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ISOMORPHIC CONE-COMPLEXES

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In this paper we show that the 1-section of a finite simplicial complex M is characterized by the topological type of the 1-section of the cone over M. This enables us to prove that a finite simplicial complex is characterized by the topological type of the 1-section of the first derived complex of its cone.

R. L. Finney [1] proved that two locally-finite simplicial complexes are isomorphic if their first derived complexes are isomorphic. J. Segal [3] making use of this showed that two locally-finite simplicial complexes are isomorphic if the 1-sections of their first derived complexes are isomorphic. He then showed in [4] that a restricted class of finite complexes are characterized by the topological type of the 1-section of their first derived complexes. In contrast to [4] the results of this paper apply without restricting the class of finite complexes.

Throughout, s_p will denote a (rectilinear) p-simplex; M will denote a finite geometric simplicial complex with r-section M^r and first derived complex M'. The cone at m over M, $m \notin M$, is denoted by mM. For more details see [2, 1.2]. We only consider complexes with at least two vertices.

LEMMA. (a) *If mM and nN are isomorphic then so are M and N.* (b) *If $(mM)^1$ and $(nN)^1$ are isomorphic then so are M^1 and N^1.*

Proof. (a) Let φ be an isomorphism of mM onto nN. If $\varphi(m) = n$ we are done so we assume $\varphi(m) \neq n$, hence we also have $\varphi^{-1}(n) \neq m$.

Given a complex K, with vertex v, the subcomplex consisting of those simplexes not having v as a vertex is denoted $K\langle v\rangle$.

We now define subcomplexes of M and N as follows:

$$M_1 = (mM)\langle\varphi^{-1}(n)\rangle \quad , \; M_2 = M\langle\varphi^{-1}(n)\rangle$$
$$N_1 = (nN)\langle\varphi(m)\rangle \quad , \; N_2 = N\langle\varphi(m)\rangle \, .$$

The following relationships are easily verified:

(1) $M_1 = mM_2, N_1 = nN_2$;

(2) $\varphi | M_2$ is an isomorphism of M_2 onto N_2 ;

(3) $\varphi | M$ is an isomorphism of M onto N_1, and $\varphi^{-1\prime} N$ is an isomorphism of N onto M_1.

Using $\approx$ to denote isomorphism we then have:

$$M \approx N_1 = nN_2 \approx mM_2 = M_1 \approx N\,.$$

Here the first and last isomorphisms follow from (3), the equalities from (1) and the middle isomorphism follows from (2) and the fact that taking cones preserves isomorphism.

A proof of part (b) is obtained by taking 1-sections at appropriate places in the above argument.

THEOREM 1. *If* $|(mM)^1|$ *and* $|(nN)^1|$ *are homeomorphic then* M^1 *and* N^1 *are isomorphic.*

Proof. For nontriviality we assume each of M and N has at least 3 vertices. Let T_M denote the set of vertices of mM whose order in $|(mM)^1|$ is not 2 and let T_N be the corresponding set in nN; then m is in T_M and n is in T_N.

Now let h be a homeomorphism of $|(mM)^1|$ onto $|(nN)^1|$. We shall modify h where necessary to get a new homeomorphism $\widetilde{h}$ which maps the vertices of $(mM)^1$ onto those of $(nN)^1$.

Clearly any homeomorphism of $|(mM)^1|$ onto $|(nN)^1|$ takes T_M onto T_N. Let $v_1, \cdots, v_r$ be the vertices of $(mM)^1$ having order 2; we show how to construct homeomorphisms $h_1, \cdots, h_r$ such that

$$h_i\colon |(mM)^1| \to |(nN)^1|$$

and for $i \leqq j$, $h_j(v_i)$ is a vertex of $(nN)^1$. Starting with h we shall construct h_1; the construction of h_i from h_{i-1} follows the same pattern and will be omitted.

For simplicity we write v rather than v_1. If $h(v)$ is a vertex of $(nN)^1$ we let $h_1 = h$. Suppose then that $h(v)$ is not a vertex. Let C be the closure in $|(mM)^1|$ of the component Q of $|(mM)^1| - T_M$ containing v; then C is either an arc with endpoints in T_M or a simple closed curve which one easily shows must be of the form $Q \cup \{m\}$.

Suppose first it is an arc with endpoints x and y. Using the fact that v has order 2 in $|(mM)^1|$ we conclude that one of x, y, say x, is m and that Q contains no vertex of mM other than v.

