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STARLIKE AND CONVEX MAPS IN BANACH SPACES

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Let X and Y be complex Banach spaces and let $B = \{x \in X: ||x|| < 1\}$. This paper concerns holomorphic maps $f: B \to Y$ which have local holomorphic inverses. That is, for each $x \in B$, there is a neighborhood $N \subset Y$ of $f(x)$ and a holomorphic function $g: N \to B$ such that $g(f(x)) = x$ and $f(g(y)) = y$ for all $y \in N$. Necessary and sufficient conditions are found which guarantee that such a map be one-to-one and map the unit ball B onto a domain which is convex or starlike with respect to 0.

1. Introduction. When $X = Y = C$ (the complex plane), it is well known that if f is holomorphic in $|z| < 1$ and satisfies $f(0) = 0$, $f'(0) \neq 0$ then f is univalent and maps the disk onto a domain which is starlike with respect to 0 if and only if $\operatorname{Re}[zf'(z)/f(z)] > 0$ when $|z| < 1$. Intuitively, this results from the fact that if we choose $z = re^{i\theta}$, $0 < r < 1$, $f(z) \neq 0$ and let $\phi(\theta)$ be a continuous branch of $\arg f(re^{i\theta})$ locally then $\partial\phi/\partial\theta = \operatorname{Re}[zf'(z)/f(z)]$. This idea does not extend readily to Banach spaces so we consider a different approach suggested by the following theorems due to M. S. Robertson [4].

THEOREM A. *Let $w(z, t) = \sum_1^\infty b_n(t)z^n$ be regular in $|z| < 1$ for $0 \leqq t \leqq 1$. Let $|w(z, t)| < 1$ for $|z| < 1, 0 \leqq t \leqq 1, w(z, 0) \equiv z$. Let ρ be a positive real number for which $w(z) = \lim_{t\to 0^+}(w(z, t) - z)/zt^\rho$ exists. Then $\operatorname{Re} w(z) \leqq 0$ for $|z| < 1$. If $w(z)$ is also analytic in $|z| < 1$ and $\operatorname{Re}(w(0)) \neq 0$, then $\operatorname{Re}(w(z)) < 0$ for $|z| < 1$.*

THEOREM B. *Let $f(z) = z + a_2z^2 + \cdots$ be regular and univalent in $|z| < 1$. For $0 \leqq t \leqq 1$ let $F(z, t)$ be regular in $|z| < 1$. Let $F(z, 0) \equiv f(z)$ and $F(0, t) \equiv 0$. Let ρ be a positive real number for which $F(z) = \lim_{t\to 0^+}(F(z, t) - F(z, 0))/zt^\rho$ exists. Let $F(z, t)$ be subordinate to $f(z)$ in $|z| < 1$ for $0 \leqq t \leqq 1$. Then $\operatorname{Re}(F(z)/f'(z)) \leqq 0, |z| < 1$. If in addition $F(z)$ is also analytic in $|z| < 1$ and $\operatorname{Re}(F(0)) \neq 0$, then $\operatorname{Re}(f'(z)/F(z)) < 0, |z| < 1$.*

As observed by Robertson, if f is holomorphic and univalent in $|z| < 1$, $f(0) = 0$, and $f(|z| < 1)$ is starlike with respect to 0 then $(1 - t)f(z)$ is subordinate to $f(z)$ (i.e., $(1 - t)f(|z| < 1) \subset f(|z| < 1)$) for each t, $0 \leqq t \leqq 1$. We may set $F(z, t) = (1 - t)f(z)$ in Theorem B and obtain $F(z) = -f(z)/z$ when $\rho = 1$. Thus we obtain the necessary condition $\operatorname{Re}[zf'(z)/f(z)] > 0$ by using the concept of subordination.

Now assume f is holomorphic in $|z| < 1$, $f(0) = 0$, $f'(z) \neq 0$ in $|z| < 1$ and $w(z) = f(z)/zf'(z)$ where $\operatorname{Re}(w(z)) > 0$ (equivalently $\operatorname{Re}[zf'(z)/f(z)] > 0$). Expanding $f^{-1}((1-t)f(z))$ in powers of t about $t = 0$, we have

$$f^{-1}((1 - tf(z)) = z - tf(z)/f'(z) + o(t) = z(1 - tw(z)) + o(t)$$

so $|f^{-1}((1-t)f(z))| < |z|$ when $0 < t < t_0$ for some $t_0 > 0$. As we shall see later, this will imply f is one-to-one and $f(|z| < 1)$ is starlike with respect to 0. This is the approach which is needed for extension of these ideas to Banach spaces.

