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THE PRINCIPLE OF SUBORDINATION APPLIED TO FUNCTIONS OF SEVERAL VARIABLES

T. J. SUFFRIDGE

In this paper we consider univalent maps of domains in $C^n (n \geqq 2)$. Let P be a polydisk in C^n. We find necessary and sufficient conditions that a function $f: P \to C^n$ be univalent and map the polydisk P onto a starlike or a convex domain. We also consider maps from

$$D_p = \{z: |z|_p < 1\} \subset C^n$$
$$|z|_p = |(z_1, z_2, \cdots, z_n)|_p = \left[\sum_{j=1}^{n} |z_j|^p\right]^{1/p}, \quad p \geqq 1 \tag{1}$$

into C^n and give necessary and sufficient conditions that such a map have starlike or convex image.

In [4] Matsuno has considered a similar problem for the hypersphere $D_2 \subset C^n$. His definition of starlikeness is different from that used in this paper, but the results show that the two definitions are equivalent. However, his definition of convex-like is not equivalent to geometrically convex.

1. Preliminary lemmas. For $(z_1, z_2, \cdots, z_n) = z \in C^n$, define $|z| = \max_{1 \leqq j \leqq n} |z_j|$. Let $E_r = \{z \in C^n: |z| < r\}$ and $E = E_1$. Let $\mathscr{P}$ be the class of mappings $w: E \to C^n$ which are holomorphic and which satisfy $w(0) = 0$, $\operatorname{Re}[w_j(z)/z_j] \geqq 0$ when $|z| = |z_j| > 0$, $(1 \leqq j \leqq n)$ where $w = (w_1, w_2, \cdots, w_n)$. The following lemmas are generalizations of Theorems A and B of Robertson [5, p. 315–317].

LEMMA 1. *Let $v(z; t): E \times I \to C^n$ be holomorphic for each $t \in I = [0, 1]$, $v(z; 0) = z$, $v(0, t) = 0$ and $|v(z; t)| < 1$ when $z \in E$. If*

$$\lim_{t \to 0^+} [(z - v(z; t))/t^\rho] = w(z) \tag{2}$$

exists and is holomorphic in E for some $\rho > 0$, then $w \in \mathscr{P}$.

Proof. The hypothesis (2) implies that $\lim_{t \to 0^+} v_j(z; t) = z_j$ (here $v(z; t) = (v_1(z; t), v_2(z; t), \cdots, v_n(z; t))$ so

$$\frac{2z_j(z_j - v_j(z; t))}{z_j + v_j(z; t)} \equiv \psi_j(z; t)$$

is holomorphic for $z \in E$, $z_j \neq 0$ $(1 \leqq j \leqq n)$. By Schwarz lemma, $|v(z; t)| \leqq |z|$ and hence $\operatorname{Re}[\psi_j(z; t)/z_j] \geqq 0$ when $|z| = |z_j| > 0$. Setting $\psi(z; t) = (\psi_1, \psi_2, \cdots, \psi_n)$, $(z \in E, z_1 z_2 \cdots z_n \neq 0)$ we observe that

$$\lim_{t \to 0^+} \psi(z; t)/t^\rho = w(z)$$

for these values of z and using continuity of w we conclude $w \in \mathscr{P}$.

LEMMA 2. *Let $f: E \to C^n$ be holomorphic and univalent and satisfy $f(0) = 0$. Let $F(z; t): E \times I \to C^n$ be a holomorphic function of z for each $t \in I = [0, 1]$, $F(z; 0) = f(z)$, $F(0, t) = 0$ and suppose $F(z; t) \prec f$ for each $t \in I$ (i.e., $F(E; t) \subset f(E)$ for each $t \in I$). Let $\rho > 0$ be such that $\lim_{t \to 0^+} F(z; 0) - F(z; t)/t^\rho = F(z)$ exists and is holomorphic. Then $F(z) = Jw$ where $w \in \mathscr{P}$. Here F and w are written as column vectors and J is the complex Jacobian matrix for the mapping f.*

