \xi \wedge \eta$ where $\xi^2 = \eta^2 = 1$ and $\xi \cdot \eta = 0$. If $[b \wedge c, \xi \wedge \eta] = 1$ then $(\xi, \eta)$ supports $2h^{1/2} = 2h_\zeta^{1/2}$ properly at $(\eta(c)b - \eta(b)c, -\xi(c)b + \xi(b)c)$. If $[b \wedge c, \xi \wedge \eta] = -1$ then $\xi^2(b) + \eta^2(b) = \delta^2$ for some $\delta > 0$. If $\eta(b) = 0$ let $b' = b/\xi(b)$ and $c' = -\xi(c)b + \xi(b)c$; if $\eta(b) \neq 0$ let $b' = b/\delta$ and $c' = -[\xi(b) + \delta^2\eta(c)]b/[\delta\eta(b)] + \delta c$. In both cases $h(b', c') = 0$ and $b' \wedge c' = b \wedge c$. If $[b \wedge c, \xi \wedge \eta] = 0$ let $\varepsilon > 0$. If $\eta(b) \neq 0$ let $b' = \varepsilon b$ and $c' = [-\eta(c)b + \eta(b)c]/[\varepsilon\eta(b)]$. Then $h(b', c') = \varepsilon^2\delta^2/4$. If $\eta(b) = 0$ and $\xi(b) = 0$ let $b' = b/\varepsilon$ and $c' = \varepsilon c$; now $h(b', c') = \varepsilon^2[\xi^2(c) + \eta^2(c)]/4$. If $\eta(b) = 0$ and $\xi(b) \neq 0$ then let $b' = \varepsilon b$ and $c' = -[\xi(c)b]/[\varepsilon\xi(b)] + c/\varepsilon$ to obtain $h(b', c') = \varepsilon^2\xi^2(b)/4$. The lemma follows by positive homogeneity.

LEMMA 3. *Let* $\lambda \geqq 1$, $k$ *be continuous on* $E$ *into* $[\lambda^{-1}, \lambda]$, $g \in \mathscr{D}$ *and* $f(a, b, c) = \max\{g(a, b, c), k(a)h_\zeta(b, c)\}$. *Then* $f \in \mathscr{D}$ *and* $Af(a, b \wedge c) = \max\{Ag(a, b \wedge c), k(a)Ah_\zeta(b \wedge c)\}$ *for all* $a, b, c \in E$.

*Proof.* That $f \in \mathscr{D}$ is evident as is the fact that $Af \geqq \max\{Ag, kAh_\zeta\}$. Choose $a, b, c$ with $b \wedge c \neq 0$. Then there exist $d$ and $e$ with $d \wedge e = b \wedge c$ and $Af(a, d \wedge e) = f(a, d, e)$, and there exist $(\rho, \sigma)$ which supports $2f_a^{1/2}$ properly at $(d, e)$, [7]. Assume, at first, that $f(a, d, e) = g(a, d, e) > k(a)h_\zeta(d, e)$. If $(\rho, \sigma)$ did not support $2g_a^{1/2}$ at $(d, e)$, then there would exist $(d_n, e_n) \to (d, e)$ such that $k(a)h_\zeta(d_n, e_n) > g(a, d_n, e_n)$ and this is impossible for large $n$. Hence $(\rho, \sigma)$ supports $2g_a^{1/2}$ properly at $(d, e)$ and $Ag(a, d \wedge e) = g(a, d, e) = f(a, d, e) = Af(a, d \wedge e)$. If $f(a, d, e) = k(a)h_\zeta(d, e) > g(a, d, e)$, a similar argument, together with the fact that $\rho \wedge \sigma = k(a)(\xi \wedge \eta)$, gives $k(a)Ah_\zeta(d \wedge e) = Af(a, d \wedge e)$. If $g(a, d, e) = k(a)h_\zeta(d, e)$, let $\varepsilon > 0$ and $\phi = \max\{(1 + \varepsilon)^2 g, k \cdot h_\zeta\}$. Obviously $((1 + \varepsilon)\rho, (1 + \varepsilon)\sigma)$ supports $2\phi_a^{1/2}$ properly at $(d, e)$ and $(1 + \varepsilon)^2 g(a, d, e) > k(a)h_\zeta(d, e)$. Hence $Af(a, d \wedge e) \leqq A\phi(a, d \wedge e) = (1 + \varepsilon)^2 Ag(a, d \wedge e)$ and the lemma follows.

Let $f \in \mathscr{A}$ and $\lambda = \lambda(f)$. We define $k$ on $E \times [T_1^* - \{0\}]$ by $1/k(a, \zeta) = \sup_{\alpha \neq 0}[a, \zeta]/f(a, \alpha)$. Then $k$ is continuous, range $k \subset [(\lambda \| \zeta \|)^{-1}, \lambda \| \zeta \|^{-1}]$, $k_a^{-1}$ is convex and

$$f(a, \alpha) = \max_{\zeta \in T_1^*} k(a, \zeta)[\alpha, \zeta] .$$

If $f(a, \alpha) = \max_{\zeta \in T^*} k(a, \zeta)[\alpha, \zeta]$ then $f$ is *simple*.

