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CLOSED VECTOR FIELDS

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We study closed vector fields on a semi-Riemannian manifold. In particular, we study the differential geometry of the submanifolds determined by a nonvanishing closed field. Expressions are computed for the Weingarten map, the mean curvature, the Riemannian curvature, and the Laplacian of the square of the length of the field. Thus we obtain a necessary and sufficient condition that the constant hypersurface of a nontrivial harmonic function be a minimal surface. We obtain conditions that imply the classical Codazzi-Mainardi equations hold. We obtain conditions that imply the existence of a representation of the manifold as a cross product in which one factor is a real line. Finally, various special cases are examined.

1. **Notation.** Let M be a connected C^∞ semi-Riemannian manifold with metric tensor $\langle\ ,\ \rangle$ and Riemannian connexion D [see Helgason 4 or Hicks 7 for definitions]. We summarize the properties of D and some associated concepts we shall use. The operator D assigns to each pair of C^∞ vector fields X and Y on an open set U of M, a C^∞ vector field $D_X Y$ called the covariant derivative of Y in the direction X. If X, Y, and Z are C^∞ fields on U and f a C^∞ function (real valued) on U then we have the following relations between vector fields on U:

$$\begin{aligned} D_X(Y+Z) &= D_X Y + D_X Z \\ D_{(X+Y)}Z &= D_X Z + D_Y Z \\ D_{fX} Y &= f D_X Y \\ D_X(fY) &= (Xf)Y + fD_X Y \\ \operatorname{Tor}(X, Y) &= D_X Y - D_Y X - [X, Y] \\ R(X, Y)Z &= D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z\ . \end{aligned}$$

We call Tor the torsion on D and R the curvature of D. Since D is Riemannian, Tor $= 0$, and D is compatible with the metric tensor, thus

$$\begin{aligned} D_X Y - D_Y X &= [X, Y] \\ X\langle Y, Z\rangle &= \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle\ . \end{aligned}$$

We extend the operator D_X, as usual, to be a complete derivation on the tensor algebra over M. If $T^{r,s}$ denotes the set of r-contravariant and s-covariant tensors on M, then D_X: $T^{r,s} \to T^{r,s}$. If $f \in T^{0,0}$, then $D_X f = Xf$. If $Y \in T^{1,0}$, then $D_X Y$ is given by the connexion. If

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$w \in T^{0,1}$, then $(D_X w)(Y) = X(w(Y)) - w(D_X Y)$. The last equality contains the seeds of what is meant by a complete derivation which we explain. Having defined D_X on functions, fields, and 1-forms if $\phi \in T^{r,s}$, $w_i \in T^{0,1}$ for $i = 1, \cdots, r$, and $Y_j \in T^{1,0}$ for $j = 1, \cdots, s$, then

$$\begin{aligned} X\phi(w_1, \cdots, w_r, Y_1, \cdots, Y_s) &= (D_X\phi)(w_1, \cdots, w_r, Y_1, \cdots, Y_s) \\ &+ \sum_i \phi(w_1, \cdots, w_{i-1}, D_X w_i, w_{i+1}, \cdots, w_r, Y_1, \cdots, Y_s) \\ &+ \sum_j \phi(w_1, \cdots, w_r, Y_1, \cdots, Y_{j-1}, D_X Y_j, Y_{j+1}, \cdots, Y_s) , \end{aligned}$$

where all terms are well-defined except the first term on the right side of the equation.

The symbol Δ will denote the general covariant differentiation operator Δ: $T^{r,s} \to T^{r,s+1}$ which is induced by D. Using the above notation, $(\Delta\phi)(w_1, \cdots, w_r, Y_1, \cdots, Y_r, X) = (D_X\phi)(w_1, \cdots, w_r, Y_1, \cdots, Y_s)$.

