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F′-SPACES AND z-EMBEDDED SUBSPACES

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A completely regular Hausdorff space is an F'-space if disjoint cozero-sets have disjoint closures. Here the theory of prime z-filters is applied to the study of F'-spaces. A z-embedded subspace is one in which the zero-sets are all intersections of the subspace with zero-sets in the larger space. It is shown that every z-embedded subspace of an F'-space is also an F'-space. Also, a new characterization of F'-spaces is obtained: Every z-embedded subspace is C^*-embedded in its closure.

F- and F'-spaces were introduced in [4] in connection with the study of finitely generated ideals in rings of continuous functions; further results on F'-spaces are found in [1] and [2].

Throughout this paper we shall use the terminology and notation of the Gillman-Jerison treatise [5]. Only completely regular Hausdorff spaces will be considered.

As noted above, a subspace Y of a space X is z-embedded in X if for every zero-set Z in Y there is a zero-set W in X such that $Z = W \cap Y$. For example, a C^*-embedded subspace is clearly z-embedded; also, a Lindelöf subspace is always z-embedded (Jerison, [9, 5.3]). Relations between z-, C^*-, and C-embedding have been given by Hager [7]. We shall find that z-embedded subspaces are of interest in problems concerning z-filters, and thus in problems concerning F'-spaces.

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1. Traces and induced z-filters.

If $Y \subseteq X$, we define the *trace* $\mathscr{F} \mid Y = \{Z \cap Y : Z \in \mathscr{F}\}$ of any z-filter $\mathscr{F}$ on X, and the *induced z-filter* $\mathscr{F}^{\#} = \{Z \in \boldsymbol{Z}(X) : Z \cap Y \in \mathscr{F}\}$ for any z-filter $\mathscr{F}$ on Y.

We now consider six basic lemmas in the calculus of traces and induced z-filters; the first two are easy to verify and the third is proved in [10].

LEMMA 1. *If $\mathscr{P}$ is a prime z-filter on Y, then $\mathscr{P}^{\#}$ is a prime z-filter on X.* [5, 4.12].

LEMMA 2. *If Y is z-embedded in X and $\mathscr{F}$ is a z-filter on Y, then $\mathscr{F} = \mathscr{F}^{\#} \mid Y$.*

LEMMA 3. *Let Y be a z-embedded subspace of X. If $\mathscr{F}$ is a*

z-filter on X every member of which meets Y, then $\mathscr{F} \mid Y$ is a z-filter on Y. If $\mathscr{F}$ is prime, then $\mathscr{F} \mid Y$ is also prime. [10, Th. 5.2].

We shall use $\mathscr{M}^p$ and $\mathscr{O}^p$ to denote the z-filters $\boldsymbol{Z}[\boldsymbol{M}^p]$ and $\boldsymbol{Z}[\boldsymbol{O}^p]$, respectively. For example, if $p \in X$, then $\mathscr{O}^p_X$ is the z-filter of all zero-set-neighborhoods of p in X. In the next two lemmas we use induced z-filters and traces to relate $\mathscr{O}^p_X$ with the corresponding z-filter on a subspace of X that contains p. The first lemma is immediate.

LEMMA 4. *If V is a neighborhood of p in X, then $\mathscr{O}^p_X = (\mathscr{O}^p_V)^\sharp$.*

LEMMA 5. *If Y is z-embedded in X, and $p \in Y$, then $\mathscr{O}^p_Y = \mathscr{O}^p_X \mid Y$.*

Proof. Clearly $\mathscr{O}^p_X \mid Y \subseteqq \mathscr{O}^p_Y$. On the other hand, if $Z \in \mathscr{O}^p_Y$, there is $W \in \mathscr{O}^p_X$ such that $W \cap Y \subseteqq Z$. Since Y is z-embedded, by Lemma 3 $\mathscr{O}^p_X \mid Y$ is a z-filter on Y, and since $W \cap Y$ is in $\mathscr{O}^p_X \mid Y$, so is Z.

