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ELEMENTARY SURGERY ALONG A TORUS KNOT

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In this paper a classification of the manifolds obtained by a (p, q) surgery along an (r, s) torus knot is given. If $|\sigma| = |rsp + q| \neq 0$, then the manifold is a Seifert manifold, singularly fibered by simple closed curves over the 2-sphere with singularities of types $\alpha_1 = s$, $\alpha_2 = r$, and $\alpha_3 = |\sigma|$. If $|\sigma| = 1$, then there are only two singular fibers of types $\alpha_1 = s$, $\alpha_2 = r$, and the manifold is a lens space $L(|q|, ps^2)$. If $|\sigma| = 0$, then the manifold is not singularly fibered but is the connected sum of two lens spaces $L(r, s) \# L(s, r)$. It is also shown that the torus knots are the only knots whose complements can be singularly fibered.

1. DEFINITIONS. A *knot* K is a polygonal simple closed curve in S^3 which does not bound a disk in S^3. A *solid torus* T is a 3-manifold homeomorphic to $S^1 \times D^2$. The boundary of T is a *torus*, a 2-manifold homeomorphic to $S \times S^1$. A *meridian* of T is a simple closed curve on ∂T which bounds a disk in T but is not homologous to zero on ∂T. *A meridianal disk* of T is a disk D in T such that $D \cap \partial T = \partial D$ and ∂D is a meridian of T. A *longitude* of T is a simple closed curve on ∂T which is transverse to a meridian of T and is null-homologous in $\overline{S^3\text{-}T}$. A *meridianlongitude pair for* T is an ordered pair (M, L) of curves such that M is a meridian of T and L is a longitude of T transverse to M. $\pi_1(\partial T) \cong Z \times Z$ with generators M and L. $qM + pL$ is the homotopy class of a simple closed curve on ∂T if and only if p and q are relatively prime.

A *torus knot of type* (r, s), denoted $K(r, s)$, is defined as follows. Let T be a standardly embedded solid torus in S^3, that is, T is isotopic to a regular neighborhood of a polygonal curve in the x-y plane. Then $\overline{S^3\text{-}T}$ is a solid torus. Let J_1 and J_2 be oriented simple closed curves on ∂T such that J_1 bounds a disk in T and J_2 bounds a disk in $\overline{S^3\text{-}T}$, that is J_1 is meridianal and J_2 is longitudinal. Identifying J_1 with $(1, 0)$ and J_2 with $(0, 1)$, let r and s be relatively prime integers, $r > s > 0$, and let $K(r, s)$ be a simple closed curve in (r, s). Then $K(r, s)$ is a torus knot of type (r, s). By Van Kampen's theorem $\pi_1(S^3\text{-}K(r,s)) \cong (a, b \mid a^r = b^s)$.

A space is a *lens space* if it contains a solid torus such that the closure of its complement is also a solid torus. Hence one way to view a lens space is as the space obtained by identifying two solid tori by a homeomorphism on the boundary.