Our study will concern *linear transformation valued tensors* on M (tensor fields of type 1, 1). For completeness, we define a linear transformation valued tensor A on an open set U of M to be a mapping that assigns to each point m in U, a linear transformation A_m: $M_m \to M_m$, where M_m is the tangent space at m. We say A is C^∞ if it maps C^∞ fields on U into C^∞ fields; then if X is a C^∞ field on U then the field $(A(X))_m = A_m(X_m)$ is C^∞ on U. We define the vector valued 2-form Tor_A by

$$\mathrm{Tor}_A (X, Y) = D_X A(Y) - D_Y A(X) - A[X, Y]$$

and let tr A and det A denote the trace and determinant functions on A, respectively.

We will use G to denote the nonsingular linear transformation induced by the metric tensor that maps M_m onto M_m^* for each m. Thus if X is in M_m then $G(X)(Y) = \langle X, Y\rangle$ for Y in M_m; or $G(X) = C_X\langle\ ,\ \rangle = \langle X,\ \rangle$ where C_X is contraction by X in the first covariant slot. We also use the symbol G for the inverse of G. Thus we think of G as a "switch map" and let the argument it is applied to tell us which map is being used. A vecter field X will be called *closed* (or *exact*) if $G(X)$ is closed (or exact), and X is *geodesic* if $D_X X = 0$. A vector X is *nonsingular* (not light-like) if $\langle X, X\rangle \neq 0$. If $\theta \in T^{r,s}$ with $r > 0$, then the *divergence of* θ is the tensor $\operatorname{div}\theta \in T^{r-1,s}$ defined by $\operatorname{div}\theta = \operatorname{tr}\Delta\theta$, where the trace is taken on the last covariant slot and last contravariant slot. If $Z_1, \cdots, Z_n$ is a base field of independent C^∞ vector fields on an open set U in M and $z_1, \cdots, z_n$ is the dual base of 1-forms, then

$$\begin{aligned} &(\operatorname{div}\theta)(w_1, \cdots, w_{r-1}, Y_1, \cdots, Y_s) \\ &\quad = \sum_{j=1}^n (\Delta\theta)(w_1, \cdots, w_{r-1}, z_j, Y_1, \cdots, Y_s, Z_j) . \end{aligned}$$

If $f \in T^{0,0}$, then the *gradient of* f, grad f, is the vector field $G(df)$, so $\langle \operatorname{grad} f, X\rangle = Xf$, and the *Laplacian of* f, $\Delta_2 f$, is the function div (grad f). A function f is *harmonic* if $\Delta_2 f = 0$, and a field T is *conservative* if div $T = 0$.

2. Operators associated with a vector field. Let T be a C^∞ vector field on M. On each tangent space M_m, we define linear maps A_T, B_T, and C_T by

$$A_T(X) = D_X T,\ B_T(X) = D_X(D_T T)\ , \quad \text{and} \quad C_T(X) = R(X, T)T\ .$$

These maps are C^∞ since D and T are C^∞. Let U be the open set of points in M where $\langle T, T\rangle$ does not vanish. On U, we define the $C^\infty(n-1)$ dimensional distribution $T^\perp$ by

$$(T^\perp)_p = [X \in M_p : \langle X, T\rangle = 0]\ .$$

From the definition of the curvature R we have

$$C_T = B_T - A_T^2 + [A_T, D_T]$$

where

$$[A_T, D_T](X) = A_T(D_T X) - D_T(A_T X)$$

and thus $[A_T, D_T]$ is a linear transformation valued tensor. By the standard symmetry properties of the four covariant Riemann Christoffel tensor, the map C_T is symmetric (self-adjoint), and we call it the *Ricci map associated with* T. The trace of C_T is the *Ricci curvature of* T, which we denote by Ric (T, T).

Following Bochner [1], we say a field T is *restrained* if $\Delta_2\langle T, T\rangle < 0$ at some point or T has constant length. Bochner has shown that every field on a compact manifold is restrained, and in the noncompact case, a field is restrained if its length attains a relative maximum at some point.

Our main interests in this study are the cases when A_T is symmetric, or equivalently, T is closed. Since the gradient of any C^∞ function is a closed field, many closed fields exist.