LEMMA 6. *For any X, and any $Y \subseteqq X$, if $\mathscr{P}$ and $\mathscr{Q}$ are prime z-filters on Y contained in the same z-ultrafilter on Y, then $\mathscr{P}^\sharp$ and $\mathscr{Q}^\sharp$ are contained in the same z-ultrafilter on X.*

Proof. If not, then $\mathscr{P}^\sharp$ and $\mathscr{Q}^\sharp$ contain distinct z-filters $\mathscr{O}^p$; hence they, and thus also $\mathscr{P}$ and $\mathscr{Q}$, have disjoint members, so that $\mathscr{P}$ and $\mathscr{Q}$ could not be contained in the same z-ultrafilter.

2. Subspaces of F'-spaces. We are now ready to use traces of z-filters to obtain our first result.

THEOREM 1. *Every z-embedded subspace of an F'-space is also an F'-space.*

Proof. According to [4, 8.13] (see also Theorem 3 below), a space T is an F'-space if and only if $\mathscr{O}^p_T$ is prime for every $p \in T$.

Let Y be z-embedded in an F'-space X. For any $p \in Y$, we have $\mathscr{O}^p_Y = \mathscr{O}^p_X \mid Y$, by Lemma 5. Since X is an F'-space, $\mathscr{O}^p_X$ is prime, and hence by Lemma 3, $\mathscr{O}^p_Y$ is also prime. Thus Y is an F'-space.

This result generalizes Corollary 1.6 and Theorem 1.11 of [1] which give the result in the case of a Lindelöf subspace or a C^*-embedded subspace. An example of a z-embedded subspace of an F'-space that is neither Lindelöf nor C^*-embedded is the subspace $X - Y$ of the space X constructed in [4, 8.14].

It is easily verified (see for example [6, 3.1]) that every cozero-set is z-embedded. Hence as an application of Theorem 1 we find that *every cozero-set in an F'-space is also an F'-space.* Thus we also obtain an immediate proof of a result in [1, § 4]: in any space, a point with an F'-neighborhood admits a fundamental system of F'-neighborhoods.

A zero-set in X need not be z-embedded in X; for example, it is easily seen that the zero-set D of the space Γ of [5, 3K] is not z-embedded.

The z-filters $\mathcal{O}^p$ may also be used to obtain other properties of F'-spaces. For example, by Lemmas 1 and 4 we see that, as noted in [1, § 4], *F' is a local property*, i.e., if every point of X has an F'-neighborhood, then X is an F'-space. Since it is clear that *any local property that is inherited by cozero-sets is also inherited by all open subspaces*, Theorem 1 also yields the following result of [1].

COROLLARY 1. [1, § 4]. *Every open subspace of an F'-space is also an F'-space.*

A space is an *F-space* if any two disjoint cozero-sets are completely separated [5, 14N.4]. Since cozero-sets are z-embedded, it is easily seen that *"cozero-set" is transitive*, i.e., is S is a cozero-set in X and T is a cozero-set in S, then T is also a cozero-set in X. Thus it is clear that a cozero-set in an F-space is also an F-space, as noted in [5, 14.26]. Hence the analog for F-spaces of the statement above on fundamental systems is also true, as noted in [1, § 4]. We note that "zero-set" is *not* transitive; for example the zero-set D above has many zero-sets that are not zero-sets of Γ. But in a *normal* space, "zero-set" *is* transitive.

It is well-known that if X is any locally-compact, σ-compact space, then $\beta X - X$ is an F-space ([5, 14.27]; see also [12, 3.3] or [11, Corollary 1]), and thus for any X, any zero-set (i.e., compact G_δ) in βX that does not meet X is an F-space [5, 14O.1]. Here is an analog for F'-spaces. *For any X, any locally compact G_δ in βX that does not meet X is an F'-space.* To see this, let Y be such a set and let $p \in Y$. Then p has a compact zero-set neighborhood Z in Y. Since Z is a G_δ in Y, it is a compact G_δ in βX, and hence an F-space. Since F' is a local property, Y is an F'-space.