Let σ be the 1-simplex of mM spanned by m and y. Applying h, we get a pair of arcs $h(|\sigma|)$ and $h(C)$ in $|(nN)^1|$ whose union is a simple closed curve containing exactly two points of T_N-namely $h(m)$ and $h(y)$. It follows that there is a vertex w of nN which lies either on $h(|\sigma|$-$\{m, y\})$ or $h(C - \{m, y\})$. In the first case we choose a self-homeomorphism k of $|(nN)^1|$ which is the identity off $h(|\sigma| \cup C)$, interchanges $h(|\sigma|)$ and $h(C)$ leaving $h(m)$ and $h(y)$ fixed, and takes $h(v)$ onto w; we define $h_1 = k \circ h$. The second case is similar-except that k is taken as the identity off $h(Q)$.

If C is a simple closed curve, $C = Q \cup \{m\}$, then Q must contain exactly two vertices of order 2, say v and w. Since $h(C) \cap T_N = \{h(m)\}$ it follows that $h(m) = n$ and $h(Q)$ contains exactly two vertices of order 2, say v' and w'. In this case we choose a self-homeomorphism k of $|(nN)^1|$ which is the identity off $h(Q)$ and takes $h(v)$ to v' and $h(w)$ to w'. The composition $h_1 = k \circ h$ has the desired properties. This completes the construction of h_1.

We let $\tilde{h} = h_r$; then $\tilde{h}$ takes each vertex of $(mM)^1$ to a vertex of $(nN)^1$. In particular $(mM)^1$ has at least as many vertices as $(nN)^1$. Since a similar construction can be made starting with h^{-1}, the number of vertices in each complex is the same. Hence the homeomorphism $\tilde{h}$ takes the vertices of $(mM)^1$ onto those of $(nN)^1$. It follows (see, for example, the argument of Theorem 3 of [4]) that $\tilde{h}$ induces an isomorphism of $(mM)^1$ onto $(nN)^1$.

Applying part (b) of the lemma, we have that M^1 and N^1 are isomorphic.

DEFINITION. An n-complex M is *full* provided, for any subcomplex K of M which is isomorphic to s_p^1, $2 \leqq p \leqq n$, K^0 spans a p-simplex of M.

THEOREM 2. *If M and N are full complexes, then they are isomorphic if $|(mM)^1|$ and $|(nN)^1|$ are homeomorphic.*

This follows from Theorem 1 and Theorem 1 of [3] which says that if M and N are full and M^1 and N^1 are isomorphic, then M and N are isomorphic.

DEFINITION. Given the cone at m over M and a subcomplex K of M we define the *tower-complex* over K (relative to mM) to be $((mK)')^1$ and we denote it by $t_m(K)$. Furthermore, we call the underlying polyhedron of $t_m(K)$ the *tower* over K (relative to mM) and denote it by $t(K)$, i.e.., $t(K) = |t_m(K)|$.

THEOREM 3. *If M and N are complexes, then M and N are isomorphic if and only if $t(M)$ and $t(N)$ are homeomorphic.*

Proof. Suppose $t(M)$ and $t(N)$ are homeomorphic. We first assume that M and N have no vertices of order 0. Then the order of each vertex of $(mM)'$ in $t_m(M)$ and of $(nN)'$ in $t_m(N)$ is at least three. So we may apply Theorem 5 of [4] to obtain an isomorphism between mM and nN. This by part (a) of the Lemma yields the desired isomorphism between M and N.

Now consider the case in which M or N has vertices of order 0. Let K denote the set of vertices of M which are of order 0 and let

L be the corresponding set for N. Let $\tilde{M} = M - K$ and $\tilde{N} = N - L$. Then

$$t(M) = t(\tilde{M}) \cup t(K)$$

and

$$t(N) = t(\tilde{N}) \cup t(L) .$$

Let h be a homeomorphism of $t(M)$ onto $t(N)$. Since $t(K)$ is the smallest connected subset of $t(M)$ that contains K, the set $h(t(K))$ is the smallest connected subset of $t(N)$ that contains $h(K)$. But $h(K) = L$, because the points of K and L are the only ones with order 1 in $t(M)$ and $t(N)$. Therefore, $h(t(K)) = t(L)$, and by taking complements we see that $h(t(\tilde{M})) = t(\tilde{N})$. Therefore, by the preceding argument, there exists an isomorphism f of $\tilde{M}$ onto $\tilde{N}$. Since h yields an isomorphism of K and L, f can be extended to an isomorphism of M and N.

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