Basic Construction: *Elementary surgery along a knot*. Let N

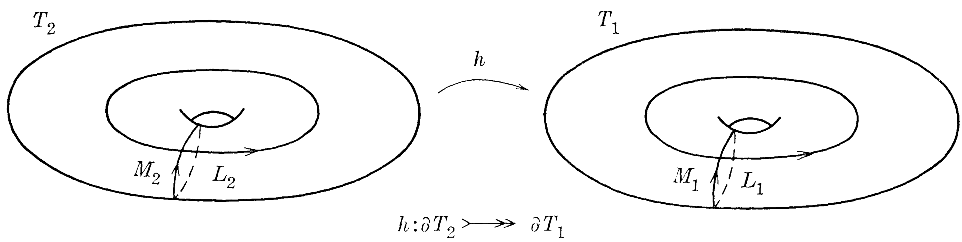

FIGURE 1

be a regular neighborhood of K, M an oriented meridianal curve for N on ∂N, and L an oriented curve on ∂N which is transverse to M and bounds an orientable surface in S^3-N. Consider $M \cap L$ as a base point for $\pi_1(\overline{S^3\text{-}N})$. Let T be a solid torus and $h: T \twoheadrightarrow N$ be a homeomorphism. Then $S^3 \cong \overline{S^3\text{-}N}\ U_{h|\partial T} T$. Now let $h_1 : \partial T \twoheadrightarrow \partial T$ be a homeomorphism with the property that h^{-1}. $h_1 : \partial T \rightarrow \partial T$ does not extend to a homeomorphism of T onto T_1. Let $\mathscr{M}^3 = \overline{S^3\text{-}N}\ U_{h_1} T$, then we say $\mathscr{M}^3$ is obtained from S^3 by performing an elementary surgery along K. The fundamental group of $\mathscr{M}^3$ is obtained by adjoining a relation of the form $L^p = M^q$ where (1) pL-qM is the image under h_1 of the boundary of a meridianal disk of T, (2) p and q are relatively prime, (3) $p \neq 0$ since we have performed an elementary surgery and we may assume that $p > 0$ since $\mathscr{M}^3\ (p, q) \cong \mathscr{M}^3(\text{-}p, \text{-}q)$. If K is unknotted, then an elementary surgery along K will yield a lens space, since the complement of the interior of a regular neighborhood of K is a solid torus and the effect of the surgery is a manifold which can be obtained by identifying two solid tori along their boundaries.

A *solid torus fibered by u, v*, denoted by $sT^3(v/u)$, is gotten from $D^2 \times I$ by rotating the top $2\pi v/u$ where $(u, v) = 1$, $0 \leq v \leq u/2$, and then identifying top and bottom. A *fiber* is denoted by F. A *cross-circle* Q is a simple closed curve meeting each F in one point. A *singularly fibered manifold* $\mathscr{M}^3$, in the sense of Seifert, is a topological 3-manifold partitioned into subsets homeomorphic to S^1, the fibers, such that each fiber has a closed neighborhood preserving homeomorphic to some $sT^3(v/u)$.

$\mathscr{M}^3$ is obtained as follows. Let B be a sphere with $g > 0$ handles (k crosscaps), cut B along a set of loops based at x_0 to get a $4g$-gon ($2k$-gon) P with sides $A_1^{-1}B_1^{-1}A_1B_1 \cdots A_g^{-1}B_g^{-1}A_gB_g(C_1C_1' \cdots C_kC_k')$ to be identified in pairs, and remove a disk D_0 around x_0 to get $\bar{P}$. $\bar{P} \times S^1$ is a 3-manifold on which we make some identifications. Let $\chi : \pi_1(B, x_0) \rightarrow \text{Aut}\ \pi_1(S^1) \cong Z_2$. Let x and x' be points on the edges of $\bar{P}$ which are identified in B, and let α be a path formed by the line segments $\overline{x_0x}$, $\overline{x'x_0}$. α is a loop in B based at x_0. Choose a base point preserving homeomorphism $x \times S^1 \rightarrow x' \times S^1$w hich induces $x([\alpha]) : \pi_1(S^1) \rightarrow$

$\pi_1(S^1)$. Identifying pairs of fibers over the edges of $\bar{P}$ by this homeomorphism gives a manifold $\overline{\mathscr{M}_0^3}$ with boundary $\partial D_0 \times S^1$. Now suppose $\partial D_0 \times S^1$ is trivially fibered by circles ω such that $[\omega] = Q_0 + bF \in \pi_1(\partial D_0 \times S^1)$ where Q_0 generates $\pi_1(\partial D_0)$ and F generates $\pi_1(S^1)$. We close $\overline{\mathscr{M}_0^3}$ with a solid torus $\mathscr{N}(F)$ by a homeomorphism $h: \partial\mathscr{N}(F) \to \partial\overline{\mathscr{M}_0^3}$ such that for M a meridian of $\mathscr{N}(F)$, $M \sim Q_0 + bF$, to obtain $\mathscr{M}_0^3 = \overline{\mathscr{M}_0^3}\ U_h \mathscr{N}(F)$. χ is called the characteristic and b the obstruction term. By removing the fibers over open disks D_i, $i = 1, \cdots, n$ in B we obtain $\overline{\mathscr{M}^3}$ with n boundary components $\partial D_i \times S^1$. Suppose $\partial D_i \times S^1$ is trivially fibered by circles ω_i such that $[\omega_i] = \alpha_i Q_i + \beta_i F_i$, where Q_i generates $\pi_1(\partial D_i)$, F_i generates $\pi_1(S^1)$, $(\alpha_i, \beta_i) = 1$, and $0 < \alpha_i < \beta_i$. By replacing the solid tori removed by $\mathscr{N}(F_i)$ such that for M_i a meridian of $\mathscr{N}(F_i)$, $M_i \sim \alpha_i Q_i + \beta_i F_i$, we obtain a closed manifold fibered by S^1 over B. F_i is a singular fiber of type α_i and has a trivial product neighborhood if and only if $\alpha_i = \pm 1$.