PROPOSITION 1. For any field T, tr $A_T =$ div T and tr $[A_T, D_T] = -T(\operatorname{div} T)$. If $T = \operatorname{grad} f$, then the Laplacian of f is the trace of A_T.

Proof. Let $Z_1, \cdots, Z_n$ be a set of nonsingular orthonormal vector fields belonging to a Riemannian normal coordinate system at a point m in M and let $w_1, \cdots, w_n$ be the dual 1-forms of this base. Thus if $e_i = \langle Z_i, Z_i\rangle$, then

$$\operatorname{tr} A_T = \Sigma e_i \langle D_{Z_i} T, Z_i\rangle = \Sigma w_i(D_{Z_i} T) = \operatorname{tr} \Delta(T)\ ,$$

and using the fact that $D_T Z_i = 0$ at m for any T,

$$\begin{aligned}\operatorname{tr}[A_T, D_T] &= \Sigma\langle A_T D_T Z_i - D_T A_T Z_i, Z_i\rangle e_i \\ &= \Sigma\langle D_{D_T Z_i} T - D_T D_{Z_i} T, Z_i\rangle e_i \\ &= -\Sigma T\langle D_{Z_i} T, Z_i\rangle e_i + \Sigma\langle D_{Z_i} T, D_T Z_i\rangle e_i \\ &= -T(\operatorname{tr} A_T) .\end{aligned}$$

PROPOSITION 2. For any field T,

$$\operatorname{Ric}(T, T) = \operatorname{tr} C_T = \operatorname{tr} B_T - \operatorname{tr} A_T^2 - T(\operatorname{div} T) .$$

Proof. Using the fields Z_i in the above proof,

$$\operatorname{tr} C_T = \Sigma\langle R(Z_i, T)T, Z_i\rangle e_i = \operatorname{Ric}(T, T) ,$$

and the rest of the proposition follows from the linearity of the trace.

PROPOSITION 3. For any field T, T has constant length if and only if (Image $A_T) \subset T^{\perp}$.

Proof. For any vector X,

$$X\langle T, T\rangle = 2\langle D_X T, T\rangle = 2\langle A_T(X), T\rangle .$$

3. The symmetric case. Throughout this section we assume T is a closed field, or equivalently, A_T is symmetric (by the following proposition).

THEOREM 1. *A field T is closed if and only if A_T is symmetric. If T is closed, then $T^{\perp}$ is integrable on U.*

Proof. If X and Y are fields, then

$$\begin{aligned}(dG(T))(X, Y) &= X\langle T, Y\rangle - Y\langle T, X\rangle - \langle T, [X, Y]\rangle \\ &= \langle D_X T, Y\rangle - \langle D_Y T, X\rangle + \langle T, D_X Y - D_Y X - [X, Y]\rangle \\ &= \langle A_T X, Y\rangle - \langle A_T Y, X\rangle ,\end{aligned}$$

since the torsion of D is zero.

If X and Y belong to $T^{\perp}$, then

$$\begin{aligned}\langle [X, Y], T\rangle &= \langle D_X Y - D_Y X, T\rangle \\ &= X\langle Y, T\rangle - \langle Y, D_X T\rangle - Y\langle X, T\rangle + \langle X, D_Y T\rangle \\ &= \langle X, A_T Y\rangle - \langle Y, A_T X\rangle = 0\end{aligned}$$

since $\langle Y, T\rangle \equiv \langle X, T\rangle \equiv 0$. Thus $T^{\perp}$ is involutive or integrable (see Chevalley [2]).

In the special case $T = \operatorname{grad} f$, then the integral manifolds of $T^{\perp}$ on U are precisely the hypersurfaces on which f is constant. We next investigate the geometry of an integral manifold M' of $T^{\perp}$ through a point m in U. Since T is normal to M', we use T to frame M' locally (see Hicks [6]). Let e be the function on U which is plus or minus one according as $\langle T, T\rangle$ is positive or negative, respectively.