It is well known that the image of the unit disk under a holomorphic map f satisfying $f'(0) \neq 0$ is convex if and only if the function $zf'(z)$ has an image which is starlike with respect to 0. The generalization of this result to Banach spaces is false. In fact, we will show that if $X = \ell^1$ then $f\colon B \to Y$ is a biholomorphic map of B onto a convex domain if and only if f is a bounded linear map having a bounded inverse (Corollary 1). We will also show (Corollary 2) that if $X = \ell^\infty$ and $f\colon B \to Y$ is a biholomorphic map of B onto a convex domain then $f = L \circ g$ where $g\colon B \to X$ is given by $g = (g_1, g_2, \cdots)$, $g_k(x) = g_k(x_k) = x_k + a_{2k}x_k^2 + \cdots$, $x = (x_1, x_2, \cdots)$ and L is a bounded linear map having a bounded inverse (i.e., f is a linear map composed with a function whose coordinates are functions of one variable only).

See [6] for extension of Robertson's Theorems A and B to C^n and see [3] and [6] for some results concerning starlikeness and convexity in C^n.

2. Starlike maps in Banach spaces. Let $B_r = \{x \in X\colon \|x\| < r\}$ and $B = B_1$. For $0 \neq x \in X$, let $T(x)$ be the collection of all continuous real linear functionals x' on X (regarded as a real linear space) satisfying $x'(x) = \|x\|$ and $x'(y) \leqq \|y\|$ for all $y \in X$. By the Hahn-Banach theorem, $T(x)$ is nonempty. Also, if $x' \in T(x)$ then $\{y \in X\colon x'(y) = \|x\|\}$ is a supporting hyperplane for the convex set $B_{\|x\|}$. Let $\mathscr{P}_0(B)$ be the class of mappings $w\colon B \to X$ which are holomorphic and satisfy $w(0) = 0$ and $x'(w(x)) \geqq 0$ when $0 \neq x \in B$ and $x' \in T(x)$. Further, let $\mathscr{P}(B)$ be the class of $w \in \mathscr{P}_0(B)$ which satisfy $x'(w(x)) > 0$ when $0 \neq x \in B$ and $x' \in T(x)$.

EXAMPLES. If $X = C$ and $z_0 \neq 0$ then $T(z_0)$ contains only the functional x' given by $x'(z) = \operatorname{Re}[|z_0|z/z_0]$ and $\mathscr{P}(B)$ consists of the class of w such that $w(z)/z$ is holomorphic in $|z| < 1$ and $\operatorname{Re}[w(z)/z] > 0$.

If $X = C^n$ with sup norm, and $0 \neq x \in X$ then $T(x)$ consists of those functionals x' given by $x'(y) = \sum_{|x_k| = \|x\|} t_k \|x\| \operatorname{Re}(y_k/x_k)$ where $t_k \geqq 0$ for each k and $\sum_{|x_k| = \|x\|} t_k = 1$. In this case, $\mathscr{P}(B)$ is the class of $w: B \to X$ which are holomorphic and satisfy $w(0) = 0$ and $\operatorname{Re}[w_j(x)/x_j] > 0$ when $\|x\| = |x_j| > 0$. We remark that $\mathscr{P}_0(B)$ in this case is the class $\mathscr{P}$ defined in [6].

If $X = \ell^p, 1 < p < \infty$, and $0 \neq x \in X$ then $T(x)$ contains only the functional x' given by $x'(y) = \operatorname{Re}(\sum |x_j|^p y_j/x_j)/\|x\|^{p-1}$ and $\mathscr{P}(B)$ consists of those holomorphic $w: B \to \ell^p$ satisfying $w(0) = 0$ and $\operatorname{Re}(\sum |x_j|^p w_j(x)/x_j) > 0$.

If $X = \ell^1$ and $0 \neq x \in X$ then $T(x)$ consists of those functionals x' given by

$$x'(y) = \operatorname{Re}\Big(\sum_{x_j \neq 0} |x_j| y_j/x_j + \sum_{x_j = 0} \alpha_j y_j\Big)$$

where α_j satisfy $|\alpha_j| \leqq 1$. In this case, $\mathscr{P}(B)$ consists of those holomorphic $w: B \to \ell^1$ satisfying $w(0) = 0$ and $\operatorname{Re}[\sum_{x_j \neq 0} |x_j| w_j(x)/x_j - \sum_{x_j = 0} |w_j(x)|] > 0$. In the finite dimensional case using the p norm, $1 \leqq p < \infty$ the class $\mathscr{P}_0(B)$ is the class $\mathscr{P}_p$ defined in [6].