THEOREM. *Let* $k$ *be as above and* $f(a, \alpha) = \max_{\zeta \in T^*} k(a, \zeta)[\alpha, \zeta]$. *Then* $g(a, b, c) = \max_{\zeta \in T^*} k(a, \zeta)h_\zeta(b, c)$ *is in* $\mathscr{D}$ *and* $f = Ag$.

*Proof.* Let $\{\zeta_p\}$ be dense in $T^*$ and $\lambda$ be as above. Let

$$g_1(a, b, c) = \max\{N_2(b, c)/\lambda, k(a, \zeta_1)h_1(b, c)\}$$

and

$$g_{p+1}(a, b, c) = \max \{g_p(a, b, c), k(a, \zeta_{p+1})h_{p+1}(b, c)\}$$

where $h_p = h_{\zeta_p}$.

By the last lemma,

$$Ag_p(a, b \wedge c) = \max \left\{\frac{N_1(b \wedge c)}{\lambda}, \max_{1 \leq m \leq p} k(a, \zeta_m)[b \wedge c, \zeta_m]\right\} \leqq f(a, b \wedge c)$$

for each $p$. Hence $\lim Ag_p \leqq f$. On the other hand, for fixed $a, b, c$ and arbitrary $\varepsilon > 0$ there exists $r$ such that $f(a, b \wedge c) < k(a, \zeta_r)[b \wedge c, \zeta_r] + \varepsilon$ and so $f = \lim Ag_p$.

A little arithmetic shows that

$$| h_p^{1/2}(r, s) - h_p^{1/2}(u, v) | \leqq || (r, s) - (u, v) || .$$

Hence $\{g_p^{1/2}\}$ is equicontinuous and $g_0 = \lim g_p$ is continuous. It is clear that $g_0 = g$ and that $g \in \mathscr{D}$. Furthermore, if $K$ and $L$ are compact subsets of $E^N$ and $T_2$, respectively, then, by a theorem of Dini, $g_p$ converges uniformly to $g$ on $K \times L$.

It remains to show that $Ag = \lim Ag_p$. Choose $a, b, c \in E$ and $\varepsilon > 0$. There exist $(b_p, c_p)$ with $N_2(b_p, c_p) \leqq \lambda Ag(a, b \wedge c)$ such that $Ag_p(a, b_p \wedge c_p) = g_p(a, b_p, c_p)$ and $b_p \wedge c_p = b \wedge c$. By passing to a subsequence, if necessary, we can suppose that there exists $(b_0, c_0)$ such that $(b_p, c_p) \to (b_0, c_0)$. Let $p$ be so large that $g_p(a, r, s) > g(a, r, s) - \varepsilon$ for $N_2(r, s) \leqq \lambda Ag(a, b \wedge c)$ and so large that $|| (b_p, c_p) - (b_0, c_0) || < \varepsilon$. Then $Ag(a, b \wedge c) = Ag(a, b_0 \wedge c_0) \leqq g(a, b_0, c_0) < g_p(a, b_0, c_0) + \varepsilon < [g_p^{1/2}(a, b_p, c_p) + \lambda^{1/2}\varepsilon]^2 + \varepsilon = [Ag_p^{1/2}(a, b_p \wedge c_p) + \lambda^{1/2}\varepsilon]^2 + \varepsilon$. Hence $Ag \leqq \lim Ag_p$, and the opposite inequality is evident.

If $\pi$ is a projection of $E$ onto a plane $P \subset E$, then there exist $\xi$ and $\eta$ in $E^*$ such that $\xi(\pi e) = \xi(e)$, $\eta(\pi e) = \eta(e)$ and $[b \wedge c, \xi \wedge \eta] \neq 0$ whenever $b$ and $c$ are linearly independent points of $P$. A computation gives $[b \wedge c, \xi \wedge \eta](\pi e) = [e \wedge c, \xi \wedge \eta]b + [b \wedge e, \xi \wedge \eta]c$ and we can identify $\pi$ with $\xi \wedge \eta$. Since we can also suppose that $\xi^2 = \eta^2 = 1$, $\xi \cdot \eta = 0$, we can identify the set of projections with the elements of $T^*$.

THEOREM 2. *Let $f \in \mathscr{A}$ and suppose that for each $a \in E$ and each $b \wedge c \neq 0$ there exists a projection $\zeta_0$ (in $T^*$) onto the plane determined by $b$ and $c$ such that $[b \wedge c, \zeta_0] > 0$ and such that $f(a, \zeta_0(d) \wedge \zeta_0(e)) \leqq f(a, d \wedge e)$ whenever $[\zeta_0(d) \wedge \zeta_0(e), \zeta_0] > 0$. Then $f$ is simple and $f(a, b \wedge c) = k(a, \zeta_0)[b \wedge c, \zeta_0]$.*

*Proof.* There exist $d$ and $e$ such that $1/k(a, \zeta_0) = [d \wedge e, \zeta_0]/f(a, d, e)$. Hence

$$\frac{1}{k(a, \zeta_0)} = \frac{[\zeta_0(d) \wedge \zeta_0(e), \zeta_0]}{f(a, d \wedge e)}$$
$$\leqq \frac{[\zeta_0(d) \wedge \zeta_0(e), \zeta_0]}{f(a, \zeta_0(d) \wedge \zeta_0(e))} = \frac{[b \wedge c, \zeta_0]}{f(a, b \wedge c)} \leqq \frac{1}{k(a, \zeta_0)} .$$

It is evident that the converse of this theorem holds.

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