THEOREM 2. *Let L be the Weingarten map on M' and take X in $(M')_m$.*

$$L(X) = [e\langle T, T\rangle]^{-3/2}[e\langle T, T\rangle A_T(X) - e\langle T, A_T(X)\rangle T]$$

and the mean curvature H of M' is given by

$$H = \operatorname{tr} L = |T|^{-1}[\operatorname{div} T - T \log |T|]$$

where $|T| = [e\langle T, T\rangle]^{1/2}$ is the length of T. Thus M' is minimal if and only if $\operatorname{div} T = T \log |T|$.

Proof. Let $N = [e\langle T, T\rangle]^{-1/2}\, T$ be the unit normal so

$$L(X) = D_X N = -[e\langle T, T\rangle]^{-3/2} e\langle A_T X, T\rangle T + [e\langle T, T\rangle]^{-1/2} A_T X\,.$$

To compute $\operatorname{tr} L$, let $Z_1, \cdots, Z_{n-1}$ be a nonsingular orthonormal base of $(M')_m$ and let $Z_n = N$. Letting $e_i = \langle Z_i, Z_i\rangle$, then

$$\begin{aligned} H = \operatorname{tr} L &= \sum_{j=1}^{n-1} \langle LZ_j, Z_j\rangle e_j \\ &= [e\langle T, T\rangle]^{-1/2} \sum_{1}^{n-1} \langle A_T Z_j, Z_j\rangle e_j\,. \end{aligned}$$

But

$$\begin{aligned} \langle A_T Z_n, Z_n\rangle e_n &= \langle D_N T, N\rangle e = \langle D_T T, T\rangle / \langle T, T\rangle \\ &= (1/2)(T\langle T, T\rangle)/\langle T, T\rangle = (1/2) T \log e\langle T, T\rangle\,. \end{aligned}$$

Hence, $H = (e\langle T, T\rangle)^{-1/2}\, [\operatorname{tr} A_T - T \log |T|]$.

COROLLARY 1. *The constant hypersurfaces of a nonconstant harmonic function are minimal surfaces if and only if the gradient of the function has constant length along its integral curves.*

Proof. Let f be harmonic and $T = \operatorname{grad} f$. Then T is closed and $\operatorname{tr} A_T = \operatorname{div} T = 0$. Hence $H = 0$ if and only if $\langle D_T T, T\rangle = 0$ or $T\langle T, T\rangle = 0$.

COROLLARY 2. *Let T be a unit field on M which is closed. Then* *the total curvature and mean curvature of the integral manifolds of* $T^{\perp}$ *are given by* $K = \det A_T$ *and* $H = \operatorname{div} T$. *Indeed,* $S = A_T$ *if and only if* T *is a unit field.*

The first corollary above suggests the definition of a *minimal harmonic function* as a harmonic function whose constant hypersurfaces are minimal surfaces. This class of harmonic functions has not been examined as yet, as far as we know, nor has the above result (Corollary 1) been proven before.

PROPOSITION 4. Let $\phi = \langle T, T\rangle$. Then $\operatorname{grad} \phi = 2D_T T$, which implies B_T is symmetric, and

$$\Delta_2\phi = 2 \operatorname{tr} B_T = 2[\operatorname{Ric}(T, T) + \operatorname{tr} A_T^2 + T(\operatorname{div} T)]$$

while

$$(\Delta^2\phi)(Z, Y) = 2\langle B_T Z, Y\rangle .$$

Proof. Consider

$$(\Delta\phi)X = X\langle T, T\rangle = 2\langle D_X T, T\rangle = 2\langle X, D_T T\rangle .$$

Hence $\operatorname{grad} \phi = 2D_T T$, and $\Delta_2\phi = \operatorname{div} \operatorname{grad} \phi = 2 \operatorname{tr} B_T$. The last expression for the Laplacian of ϕ follows from Proposition 2.

Finally,

$$(\Delta^2\phi)(Z, Y) = [D_Y(\Delta\phi)]Z = 2Y\langle Z, D_T T\rangle - 2\langle D_Y Z, D_T T\rangle = 2\langle Z, B_T Y\rangle .$$

We have immediately a slight generalization of a result of Bochner [1].