Proof. Since $F(z; t) \prec f$ for each $t \in I$, there exists $v: E \times I \to E$ such that $f(v(z; t)) = F(z; t)$ where $|v(z; t)| \leqq |z|$. Writing f as a column vector we have $f(v(z; t)) = f(z) + J(v(z; t) - z) + R(v(z; t), z)$ where $|R(\zeta, z)|/|\zeta - z| \to 0$ as $|\zeta - z| \to 0$. Hence

$$\frac{F(z; 0) - F(z; t)}{t^\rho} = J\left(\frac{z - v(z; t)}{t^\rho}\right) - \frac{R(v(z; t), z)}{t^\rho}$$

and the lemma follows from Lemma 1.

2. Starlike and convex mappings of the polydisk.

DEFINITION. A holomorphic mapping $f: E \to C^n$ is starlike if f is univalent, $f(0) = 0$ and $(1 - t)f \prec f$ for all $t \in I$.

THEOREM 1. *Suppose $f: E \to C^n$ is starlike and that J is the complex Jacobian matrix of f. There exists $w \in \mathscr{P}$ such that $f = Jw$ where f and w are written as column vectors.*

Proof. Apply Lemma 2 with $F(z; t) = (1 - t)f(z)$. Then

$$f(z) = \lim_{t \to 0^+} \frac{f(z) - (1 - t)f(z)}{t} = \lim_{t \to 0^+} \frac{F(z; 0) - F(z; t)}{t}$$

and the theorem follows from Lemma 2.

We now consider the conclusion of Theorem 1 in component form. Let J_j be the matrix obtained by replacing the jth column in J by the column vector f, $1 \leqq j \leqq n$. Then the jth component w_j of w is $\det(J_j)/\det J$. Theorem 1 therefore says that if f is starlike then $\operatorname{Re}[\det(J_j)/z_j \det J] \geqq 0$ when $|z| = |z_j| > 0$. Also,

$$(3) \qquad f_j = \frac{\partial f_j}{\partial z_1} w_1 + \frac{\partial f_j}{\partial z_2} w_2 + \cdots + \frac{\partial f_j}{\partial z_n} w_n, \qquad 1 \leqq j \leqq n$$

and equating coefficients in the power series using (3) we find

$$w_j(z) = z_j + \text{terms of total degree 2 or greater} .$$

Now suppose $|z^{(0)}| = |z_j^{(0)}| > 0$ and let α_k, $(1 \leqq k \leqq n)$ be such that $z_k^{(0)} = \alpha_k z_j^{(0)}$. Then $|\alpha_k| \leqq 1$, $(1 \leqq k \leqq n)$. Consider $w_j(z)/z_j = u(z_j)$ where z is restricted to the set,

$$z = (\alpha_1, \alpha_2, \cdots, \alpha_n) z_j , \qquad |z_j| < 1 .$$

Then $\operatorname{Re} u(z_j) \geqq 0$, $0 < |z_j| < 1$ and $u(z_j) \to 1$ as $z_j \to 0$. Since $\operatorname{Re} u(z_j)$ is a harmonic function of z_j, we conclude $\operatorname{Re} u(z_j) > 0$, $|z_j| < 1$ and

$$\text{(4)} \qquad \operatorname{Re} [w_j(z)/z_j] > 0 \quad \text{when} \quad |z| = |z_j| > 0 .$$

We now prove the converse of Theorem 1.

THEOREM 2. *Suppose* $f: E \to C^n$ *is holomorphic,* $f(0) = 0$, J *is nonsingular and that*

$$\text{(5)} \qquad f(z) = Jw, \; w \in \mathscr{P} .$$

Then f *is starlike.*

Proof. Since $\det J \neq 0$ when $z = 0$, f is univalent in a neighborhood of 0. It is clear that $\{r: 0 \leqq r \leqq 1 \text{ and } f \text{ is univalent in } E_r\} = A$ is a closed subset of $[0, 1]$. We will show that A is also open and that if f is univalent in E_r then $f(E_r)$ is starlike with respect to 0.