The fundamental group of $\mathscr{M}^3$ is given in terms of the (α_i, β_i), b, and χ by Van Kampen's theorem.

$$\begin{aligned}\pi_1(\mathscr{M}^3) = (A_i, B_i, (C_i), Q_0, Q_1, \cdots, Q_n, F \mid & \prod_{i=1}^{g}[A_i, B_i]Q_1 \cdots Q_n Q_0 = 1 \\ & (\prod_{i=1}^{k} C_i^2 Q_1 \cdots Q_n Q_0 = 1) \\ A_i^{-1}FA_i = F^{\chi(A_i)},\ B_i^{-1}FB_i = F^{\chi(i)},\ & (C_i^{-1}FC_i = F^{\chi(C_i)}), \\ [F, Q_i] = 1,\ Q_0F^b = 1,\ Q_i^{\alpha_i}F^{\beta_i} = 1). &\end{aligned}$$

2. Fibering the complement of a knot.

THEOREM 2. *The complement of a knot K can be singularly fibered in the sense of Seifert if and only if K is a torus knot.*

Proof. Let $K(r, s)$ be a torus knot lying on a standardly embedded torus in S^3. The diagram illustrates the case $r = 3, s = 2$.

We have a fibering of $S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\}$ given by $(z_1, z_2) = (z_1\lambda^s, z_2\lambda^r)$ for $\lambda \in S^1$ (that is, a partition of S^3 into orbits S^1) over $B = S^2$ with the unit circle as a singular fiber of type $\alpha_1 = s$ and the z-axis as a singular fiber of type $\alpha_2 = r$. Each nonsingular fiber is an (r, s) torus knot. If we remove a regular neighborhood of the torus knot, we have $\overline{S^3\text{-}\mathscr{N}(K)}$ singularly fibered.

Suppose $\overline{\mathscr{M}^3} = \overline{S^3\text{-}\mathscr{N}(K)}$ is singularly fibered. Let $F \sim mL + nM$ where F is a fiber on $\partial\overline{\mathscr{M}^3}$ and (M, L) is a meridian-longidude pair for $\mathscr{N}(K)$. If $m \neq 0$, then $M \not\sim F$ on $\partial\overline{\mathscr{M}^3}$. Hence, there exists a singularly fibered solid torus $sT^3(v/u)$ and a fiber preserving homeomorphism $h: \partial sT^3 \to \partial\overline{\mathscr{M}^3}$ which takes a meridian of sT^3 to M by Lemma 6 of Seifert [4]. Hence, $\overline{\mathscr{M}^3}\ U_h sT^3 = S^3$ and S^3 is singularly fibered with K as a fiber of multiplicity m.