In particular, *if X is σ-compact, and locally compact at infinity* (i.e., $\beta X - X$ is locally compact, see [8, p. 94]), *then $\beta X - X$ is an F'-space.*

For example, the space Σ of [5, 4M] is σ-compact but not locally compact. According to [8, 3.1], a space X is locally compact at infinity if and only if the set $R(X)$, of all points of X at which X is not locally compact, is compact. Since $R(\Sigma) = \{\sigma\}$, Σ is locally compact at infinity; hence $\beta\Sigma - \Sigma$ is an F'-space. However, since $\beta\Sigma - \Sigma$ is an open subspace of $\beta\mathbf{N} - \mathbf{N}$, this is a special case of Corollary 1.

For an application not covered by Corollary 1, we consider the following.

EXAMPLE. Let $\Lambda_0 = \beta\mathbf{R} - \mathbf{N}$. A moment's reflection shows that Λ_0 is σ-compact and that $R(\Lambda_0) = \beta\mathbf{N} - \mathbf{N}$; hence $\beta\Lambda_0 - \Lambda_0$ is an F'-space. This example also shows the usefulness of [8, 3.1] in a situation in which it is not convenient to examine $\beta X - X$ directly.

The analog of Corollary 1 for F-spaces is not settled. However, under the continuum hypothesis it is shown in [3, 4.2] that all open subsets of the particular F-spaces $\beta\mathbf{R} - \mathbf{R}$ and $\beta\mathbf{N} - \mathbf{N}$ are also F-spaces.

As to *closed* subspaces, it is trivial that a closed subspace of a *compact* F-space is also an F-space, since it is C^*-embedded [5, 14.26]. For *locally compact* F-spaces we have the following.

COROLLARY 2. *Every closed subspace of a locally compact F-space is an F'-space.*

Proof. Let X be a locally compact F-space and G a closed subspace. It is shown in [5, 14.25] that X is an F-space if and only if βX is an F-space (this also follows immediately from Lemmas 1 and 3 using the relations $\mathcal{O}^p_{\beta X} = (\mathcal{O}^p_X)^\sharp$ and $\mathcal{O}^p_X = \mathcal{O}^p_{\beta X} | X$ which follow from [5, 7.12(a)]). Hence βX is a compact F-space and thus $\mathrm{cl}_{\beta X} G$ is an F-space. Also, X is open in βX and hence $G = X \cap \mathrm{cl}_{\beta X} G$ is an open subspace of $\mathrm{cl}_{\beta X} G$. Hence G is an F'-space by Corollary 1.

3. Continuous images. Our z-filters also yield a simple proof of the following result, which is essentially the content of the lemma in [2].

THEOREM (*Comfort-Ross*). *An open continuous image of an F'-space is also an F'-space.*

Proof. Let $\tau : X \to Y$ be an open continuous mapping of an F'-space X onto a space Y. For any $p \in X$, since $\mathcal{O}^p_X$ is prime, so is its sharp-image $\tau^\sharp \mathcal{O}^p_X$ [5, 4.12], and hence any z-filter containing $\tau^\sharp \mathcal{O}^p_X$ is also prime [5, 2.9]. If $Z \in \tau^\sharp \mathcal{O}^p_X$, then $\tau^{\leftarrow}[Z]$ is a neighborhood of p, so that Z is a neighborhood of τp; hence $\tau^\sharp \mathcal{O}^p_X \subseteqq \mathcal{O}^{\tau p}_Y$, and thus $\mathcal{O}^{\tau p}_Y$ is prime. Hence Y is an F'-space.