COROLLARY 1. *Let* T *be a closed field such that* $\operatorname{div} T$ *is constant along the integral curves of* T. *If* T *is restrained, then* $\operatorname{Ric}(T, T) < 0$ *at some point of* M *or* $\operatorname{Ric}(T, T) \leqq 0$ *on all of* M. *On a compact manifold whose Ricci curvature is always positive there can be no nontrivial closed field* T *with* $T(\operatorname{div} T) = 0$. *On a compact manifold whose Ricci curvature is nonnegative any nontrivial closed field* T *with* $T(\operatorname{div} T) = 0$ *must be a global parallel field with constant length, zero Ricci curvature, and* $A_T = 0$ (see Proposition 6).

Proof. In these cases,

$$\operatorname{Ric}(T, T) = (1/2)\Delta_2\phi - \operatorname{tr} A_T^2$$

which proves the first two statements immediately. If T is restrained, as in the last statement, then we force $\operatorname{Ric}(T, T) \equiv 0$ and T to have constant length since $R(T, T) < 0$ at any point is impossible. Thus ϕ

is constant, $\Delta_2\phi = 0$, and $\operatorname{tr} A_T^2 = 0$ which implies all the eigenvalues of A_T are zero, so $A_T = 0$.

COROLLARY 2. *A nontrivial closed field has constant length on a semi-Riemannian manifold if and only if its integral curves are geodesics.*

Proof. This is trivial since $\operatorname{grad}\phi = 2D_T T$.

The following result applies to any vector field.

PROPOSITION 5. The integral curves of a field T are reparameterizations of geodesics if and only if $D_T T = gT$ for some real valued C^∞ function g.

Proof. If the field fT is geodesic (f never vanishes), $0 = D_{fT} fT = f[(Tf)T + fD_T T]$ and $g = -T(\log f)$. Conversely, if $D_T T = gT$ then along each integral curve of T we need only solve the linear equation $(Tf) + fg = 0$ to obtain f for which fT is geodesic.

COROLLARY. *If T is closed, nonvanishing, and $D_T T = gT$ then* $\operatorname{Ric}(T) = g \operatorname{div} T - \operatorname{tr} A_T^2 + T(g - \operatorname{div} T)$.

We now study the case when T has constant length on the hypersurfaces M'.

THEOREM 3. *The following four statements are equivalent on the set U:*

(a) *A_T is invariant on $T^\perp$.*

(b) *T has constant length on any M'.*

(c) *$D_T T$ is orthogonal to $T^\perp$.*

(d) *$[T, X]$ is in $T^\perp$ if X in $T^\perp$.*

Proof. If X is in $T^\perp$ then $X\langle T, T\rangle = 2\langle A_T X, T\rangle = 2\langle X, D_T T\rangle$ which shows (a), (b), and (c) are equivalent. Also

$$\begin{aligned}\langle A_T X, T\rangle = \langle X, A_T T\rangle &= T\langle X, T\rangle - \langle D_T X, T\rangle \\ &= -\langle D_X T + [T, X], T\rangle ,\end{aligned}$$

where we extend X to be a C^∞ field in $T^\perp$. Hence $2\langle A_T X, T\rangle = \langle [X, T], T\rangle$ which shows (a) is equivalent to (d).

THEOREM 4. *If one of the statements in Theorem 3 holds and T does not vanish, then the integral curves of T are reparameterizations of geodesics,* $\operatorname{grad}\phi = 2D_T T = (T \log e\phi)T$, *and the vector* $\operatorname{grad}\phi$ *has* *constant length on M', i.e.* $T \log \phi$ *is constant on M'. Moreover, the mean curvature of M' is constant if and only if* $\operatorname{div} T$ *is constant on M'.*

Proof. Letting $\operatorname{grad} \phi = fT$ then $T\phi = 2\langle D_T T, T\rangle = \langle fT, T\rangle = f\phi$. If $T \neq 0$, then $\phi \neq 0$, so $f = (T\phi)/\phi = T \log e\phi$. The integral curves of T are reparameterizations of geodesic by Proposition 5.