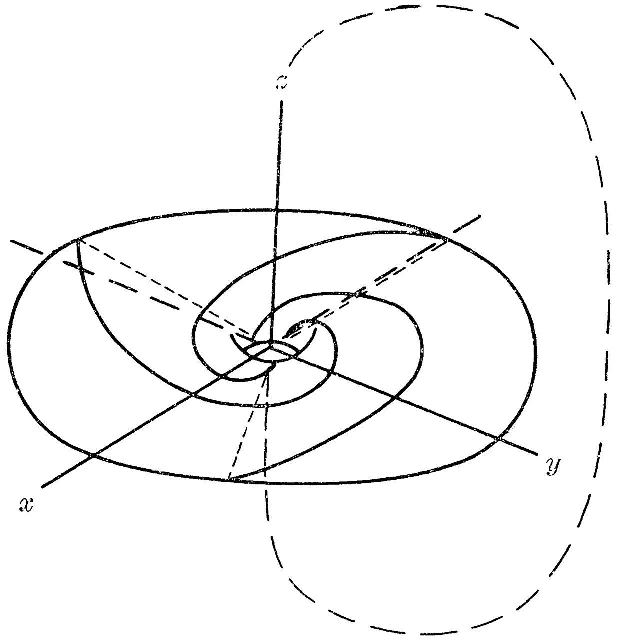

FIGURE 2

If $m \neq \pm 1$, then K is a singular fiber and hence unknotted. If $m = \pm 1$, then K is an ordinary fiber and hence a torus knot. If $m = 0$, $F \sim nM$ where M generates $H_1\,(\overline{S^3 - \mathscr{N}\,(K)}) \simeq Z$. But if $\overline{\mathscr{M}^3 = S^3 - \mathscr{N}\,(K)}$ is singularly fibered, then

$$\begin{aligned}
\pi_1(\overline{\mathscr{M}^3}) = (A_i,\, B_i,\, (C_i),\, Q_0,\, Q_1,\, \cdots,\, Q_n,\, F \,|\, &\prod_{i=1}^{g}[A_i,\, B_i]Q_1 \cdots Q_nQ_0 = 1 \\
&(\prod_{i=1}^{k} C_i^2Q_1 \cdots Q_nQ_0 = 1) \\
A_i^{-1}FA_i = F^{\chi(A_i)},\; B_i^{-1}FB_i = F^{\chi(B_i)},\; &(C_i^{-1}FC^i = F^{\chi(C_i)}) \\
[F,\, Q_i] = 1,\; Q_0F^b = 1,\; &Q_i^{\alpha_i}F^{\beta_i} = 1,\, 1 \leq i \leq n-1) \\
\simeq (A_i,\, B_i,\, (C_i),\, Q_1,\, \cdots,\, Q_{n-1},\, F \,|\, &A_i^{-1}FA_i = F^{\chi(A_i)},\; B_i^{-1}FB_i = F^{\chi(B_i)}, \\
&(C_i^{-1}FC_i = F^{\chi(C_i)}) \\
[F,\, Q_i] = 1,\, &Q_i^{\alpha_i}F^{\beta_i} = 1,\, 1 \leq i \leq n-1).
\end{aligned}$$

Abelianizing, we see that $g = 0$ $(k = 0)$. Setting $F = 1$, we see that $i = 1$ unless $n = \pm 1$ in which case $\alpha_i = \pm 1$, a contradiction. Hence $\pi_1\,(\overline{\mathscr{M}^3}) = (Q_1,\, F \,|\, Q_1^{\alpha_1}F^{\beta_1} = 1)$ and K is a torus knot of type $(\alpha_1,\, \beta_1)$.

NOTE: Theorem 2 can also be proved with results from [1] and [5].

3. The fibered manifolds obtained by elementary surgery along a torus knot.

PROPOSITION 3.1. *If an elementary surgery of type* $(p,\, q)$ *is per-*

formed along $K(r, s)$ *and* $|\sigma| = |rsp + q| \neq 0$, *then the manifold obtained is singularly fibered with fibers of multiplicities* $\alpha_1 = s$, $\alpha_2 = r$, *and* $\alpha_3 = |\sigma| = |rsp + q|$.