Let $r > 0$ be such that f is univalent in E_r, $(0 < r < 1)$. Let z be fixed, $|z| \leqq r$ and let $v(z; t)$ be such that $f(v(z; t)) = (1 - t)f(z)$, $-\varepsilon < t < t_0$ where ε is small and positive and $t_0 > 0$. This is possible since $\det J \neq 0$.

Then

$$\text{(6)} \qquad \begin{aligned} v(z; t) &= v(z; 0) + J^{-1} \cdot (-f(z)) \cdot t + g(t) \\ &= z - J^{-1} \cdot J \cdot w \cdot t + g(t) \\ v(z; t) &= z - tw + g(t) \end{aligned}$$

by (5). Here $|g(t)|/t \to 0$ as $t \to 0$. Using (4), we conclude $|v(z; t)|$ is a strictly decreasing function of t. Hence each point of the ray $(1 - t)f(z)$, $0 < t \leqq 1$ is the image of a point $v(z; t) \in E_r$ for each z such that $|z| \leqq r$. We conclude that $f(E_r)$ is starlike with respect to 0. We now show A is open. Observe that f is one-to-one in the closed polydisk $\bar{E}_r$ for if $|z| \leqq |\zeta| = r$, $z \neq \zeta$ and $f(z) = f(\zeta)$ then by (6) and (4) we can conclude that for t positive and sufficiently small there are functions $v(\zeta; t)$, $v(z; t)$ such that $v(\zeta; t), v(z, t) \in E_r$, $v(\zeta; t) \neq v(z; t)$ and

$f(v(z; t)) = (1 - t)f(z) = (1 - t)f(\zeta) = f(v(\zeta, t))$ which is a contradiction.

We now define a continuous nonnegative function $\phi: E \times E \to R$ (R is the real numbers) such that $\phi(z, \zeta) = 0$ if and only if $f(z) = f(\zeta)$, $z \neq \zeta$. We show that ϕ is positive on the closed set $\bar{E}_r \times \bar{E}_r$ and hence has a positive minimum on this set. This will imply f is univalent in $E_{r+\varepsilon}$ for some $\varepsilon > 0$ and hence A is open. For $z, \zeta \in E$, define $G(z, \zeta) = \det(a_{ij})$ where

$$a_{ij} = \begin{cases} \dfrac{f_i(z_1, z_2, \cdots, z_j, \zeta_{j+1} \cdots, \zeta_n) - f_i(z_1, z_2, \cdots, z_{j-1}, \zeta_j, \cdots, \zeta_n)}{z_j - \zeta_j}, & (z_j \neq \zeta_j) \\ \dfrac{\partial f_i}{\partial z_j}(z_1, z_2, \cdots, z_j, \zeta_{j+1}, \cdots, \zeta_n), & (z_j = \zeta_j) \end{cases}$$

and $f = (f_1, f_2, \cdots, f_n)$.

Now set $\phi(z, \zeta) = |G(z, \zeta)| + \sum_{j=1}^{n} |f_j(z) - f_j(\zeta)|$. Then $\phi(z, z) = |\det(J(z)| > 0$ while

$$\phi(z, \zeta) > 0 \quad \text{when} \quad f(z) \neq f(\zeta) .$$

If $f(z) = f(\zeta)$ for some $z, \zeta \in E, z \neq \zeta$ then the columns of $G(z, \zeta)$ are not linearly independent so $G(z, \zeta) = 0$ and $\phi(z, \zeta) = 0$. The proof is now complete.

THEOREM 3. *Suppose $f: E \to C^n$ is holomorphic, $f(0) = 0$ and that J is nonsingular for all $z \in E$. Then f is a univalent map of E onto a convex domain if and only if there exist univalent mappings f_j $(1 \leqq j \leqq n)$ from the unit disk in the plane onto convex domains in the plane such that $f(z) = T(f_1(z_1), f_2(z_2) \cdots, f_n(z_n))$ where T is a nonsingular linear transformation.*

Proof. It is clear that if f satisfies the conditions given in the theorem, then f is univalent and $f(E)$ is convex. We will prove the converse.