The following lemmas generalize Robertson's Theorems A and B to Banach spaces. They also include Lemmas 1–4 of [6].

LEMMA 1. *Let $v(x, t): B \times I \to B$ be holomorphic for each $t \in I = [0, 1]$, $v(0, t) = 0$ and $v(x, 0) = x$. If $\lim_{t \to 0^+} [(x - v(x, t))/t] = w(x)$ exists and is holomorphic in B then $w \in \mathscr{P}_0(B)$.*

Proof. Let $0 \neq x \in B$ and $x' \in T(x)$. By Schwarz lemma, $\|v(x, t)\| \leqq \|x\|$ so $x'[(x - v(x, t))/t] = (\|x\| - x'(v(x, t)))/t \geqq (\|x\| - \|v(x, t)\|)/t \geqq 0$ and the desired result follows by continuity of x'.

The following example shows that there are nontrivial cases in which the limit function w of Lemma 1 is not in the class $\mathscr{P}(B)$.

EXAMPLE. Let $X = C^2$ with $\|x\|^2 = |x_1|^2 + |x_2|^2$. For $0 \leqq t \leqq 1$, let $v(x, t): B \to B$ be the restriction of the linear map having matrix

$$\begin{pmatrix} \sqrt{1 - t^2} & t \\ -t & \sqrt{1 - t^2} \end{pmatrix}.$$

Then the limit function w of Lemma 1 is given by $w(x) = (-x_2, x_1)$. If $x' \in T(x)$ then $x'(y) = \operatorname{Re}(\bar{x}_1 y_1 + \bar{x}_2 y_2)/\|x\|$ so $x'(w(x)) = \operatorname{Re}(-\bar{x}_1 x_2 + x_2 \bar{x}_1)/\|x\| \equiv 0$.

LEMMA 2. *Let $f: B \to Y$ be a biholomorphic map (holomorphic with a holomorphic inverse) of B onto an open set $f(B) \subset Y$ and let $f(0) = 0$. Let $F(x, t): B \times I \to Y$ be a holomorphic function of x for each $t \in I$, $F(x, 0) = f(x)$, $F(0, t) = 0$ and suppose $F(B, t) \subset f(B)$ for each $t \in I$. Further, suppose $\lim_{t \to 0^+}[(F(x, 0) - F(x, t))/t] = F(x)$ exists and is holomorphic. Then $F(x) = Df(x)(w(x))$ where $w \in \mathscr{P}_0(B)$ ($Df(x)$ is the Fréchet derivative of f at x).*

Proof. Since $F(B, t) \subset f(B)$ for each $t \in I$, there exists $v: B \times I \to B$ $(v(x, t) = f^{-1}(F(x, t)))$ holomorphic for each $t \in I$ such that $f(v(x, t)) = F(x, t)$ where $\|v(x, t)\| \leqq \|x\|$. Fix x, $\|x\| > 0$. Then

$$f(v(x, t)) = f(x) + Df(x)(v(x, t) - x) + R(v(x, t), x)$$

where $\|R(y, x)\|/\|y - x\| \longrightarrow 0$ as $\|y - x\| \longrightarrow 0$. Therefore

$$\frac{F(x, 0) - F(x, t)}{t} = Df(x)\left(\frac{x - v(x, t)}{t} - \frac{R(v(x, t), x)}{t}\right).$$

We wish to conclude $R(v(x, t), x)/t \to 0$ as $t \to 0^+$ so that $\lim_{t\to 0^+}((x - v(x, t))/t)$ exists and an application of Lemma 1 completes the proof. We first show that $\|x - v(x, t)\|/t$ is bounded as $t \to 0^+$. Suppose for some sequence $\{t_n\}$, $t_n \to 0^+$ and $\|x - v(x, t_n)\|/t_n \to \infty$. We have

$$F(x) = \lim_{n\to\infty}\left\{\left[Df(x)\left(\frac{x - v(x, t_n)}{\|x - v(x, t_n)\|}\right) - \frac{R(v(x, t_n), x)}{\|x - v(x, t_n)\|}\right]\frac{\|x - v(x, t_n)\|}{t_n}\right\}$$

and since $R(v(x, t_n), x)/\|x - v(x, t_n)\| \to 0$ we must also have $Df(x)((x - v(x, t_n))/\|x - v(x, t_n)\|) \to 0$. This contradicts the assumption f^{-1} is holomorphic so such a sequence cannot exist, i.e., $\|x - v(x, t)\|/t$ is bounded as $t \to 0^+$. The desired conclusion now clearly follows.