We note that a closed continuous image of an F'-space need not be an F'-space. For example, if X is the open unit disk in the plane, and the compactification BX is the closed disk, then the unit circle $BX - X$ is a closed continuous image of the F'-space $\beta X - X$, but is not an F'-space, since a metrizable F-space must be discrete [5, 14N.3].

4. Induced mappings. In attempting to extend Theorem 1 to the case that X is an F'-space and $\tau : Y \to X$ is a continuous mapping of Y into X, a reasonable condition which generalizes z-embedding is that for every zero-set Z in Y there is a zero-set W in X such that $Z = \tau^{\leftarrow}[W]$. In this case Y is also an F'-space; however, the following result, an analog of [5, Th. 10.3(b)], shows that this situation is essentially the same as that of Theorem 1.

THEOREM 2. *Let $\tau : Y \to X$ be a continuous mapping of Y into X, and τ' the induced mapping $W \to \tau^{\leftarrow}[W]$ of $\boldsymbol{Z}(X)$ into $\boldsymbol{Z}(Y)$. Then τ' is onto $\boldsymbol{Z}(Y)$ if and only if τ is a homeomorphism whose image is z-embedded in X.*

Proof. For any zero-set W in X we have $\tau^{\leftarrow}[W] = \tau^{\leftarrow}[W \cap \tau[Y]]$, where $W \cap \tau[Y]$ is a zero-set in $\tau[Y]$. Thus in proving the necessity we may assume that τ is onto X. Any two distinct points p_1 and p_2 of Y have disjoint zero-set-neighborhoods of the form $\tau^{\leftarrow}[W_1]$ and $\tau^{\leftarrow}[W_2]$, where W_1 and W_2 are zero-sets in X; it follows that W_1 and W_2 are disjoint and hence $\tau p_1 \neq \tau p_2$. Thus τ is one-to-one. In both Y and X the closure of a set is the intersection of the zero-sets containing it. It follows that for any subset E of Y, we have $\mathrm{cl}_Y E = \tau^{\leftarrow}[\mathrm{cl}_X \tau[E]]$. Thus $\tau[\mathrm{cl}_Y E] = \mathrm{cl}_X \tau[E]$, and τ is a homeomorphism. The sufficiency is clear.

5. Characterization of F'-spaces. We now give a characterization of F'-spaces in terms of z-embedded subspaces (see condition (4) below), and include for convenience several other known characterizations. Characterization (5) is due to Comfort, Hindman, and Negrepontis [2, Th. 1.1], while the others are from [4] and [5].

THEOREM 3. *For any X, the following are equivalent.*

(1) *For every $p \in X$, the ideal O^p [resp. z-filter $\mathscr{O}^p$] is prime.*

(2) *The prime ideals [resp. prime z-filters] contained in any given fixed maximal ideal [resp. fixed z-ultrafilter] form a chain.*

(3) *Given $p \in X$ and $f \in C(X)$, there is a neighborhood of p on which f does not change sign.*

(4) *Every z-embedded subspace is C^*-embedded in its closure.*

(5) *Every cozero-set is C^*-embedded in its closure.*

(6) *For each $f \in C(X)$, $\operatorname{pos} f$ and $\operatorname{neg} f$ have disjoint closures.*

(7) *Disjoint cozero-sets have disjoint closures* (i.e., *X is an F'-space*).

Proof. As in [5, 14.25], the equivalence of (1), (2), and (3) follows directly from [5; 7.15, 14.8(a), 14.2(a), 2.8, 2.9].