Letting X be a C^∞ field in $T^\perp$, then

$$Xf = XT(\log \phi) = [X, T] \log \phi + T(X \log \phi) = 0$$

since $[X, T]$ is in $T^\perp$ and ϕ is constant on M'.

The last statement of the conclusion follows from Theorem 2.

COROLLARY. *If* $\operatorname{grad} \phi$ *does not vanish on M, then the hypersurfaces M' are precisely the constant hypersurfaces of ϕ if and only if one of the statements in Theorem* 3 *is true.*

We return to the study of the geometry of the hypersurface M'. Recall the fact that if L is the Weingarten map of an oriented nonsingular hypersurface in a semi-Riemannian manifold, then the Codazzi-Mainardi equations hold on the hypersurface if and only if $\mathrm{Tor}_L = 0$. In the following theorem, we write $A_T = \Delta T$ which is admissable by the identification of linear transformations with tensors of type 1,1.

THEOREM 5. *On the set U, the following three statements are equivalent:*

(a) *The Codazzi-Mainardi equations hold on M'.*

(b) $\mathrm{Tor}_{\Delta T} = 0$ *on vectors in $T^\perp$.*

(c) $R(X, Y)T = 0$ *for all X, Y in $T^\perp$.*

Proof. Let D' be the induced Riemannian covariant differentiation on M', thus for fields X and Y in $T^\perp$,

$$D_X Y = D'_X Y - \langle LX, Y\rangle rN$$

by the Gauss equation (see Hicks [7]), where $r = \langle N, N\rangle = e$.

Using the Gauss equation and Theorem 2, a straightforward computation yields,

$$\begin{aligned}\mathrm{Tor}_L(X, Y) &= D'_X(LY) - D'_Y(LX) - L([X, Y]) \\ &= [e\langle T, T\rangle]^{-1/2}\,\mathrm{Tor}_{\Delta T}(X, Y) \\ &\quad - [e\langle T, T\rangle]^{-3/2} e\langle \mathrm{Tor}_{\Delta T}(X, Y), T\rangle T \\ &= |T|^{-1}\,\mathrm{Tor}_{\Delta T}(X, Y)\,,\end{aligned}$$

since $\mathrm{Tor}_{\Delta T}(X, Y) = D_X A_T Y - D_Y A_T X - A_T[X, Y] = R(X, Y)T$ and $\langle R(X, Y)T, T\rangle = 0$ by the skew-symmetry of the covariant Riemann-Christoffel curvature tensor. Thus $\mathrm{Tor}_{\Delta T}(X, Y)$ has no component orthogonal to M' and the conclusion now follows.

THEOREM 6. *On the set U, let P be a two dimensional subspace of M' with nonsingular orthonormal base X, Y Then*

$$K(P) = K'(P) \\ - [e\langle T, T\rangle\langle X, X\rangle\langle Y, Y\rangle]^{-1}[\langle A_T X, X\rangle\langle A_T Y, Y\rangle - \langle A_T X, Y\rangle^2]$$

relates the Riemannian curvature of P with respect to M and M'.

Proof. The general Gauss curvature equation (see Hicks [6]) states that

$$\tan R(X, Y)Z = R'(X, Y)Z - r(\langle LY, Z\rangle LX - \langle LX, Z\rangle LY) .$$

Using Theorem 2, a straightforward computation yields the result.

COROLLARY. *If M is Riemannian and $T = \operatorname{grad} f$, m in U, and $x, y, \cdots$ are a set of Riemann normal coordinates at m such that $\partial/\partial x$ and $\partial/\partial y$ span the subspace P in M'_m, then*

$$K(P) = K'(P) - \left[\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right]\Big/ |\operatorname{grad} f|^2$$

at m.

Proof. Let $X = \partial/\partial x$ and $Y = \partial/\partial y$. Then $\langle A_T X, Y\rangle_m = \langle D_X T, Y\rangle = X\langle T, Y\rangle = X_m(Yf)$ since $(D_X Y)_m = 0$.