Suppose f is a univalent map of E onto a convex domain. Let $A = (A_1, A_2, \cdots, A_n)$ where $A_j \geqq 0$ $(1 \leqq j \leqq n)$ and let

$$A_t(z) = (z_1 e^{iA_1 t}, z_2 e^{iA_2 t}, \cdots, z_n e^{iA_n t})$$

where $-1 \leqq t \leqq 1$. Then

$$F(z; t) = 1/2[f(A_t(z)) + f(A_{-t}(z))] \prec f \qquad 0 \leqq t \leqq 1$$

and $F(z; t)$ satisfies the hypotheses of Lemma 2 with $\rho = 2$. Using the same notation as in Lemma 2, we have

$$F(z) = (F_1, F_2, \cdots, F_n)$$

$$(7) \qquad 2F_j = \sum_{k=1}^{n} A_k^2\left(z_k^2 \frac{\partial^2 f_j}{\partial z_k^2} + z_k \frac{\partial f_j}{\partial z_k}\right) + 2\sum_{k=2}^{n}\sum_{l=1}^{k-1} A_k A_l z_k z_l \frac{\partial^2 f_j}{\partial z_l \partial z_k}$$

and also $F = Jw$, $w \in \mathscr{P}$. Hence we find that $w_j = \det J^{(j)}/\det J$ where $J^{(j)}$ is obtained from J by replacing the jth column by F written as a column vector. Fix k, $1 \leqq k \leqq n$ and choose $A_k = 1$, $A_l = 0$, $l \neq k$, $1 \leqq l \leqq n$. Suppose $|z| = |z_j| > 0$, $j \neq k$ and $z_k = 0$. Then $w_j/z_j = 0$ and since $\operatorname{Re}(w_j/z_j) \geqq 0$ when $|z| = |z_j| > 0$ we must have $w_j \equiv 0$. We have therefore shown that for $1 \leqq j \leqq n$ and $1 \leqq k \leqq n$ we have

$$(8) \qquad z_k^2 \frac{\partial^2 f_j}{\partial z_k^2} + z_k \frac{\partial f_j}{\partial z_k} = \frac{\partial f_j}{\partial z_k}\psi_k$$

where $\operatorname{Re}[\psi_k(z)/z_k] \geqq 0$ when $|z| = |z_k| > 0$. With k as before, fix l, $1 \leqq l \leqq n$, $l \neq k$ and choose $A_k = 1$, $A_l = \varepsilon > 0$ and $A_m = 0$, $1 \leqq m \leqq n$, $m \neq k, l$.

Using (8) we conclude

$$w_j = \varepsilon \frac{z_k z_l G_j}{\det J} + O(\varepsilon^2) \qquad (j \neq k)$$

where G_j is obtained from $\det J$ by replacing the jth column by the column $\partial^2 f_m/\partial z_l \partial z_k (1 \leqq m \leqq n)$. Hence $\operatorname{Re}[z_k z_l/z_j \cdot G_j/\det J] \geqq 0$ when $|z| = |z_j| > 0$. Since $\operatorname{Re}[z_k z_l/z_j \cdot G_j/\det J] = 0$ when $z_k z_l = 0$ we see that $G_j \equiv 0$ for each j, $1 \leqq j \leqq n$.