Proof. In performing the surgery, we remove a fiber neighbornood of a nonsingular fiber K to obtain $S^3 - \mathcal{N}(K)$ and then close $\overline{S^3 - \mathcal{N}(K)}$ with sT^3 such that $M' \sim pL - qM$ where M' is a meridian of sT^3, L is a longitude of $\mathcal{N}(K)$, and M is a meridian of $\mathcal{N}(K)$. If F is a fiber on $\partial\, \mathcal{N}(K)$ in $\overline{S^3 - \mathcal{N}(K)}$, F loops around the z-axis r times, but the z-axis $\sim sM$ in $\overline{S^3 - \mathcal{N}(K)}$, so $F \sim rsM$ in $\overline{S^3 - \mathcal{N}(K)}$, $F - rsM \sim 0 \sim L$ in $\overline{S^3 - \mathcal{N}(K)}$, and $M' \sim pL - qM \sim p(F - rsM) - qM = pF - (rsp + q)M$. Since M is a crosscircle on $\partial\mathcal{N}(K)$, sT^3 contains a singular fiber of multiplicity $|rsp + q| = |\sigma|$. If $|\sigma| \neq 1$ or 0, the 3-manifold obtained is a Seifert fiber space with three singular fibers of multiplicities $\alpha_1 = s$, $\alpha_2 = r$, and $\alpha_3 = |\sigma|$. The space is topologically a product of a disk with 3 holes and S^1 if we remove regular neighborhoods of the z-axis, unit circle, $K(r, s)$, and an additional nonsingular fiber. If $\alpha_3 = |\sigma| = 1$, $u = 1$ and $v = 0$. The sT^3 added is nonsingularly fibered, so the resultant manifold has only two nonsingular fibers of types $\alpha_1 = s$ and $\alpha_2 = r$.

Assuming a given fixed orientation on $\mathcal{M}(p, q)$, we can determine the β_i and the obstruction term b in terms of p. $H_1(\mathcal{M}(p, q))$ is cyclic of order $b\alpha_1\alpha_2\alpha_3 + \beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3 > 0$ ($b\alpha_1\alpha_2 + \beta_1\alpha_2 + \alpha_1\beta_2$ for $|\sigma| = 1$); on the other hand $H_1(\mathcal{M}(p, q))$ is cyclic of order $|q| = rsp \mp \sigma$. Equating $b\alpha_1\alpha_2\alpha_3 + \beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3$ ($b\alpha_1\alpha_2 + \beta_1\alpha_2 + \alpha_1\beta_2$ for $|\sigma| = 1$) and $q = rsp \mp \sigma$, we can solve for the β_i and b. For example, if $(r, s) = (3, 2)$ and $\sigma = 5$, then the Seifert manifolds obtained are given by the following symbols:

$$(\mathcal{O}, o, 0 \mid p\text{-}6/5;\ 2, 1;\ 3, 1;\ 5, 1) \text{ if } p \equiv 1 \pmod 5$$
$$(\mathcal{O}, o, 0 \mid p\text{-}7/5;\ 2, 1;\ 3, 1;\ 5, 2) \quad p \equiv 2 \pmod 5$$
$$(\mathcal{O}, o, 0 \mid p\text{-}8/5;\ 2, 1;\ 3, 1;\ 5, 3) \quad p \equiv 3 \pmod 5$$
$$(\mathcal{O}, o, 0 \mid p\text{-}9/5;\ 2, 1;\ 3, 1;\ 5, 4) \quad p \equiv 4 \pmod 5.$$

If $|\sigma| = 1$, then the manifold is a lens space $L(|q|, x)$. The Seifert invariants do not determine x; we determine x in the next proposition.

PROPOSITION 3.2. *If an elementary surgery of type* (p, q) *is performed along* $K(r, s)$ *and* $|\sigma| = |rsp + q| = 1$, *then the manifold is a lens space* $L(|q|, ps^2)$.

Proof. Let T_1 be a standardly embedded torus in S^3 as shown below and let T_2 be $\overline{S^3 - T_1}$. Let (M_1, L_1) be a standard meridian-longitude pair for T_1, $(M_2, L_2) = (L_1, M_1)$ for T_2. $K \sim F \sim rM_1 + sL_1$.

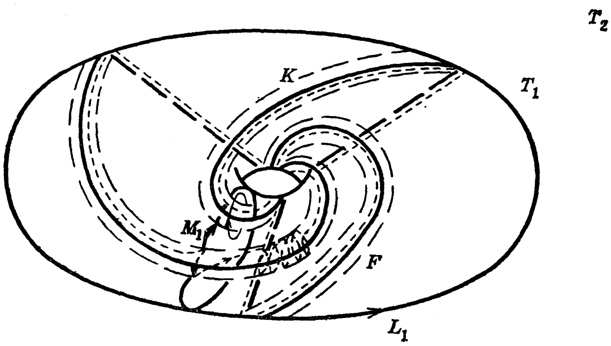

FIGURE 3.1

We remove $\mathcal{N}(K)$ so that T_2 is still a solid torus and replace it with sT^3 such that $M' \sim pL - qM \sim pF \mp M$ ($\sigma = \pm 1$) and so $L' \sim F$. $sT^3 U T_1$ is a solid torus T_3 ($sT^3 \cap T_1 \simeq S^1 \times I$) since a longitude of sT^3, $L' \sim F$. Let M_3 be a meridian of T_3. We want to determine x such that $M_3 \sim |q| L_2 + xM_2$.