We now show the tensor $\mathrm{Tor}_{\Delta T}$ represents a condition on the holonomy of the distribution $T^\perp$.

THEOREM 7. *Let M be Riemannian, complete, connected, and simply connected. Let T be a nonvanishing closed field such that ΔT has no torsion. Then M is diffeomorphic to a product $M' \times R$, where M' is the $(n-1)$ dimensional integral submanifold of $T^\perp$ through a point m in M and R is the real line. Hence the orbit space M/T is diffeomorphic to M'.*

Proof. Since M is simply connected its restricted homogeneous holonomy group is equal to its homogeneous holonomy group H. The Lie algebra of H is generated by the linear transformations $R(X, Y)$ on M_m for all vectors X and Y in M_m (see Nomizu [8]). Since $\mathrm{Tor}_{\Delta T} = 0$, $R(X, Y)T = 0$ for all X and Y hence $R(X, Y)$ is invariant on $T^\perp$.

Since H is contained in the special orthogonal group $\mathrm{SO}(n, R)$, which is compact, the exponential map is onto. If h is in H, then $h = \exp R(X, Y)$ for some X and Y in M_m, and thus $h(T^\perp)$ is contained in $T^\perp$. We now apply the result of DeRham [3] to get $M = M' \times N$. Since M is Riemannian and complete, N is diffeomorphic to the real line or the one dimensional torus. Since M is simply connected, N is diffeomorphic to R.

4. **Special cases.** We conclude with some special cases that follow from the above results. We will always assume the field T is nontrivial, nonsingular, and closed.

PROPOSITION 6. *If $A_T \equiv 0$, then T is a geodesic field with constant length, zero divergence, and zero Ricci curvature. If T lies in the plane section P then $K(P) = 0$. Thus there is no pair of conjugate points along the geodesics determined by T. The distribution $T^\perp$ is integrable and its integral manifolds M' are flatly imbedded in M (i.e. $L \equiv 0$ on M'). Hence M' is a geodesic submanifold of M. If M is Riemannian, complete, and simply connected, then M is isometric to the product $M' \times R$.*

PROPOSITION 7. *If $B_T \equiv 0$ and T is geodesic then T has constant length c and $\mathrm{Ric}(T) = -\mathrm{tr}\, A_T^2 - T(\mathrm{div}\, T)$. When M' is defined it has total curvature zero and mean curvature $(1/c)\, \mathrm{div}\, T$. If M' is defined and flat everywhere, then $A_T \equiv 0$ and Proposition 6 is applicable.*

PROPOSITION 8. *If $B_T \equiv 0$ and the integral curves of T are reparameterizations of geodesics with $D_T T = gT$, then at points where g and T do not vanish, M' is flat and the Ricci curvature of T is zero.*

In proving Proposition 8 one shows at points in U where g does not vanish then $A_T T = D_T T = (\mathrm{div}\, T)T$ by applying Proposition 6 to $D_T T$. Furthermore, at such points $0 = B_T T = [T(\mathrm{div}\, T) + (\mathrm{div}\, T)^2]T$ so $\mathrm{tr}\, A_T^2 = (\mathrm{div}\, T)^2 = -T(\mathrm{div}\, T)$ and $\mathrm{Ric}\,(T) = 0$.

PROPOSITION 9. *If $B_T \equiv 0$ and the integral curves of T are not reparameterizations of geodesics, then Proposition 6 may be applied to $D_T T$. Moreover $T^2\langle T, T\rangle$ is constant, hence there can be at most one point on each integral curve of T where the length of T has a critical point. If the integral curves of T are parametrically complete (defined for all parameter values), then M cannot be compact.*

Notice in Proposition 9 the length of T is not constant along any of its integral curves, for $0 = T\langle T, T\rangle = 2\langle D_T T, T\rangle$ implies $D_T T = gT$ by Theorem 4, which implies the integral curves of T are geodesics by Proposition 5.

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