If $X = C$ and $w \in \mathscr{P}_0(B)$ then $\operatorname{Re}(w(z)/z) \geqq 0$ when $|z| < 1$. It is then easy to see that for $|\alpha| < 1$, $w(\alpha z)/\alpha \in \mathscr{P}_0(B)$ and in fact $w \in \mathscr{P}(B)$ unless $\operatorname{Re} w'(0) = 0$ and in this case w is constant. This result takes the following form in normed linear spaces.

LEMMA 3. *If $w \in \mathscr{P}_0(B)$ and $|\alpha| < 1$ then*

$$\frac{1}{\alpha}w(\alpha x) \in \mathscr{P}_0(B) \ \left(\frac{1}{\alpha}w(\alpha x) \textit{ is understood to be the limit value } Dw(0)(x) \textit{ when } \alpha = 0\right).$$

Further, if $x' \in T(x)$, $0 < \|x\| < 1$ then $x'(w(x)) = 0$ if and only if $x'(Dw(0)(x)) = 0$ and in this case $x'(1/\alpha \cdot w(\alpha x)) \equiv 0$ when $|\alpha| < 1/\|x\|$.

Proof. For $0 < |\alpha| < 1$ and $x' \in T(x)$, define x'_α by $x'_\alpha(y) = x'(|\alpha|y/\alpha)$ for all $y \in X$. Then $x'_\alpha(\alpha x) = \|\alpha x\|$ and $x'_\alpha(y) \leqq \|y\|$ for all $y \in X$ so $x'_\alpha \in T(\alpha x)$. Therefore,

$$0 \leqq \frac{1}{|\alpha|}x'_\alpha(w(\alpha x)) = x'(w(\alpha x)/\alpha)$$

and by continuity of x', $x'(Dw(0)(x)) \geqq 0$. Thus we have shown $w(\alpha x)/\alpha \in \mathscr{P}_0(B)$ when $|\alpha| < 1$.

Also, $x'(y) = \operatorname{Re}[x'(y) - ix'(iy)]$ is the real part of a continuous complex linear functional so $x'(w(\alpha x)/\alpha)$ is a nonnegative harmonic function of α for fixed x, $|\alpha| < 1/||x||$. Therefore $x'(w(\alpha x)/\alpha) > 0$ or $x'(w(\alpha x)/\alpha) \equiv 0$ (x fixed). This completes the proof.

DEFINITION 1. A holomorphic map $f: B \to Y$ is starlike if f is one-to-one, $f(0) = 0$ and $(1 - t)f(B) \subset f(B)$ for all $t \in I$.

THEOREM 1. *Suppose $f: B \to Y$ is starlike and that f^{-1} is holomorphic on $f(B)$ open $\subset Y$. There exists $w \in \mathscr{P}(B)$ such that*

$$f(x) = Df(x)(w(x)) \; . \tag{1}$$

Proof. Apply Lemma 2 with $F(x, t) = (1 - t)f(x)$ to obtain (1) with $w \in \mathscr{P}_0(B)$. Since

$$\frac{1}{\alpha}f(\alpha x) = Df(\alpha x)\Big(\frac{1}{\alpha}w(\alpha x)\Big)$$

for $|\alpha| < 1$, letting $\alpha \to 0$ we have

$$Df(0)(x) = Df(0)(Dw(0)(x))$$

so $Dw(0)(x) = x$ since f^{-1} holomorphic implies $Df(0)$ is invertible. Using Lemma 3 we conclude $w \in \mathscr{P}(B)$.

We have the following "converse" to Theorem 1.

THEOREM 2. *Let $f: B \to Y$ be holomorphic and $f(0) = 0$. Assume $Df(x)$ has a bounded inverse for each $x \in B$ (hence f^{-1} exists and is holomorphic in a neighborhood of each point of $f(B)$) and that for some $w \in \mathscr{P}(B)$, $f(x) = Df(x)(w(x))$. Suppose further that for each r, $0 < r < 1$, there exists $M(r)$ such that $||[Df(x)]^{-1}|| \leqq M(r)$ when $||x|| \leqq r$. Then f is starlike.*

Proof. We first observe that $f(x) \neq 0$ if $x \neq 0$ for $f(x) = 0$ implies $Df(x)(w(x)) = 0$ so $w(x) = 0$ since $Df(x)$ is invertible. But $x' \in T(x)$ implies $x'(w(x)) > 0$ so $w(x) \neq 0$ when $x \neq 0$.