Since the system of equations

$$\sum_{j=1}^{n} \frac{\partial f_m}{\partial z_j}\phi_j = \frac{\partial^2 f_m}{\partial z_l \partial z_k} \qquad 1 \leqq m \leqq n$$

has solution

$$\phi_j = \frac{G_j}{\det J} = 0 \qquad 1 \leqq j \leqq n$$

we conclude

$$\frac{\partial^2 f_m}{\partial z_l \partial z_k} = 0 \qquad 1 \leqq m \leqq n\,.$$

This implies

$$(9) \qquad f_m(z) = \sum_{j=1}^{n} a_{j,m}\phi_{j,m}(z_j) \qquad 1 \leqq m \leqq n$$

where $\phi_{j,m}$ is analytic on the unit disk in the complex plane. Using

(8) we conclude $\phi_{j,m} = \phi_{j,k}$ $(1 \leqq m, k \leqq n)$ provided the constants $a_{j,m}$ in (9) are appropriately chosen. The theorem now follows readily from (8).

EXAMPLE 1. Let $f: E \to C^2$ be given by $f(z) = (z_1 + az_2^2, z_2)$ where a is a complex number, $a \neq 0$. Clearly f is univalent. Letting $f = Jw$, we find $w_1 = z_1 - az_2^2$, $w_2 = z_2$ so f is starlike provided $|a| < 1$. Note that Theorem 3 implies the suprising result that none of the sets $f(E_r)$ is convex $(1 > r > 0)$.

EXAMPLE 2. Let $f: E \to C^2$ be given by $f(z) = (z_1 g(z), z_2 g(z))$, $g: E \to C$ where g is holomorphic, $0 \notin g(E)$. Then $f = Jw$ implies

$$w_1/z_1 = w_2/z_2 = 1 + \left[z_1\frac{\partial g}{\partial z_1} + z_2\frac{\partial g}{\partial z_2}\right]\Big/ g \tag{10}$$

and f is starlike if and only if $\operatorname{Re}(w_1(z)/z_1) \geqq 0$, $z \in E$. Conversely, one can show that if $f: E \to C^2$ is holomorphic, $f = Jw$ where $w \in \mathscr{P}$ and $w_1/z_1 = w_2/z_2$ then there exists $g: E \to C$, g holomorphic, $0 \notin g(E)$ such that (10) holds and $f = ((a_1 z_1 + a_2 z_2)g, (b_1 z_1 + b_2 z_2)g)$, $(a_1 b_2 \neq a_2 b_1)$. In these cases the intersection of the polydisk E with an analytic plane $\alpha z_1 + \beta z_2 = 0$ maps into an analytic plane $\delta f_1 + \gamma f_2 = 0$. Interesting choices of g are $g(z) = (1 - z_1 z_2)^{-1}$ and $g(z) = [(1 - z_1)(1 - z_2)]^{-1}$.

3. Extension to convex and starlike maps of D_p. Since the details of the proofs for the results in this section are similar to those in §'s 2 and 3, we omit the details. We wish to find lemmas which apply to D_p (D_p is defined in equation (1)) in the same way that Lemmas 1 and 2 apply to the polydisk. The crucial point is that given equation (6) with $0 \neq z \in D_p$ we wish to conclude

$$|v(z; t)|_p \leqq |z|_p \quad \text{when} \quad 0 < t < \varepsilon$$

for some $\varepsilon > 0$. This will be true provided $\sum_{j=1}^n |z_j - tw_j|^p < \sum_{j=1}^n |z_j|^p$ for t sufficiently small. That is

$$\sum_{\substack{j=1\\ z_j \neq 0}}^n |z_j|^p (1 - 2t \operatorname{Re} w_j/z_j + t^2 |w_j/z_j|^2)^{p/2} + \sum_{z_j = 0} t^p |w_j|^p < \sum_{j=1}^n |z_j|^p$$

or

$$t\Big(\sum_{\substack{j=1\\ z_j \neq 0}}^n -p \operatorname{Re} |z_j|^p \operatorname{Re}(w_j/z_j) + \sum_{z_j=0} t^{p-1} |w_j|\Big) < 0$$

when t is sufficiently small, $t > 0$. Hence we define $\mathscr{P}_p$ for $p \geqq 1$ by $w \in \mathscr{P}_p$ if $w: D_p \subset C^n \to C^n$, $w(0) = 0$, w holomorphic and

$$
\begin{aligned}
&\operatorname{Re} \sum_{j=1}^{n} w_j \cdot |z_j|^p / z_j \geqq 0 \qquad \text{if } p > 1 \\
&\operatorname{Re} \sum_{\substack{j=1 \\ z_j \neq 0}}^{n} w_j \cdot |z_j| / z_j - \sum_{z_j = 0} |w_j| \geqq 0 \qquad \text{if } p = 1 ,
\end{aligned} \tag{11}
$$