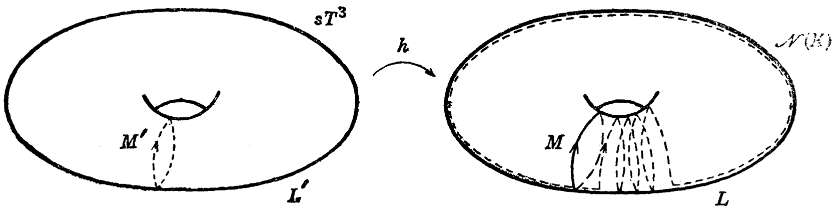

FIGURE 3.2

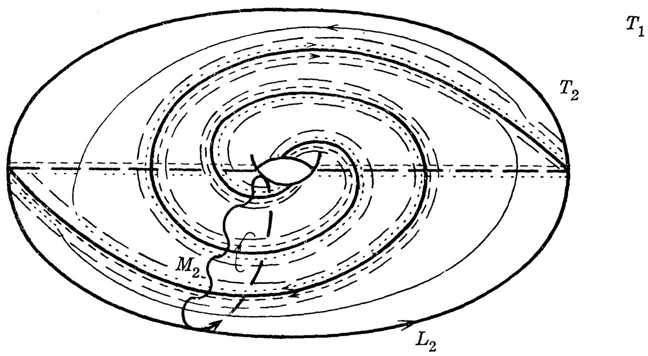

FIGURE 3.3

Now $M' \sim pF \mp M \sim p(rM_1 + sL_1) \mp M = prM_1 + psL_1 \mp M$
also $M_2 \sim L_1 - rM$, $L_2 \sim M_1 + sM$
and $M_3 \sim M_1 \mp sM' \sim M_1 \mp s(prM_1 + psL_1 \mp M) = (1 \mp rsp)M_1$

$$\mp ps^2L_1 + sM \sim (1 \mp rsp)\ (L_2 - sM) \mp ps^2(M_2 + rM) + sM$$
$$= (1 \mp rsp)L_2 - sM \pm rs^2pM \mp ps^2M_2 \mp rs^2pM + sM$$
$$= |q| L_2 \mp ps^2M_2$$

so we have $L(|q|, ps^2)$. The diagrams illustrate the case $r = 3$, $s = 2$, $\sigma = 1$, $q = -(2)(3) + 1 = -5$, and $x = -2(2)$.

REMARK. Distinct surgeries along a given torus knot yield distinct lens spaces; however, the same lens space may be obtained by surgering different torus knots. For example, a $(2, 11)$ surgery on $K(3, 2)$ gives $L(11, 8)$, a $(1, 11)$ surgery on $K(5, 2)$ gives $L(11, 4)$ which is homeomorphic to $L(11, 8)$, but a $(1, 11)$ surgery on $K(4, 3)$ gives $L(11, 9)$ which is not homeomorphic to $L(11, 8)$.

4. The nonfibered, nonprime manifolds.

PROPOSITION 4. *If an elementary surgery of type (p, q) is performed along $K(r, s)$ and $|\sigma| = |rsp + q| = 0$, then the manifold obtained is the connected sum of two lens spaces $L(r, s)\# L(s, r)$ and is not singularly fibered.*

Proof. If $|\sigma| = |rsp + q| = 0$, then $p = 1$, since p and q are relatively prime, $p > 0$, and $r > s > 0$. By Kneser's conjecture the manifold obtained is a connected sum since the fundamental group is a free product $\pi_1(\mathscr{M}(p, q)) \simeq (a, b \mid a^r = b^s,\ a^r = 1)$.