(2) implies (4). Let Y be z-embedded in X. According to [5, 6.4], Y is C^*-embedded in $\operatorname{cl} Y$ if every point of $\operatorname{cl} Y$ is the limit of a *unique* z-ultrafilter on Y. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be z-ultrafilters on Y converging to the same point p in $\operatorname{cl} Y$. By Lemma 1 the induced z-filters $\mathscr{M}_1^\#$ and $\mathscr{M}_2^\#$ are prime. Let $Z \in \mathscr{M}_1^\#$; thus $Z \cap Y \in \mathscr{M}_1$. If V is any neighborhood of p in X, then $V \cap \operatorname{cl} Y$ contains some member of $\mathscr{M}_1$ [5, 6.2]; hence $V \cap \operatorname{cl} Y$ meets $Z \cap Y$ and thus $V \cap Z \neq \emptyset$. It follows that $p \in Z$. Thus $\mathscr{M}_1^\#$ is contained in the z-ultrafilter $\mathscr{M}_X^p$, and similarly $\mathscr{M}_2^\#$. By hypothesis, $\mathscr{M}_1^\#$ and $\mathscr{M}_2^\#$ are comparable. If, say, $\mathscr{M}_1^\# \subseteqq \mathscr{M}_2^\#$, then since Y is z-embedded, we have by Lemma 2, $\mathscr{M}_1 = \mathscr{M}_1^\# | Y \subseteqq \mathscr{M}_2^\# | Y = \mathscr{M}_2$, so that $\mathscr{M}_1 = \mathscr{M}_2$. Hence Y is C^*-embedded in $\operatorname{cl} Y$.

(4) implies (5). As noted in § 2, every cozero-set is z-embedded.

(5) implies (6). Put $T = \operatorname{cl}_X(\operatorname{pos} f \cup \operatorname{neg} f)$. Put $g = 1$ on $\operatorname{pos} f$ and $g = -1$ on $\operatorname{neg} f$, and extend g to $h \in C^*(T)$. Since $h = 1$ on $\operatorname{cl}_X(\operatorname{pos} f)$ and $h = -1$ on $\operatorname{cl}_X(\operatorname{neg} f)$, these closures are disjoint.

(6) implies (7). If $X - \boldsymbol{Z}(f)$ and $X - \boldsymbol{Z}(g)$ are disjoint, then $X - \boldsymbol{Z}(f) \subseteqq \operatorname{pos}(f^2 - g^2)$ and $X - \boldsymbol{Z}(g) \subseteqq \operatorname{neg}(f^2 - g^2)$.

(7) implies (1). If Z and W are zero-sets with $Z \cup W = X$, then $X - Z$ and $X - W$ are disjoint cozero-sets and thus have disjoint closures. Hence $\operatorname{int} Z \cup \operatorname{int} W = X$, and thus $Z \in \mathscr{O}^p$ or $W \in \mathscr{O}^p$. By [5, 2E], $\mathscr{O}^p$ is prime.

We may use Theorem 3 to obtain an alternative proof of Theorem 1 as follows. Let Y be z-embedded in an F'-space X. Let T be a z-embedded subspace of Y. Then T is z-embedded in X, and thus C^*-embedded in $\operatorname{cl}_X T$, hence in $\operatorname{cl}_Y T$. Thus Y is an F'-space. Still another instructive proof may be based on condition (2) and Lemmas 6 and 2.

Theorem 3 also yields the following extension of [1, Th. 1.8]. *Any F'-space with a dense normal z-embedded subspace is an F-space.* The proof given in [1] serves here as well.

The above characterization of F'-spaces in terms of z-embedded subspaces has an analog for F-spaces, [7]; it may also be obtained from our characterization of F'-spaces as follows.

COROLLARY (*Hager*). *A space X is an F-space if and only if every z-embedded subspace is C^*-embedded.*

Proof. According to [5, 14.25], X is an F-space if and only if every cozero-set is C^*-embedded in X. Since a cozero-set is z-embedded, the sufficiency is clear. Now let X be an F-space and Y a z-embedded subspace. Since X is z-embedded in βX, so is Y. Since βX is an F-space, it follows from Theorem 3 that Y is C^*-embedded in $\mathrm{cl}_{\beta X} Y$. The latter space is compact, hence C^*-embedded in βX. Thus Y is C^*-embedded in βX, hence in X.

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