$z \in D_p$, $w = (w_1, w_2, \cdots, w_n)$.

We now have the following lemmas and theorems which correspond to the lemmas and theorems of §§ 2 and 3.

LEMMA 3. *Let* $v(z; t)\colon D_p \times I \to C^n$ *be holomorphic for each* $t \in I$, $v(z, 0) = z$, $v(0, t) = 0$ *and* $|v(z; t)|_p < 1$ *when* $z \in D_p$. *If*

$$\lim_{t \to 0^+} [(z - v(z; t))/t^\rho] = w(z)$$

exists and is holomorphic in D_p *for some* $\rho > 0$, *then* $w \in \mathscr{P}_p$.

LEMMA 4. *Let* $f\colon D_p \to C^n$ *be holomorphic and univalent and satisfy* $f(0) = 0$. *Let* $F(z; t)\colon D_p \times I \to C^n$ *be a holomorphic function of* z *for each* $t \in I$, $F(z, 0) = f(z)$, $F(0; t) = 0$ *and suppose* $F(z; t) \prec f$ *for each* $t \in I$. *Let* $\rho > 0$ *be such that* $\lim_{t \to 0^+}(F(z; 0) - F(z; t))/t^\rho = F(z)$ *exists and is holomorphic. Then* $F(z) = Jw$ *where* $w \in \mathscr{P}_p$.

THEOREM 4. *If* $f\colon D_p \to C^n$ *is starlike then there exists* $w \in \mathscr{P}_p$ *such that* $f = Jw$. *Conversely, if* $f\colon D_p \to C^n$, $f(0) = 0$, J *is nonsingular and* $f = Jw$, $w \in \mathscr{P}_p$ *then* f *is starlike.*

THEOREM 5. *Let* $f\colon D_p \to C^n$, $f(0) = 0$ *and suppose* J *is nonsingular. Then* $f(D_p)$ *is convex if and only if* $F = Jw$ *where* $w \in \mathscr{P}_p$ *for each choice of* $A = (A_1, A_2, \cdots, A_n)$, $A_j \geqq 0$ $(1 \leqq j \leqq n)$ *and* F *is given by* (7) *with* $z \in D_p$.

Now set $p = 2$. It is easy to see that Theorem 4 above is equivalent to Matsuno's Theorem 1 [4, p. 91]. Consider $f\colon D_2 \to C^2$ given by $f(z) = (z_1 + az_2^2, z_2)$. Theorem 5 shows that $f(D_2)$ is convex if and only if $|a| \leqq 1/2$ while Matsuno's Lemma 3 [4, p. 94] implies f is convex-like if and only if $|a| \leqq 3\sqrt{3}/4$. This shows that convex-like is not equivalent to geometrically convex.

REFERENCES

1. S. Bergman, *The kernel function and conformal mapping*, Mathematical Surveys, Vol. V., Amer. Math. Soc., New York, 1950.
2. S. Bochner and W. T. Martin, *Several complex variables*, Princeton Univ. Press, 1948.

3. R. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.
4. Takeshi Matsuno, *Star-like theorems and convex-like theorems in the complex vector space*, Sci. Rep. Tokyo, Kyoiku Daigaku, Sect. A **5** (1955), 88-95.
5. M. S. Robertson, *Applications of the subordination principle to univalent functions*, Pacific J. Math. **11** (1961), 315-324.

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UNIVERSITY OF KENTUCKY
LEXINGTON, KENTUCKY