Let S^3 be the union of two solid tori T_1 and T_2, (M_1, L_1) a standard meridian-longitude pair for T_1, $(M_2, L_2) = (L_1, M_1)$ for T_2, K an (r, s) curve on T_1. Let $\mathscr{N}(K)$ be a regular neighborhood of the knot with meridian-longitude pair (M, L). We remove $\mathscr{N}(K)$ from S^3 forming a depression along K in each of T_1 and T_2 but leaving each a solid torus.

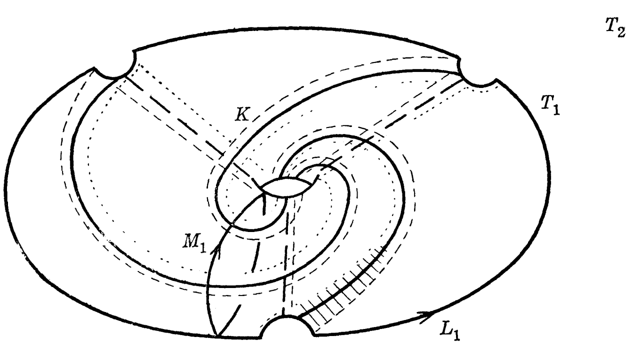

FIGURE 4.1

We sew back a solid torus sT^3 with meridian M' so that $M' \sim L\text{-}qM \sim K$. A meridian goes to one edge of the depression; another meridian goes to the other edge since they are parallel. Thus we may assume that the ∂sT^3 between two meridians is sewn to each half of the picture. Each half would be a lens space except that a 3-cell is missing—the 3-cell which is the other half of sT^3.

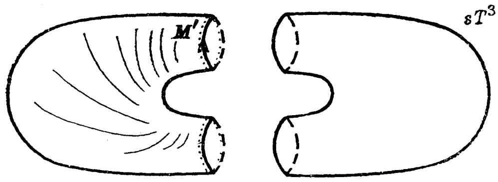

FIGURE 4.2

We now consider how the two halves of the picture are identified. The boundaries of T_1 and T_2 outside of the depression are identified, as are the meridianal disks of sT^3. The boundaries are annuli and the disks are sewn to them so as to make 3-spheres. Filling in these 3-spheres would give $L(r, s)$ and $L(s, r)$ since $M' \sim F \sim rM_1 + sL_1 \sim sM_2 + rL_2$. Hence the manifold obtained is $L(r, s) \# L(s, r)$.

5. Conjectures. A natural question to ask is whether Seifert manifolds can be obtained by elementary surgery along a knot other than a torus knot. We conjecture that the answer to this question is "no" in light of the following information:

1. If the fundamental group of a Seifert manifold is infinite, then the subgroup generated by the fiber is an in finite cyclic normal subgroup, the center of the group in case the characteristic is trivial [4].

2. All the known finite fundamental groups of closed 3-manifolds are groups of Seifert manifolds. All the possible finite fundamental groups have a nontrivial center. In case the order of the group is even, the unique element of order 2 lies in the center. In case the order of the group is odd, the group is cyclic and the center is the whole group [3].

3. Waldhausen has a partial converse to 1. If $\mathscr{M}^3$ is an irreducible 3-manifold such that $\pi_1(\mathscr{M}^3)$ has a nontrivial center and either $H_1(\mathscr{M}^3)$ is infinite or $\pi_1(\mathscr{M}^3)$ is a nontrivial free product with amalgamation, then $\mathscr{M}^3$ is a Seifert manifold [5].

4. Burde and Zieschang have shown that if the fundamental group of the complement of a knot has a nontrivial center, then the knot is a torus knot and the center is infinite cyclic [1].

Conjecture 1. If $\mathscr{M}^3$ is a Seifert manifold and $\mathscr{M}^3$ is obtained by elementary surgery along a knot K, then K is a torus knot.

Conjecture 2. If $\mathscr{M}^3$ is a lens space obtained by elementary surgery along a knot K, then K is a torus knot.

Conjecture 3. If $\mathscr{M}^3$ is obtained by elementary surgery along a knot K and $\pi_1(\mathscr{M}^3)$ is finite, then K is a torus knot.

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