For $0 \neq x \in B$, let N_x be a neighborhood of $f(x)$ in which f^{-1} exists as a holomorphic function. Let $v(x, t) = f^{-1}((1 - t)f(x))$ for $-t_0 < t < t_1$ where t_0 and t_1 are positive such that $(1 - t)f(x) \in N_x$ when $-t_0 < t < t_1$. We wish to show $||v(x, t)||$ is strictly decreasing as a function of t. Note that

$$\begin{aligned} v(x, t) &= v(x, 0) + [Df(x)]^{-1}(-tf(x)) + o(t) \\ &= x - tw(x) + o(t) \end{aligned}$$

so for $x' \in T(x)$ we have

$$\begin{aligned}||v(x, t)|| &\geqq x'(v(x, t)) = x'(x) - tx'(w(x)) + o(t) \\ &= ||x|| - tx'(w(x)) + o(t) > ||x||\end{aligned}$$

when t is negative, $|t|$ sufficiently small. But we may apply this result at $y = v(x, t)$ (so $v(y, \tau) = f^{-1}((1 - \tau)(1 - t)f(x)))$ to conclude that $||v(x, t)||$ is strictly decreasing in $-t_0 < t < t_1$.

For $0 \leqq t \leqq 1$, let $A_t = \{(1 - \tau)f(x)\colon 0 \leqq \tau \leqq t\}$ and let $T = \{t \in [0, 1]$: there exists a neighborhood N_t of A_t such that f^{-1} exists as a holomorphic function on N_t such that $f^{-1}(f(x)) = x\}$. It is easy to see that T is open and nonempty. We wish to show that T is closed so $T = [0, 1]$. Let $0 < t_2 \leqq 1$ where $0 \leqq t < t_2$ implies $t \in T$. We wish to show $t_2 \in T$. Define

$$L_\tau = \int_0^\tau ||[Df(f^{-1}((1 - t)f(x)))]^{-1}(-f(x))||\, dt\,. \tag{2}$$

Computing the derivative of $f^{-1}((1 - t)f(x))$ with respect to t, we see that the integral (2) gives the length of the arc $f^{-1}\{(1 - t)f(x)\colon 0 \leqq t \leqq \tau\}$. Note also that for $0 \leqq \tau < t_0$, $||f^{-1}((1 - t)f(x))|| \leqq ||x||$ in the integrand since $||f^{-1}((1 - t)f(x))||$ decreases with t. Hence by hypothesis, the integrand is continuous and bounded by $M(||x||)\,||f(x)||$ so the integral exists for $0 \leqq \tau \leqq t_2$. Further, if $0 \leqq \tau_1 < \tau_2 < t_2$ we have

$$\begin{aligned}L_{\tau_2} - L_{\tau_1} &= \int_{\tau_1}^{\tau_2} ||[Df(f^{-1}((1 - t)f(x)))]^{-1}(-f(x))||\, dt \\ &\geqq ||f^{-1}((1 - \tau_2)f(x)) - f^{-1}((1 - \tau_1)f(x))||\,.\end{aligned}$$

Let $\{\tau_k\}_{k=1}^\infty$ be an increasing sequence of values of τ such that $L_{\tau_k} = (1 - 2^{-k})L_{t_2}$. Then

$$\begin{aligned}L_{t_2}|2^{-m} - 2^{-n}| &= |L_{\tau_m} - L_{\tau_n}| \\ &\geqq ||f^{-1}((1 - \tau_m)f(x)) - f^{-1}((1 - \tau_n)f(x))||\end{aligned}$$

so $\{f^{-1}((1 - \tau_k)f(x))\}_{k=1}^\infty$ is a Cauchy sequence which converges to some value $y \in X$ such that $||y|| \leqq ||x||$. Clearly $f(y) = (1 - t_2)f(x)$. Let G be a neighborhood of $f(y)$ such that f^{-1} exists uniquely in G as a holomorphic function and $f^{-1}(f(y)) = y$. Since some of the values $(1 - t)f(x)$ lie in G for $t < t_2$, the values of f^{-1} in G and in N_t must agree for these values of t. Therefore, $(\bigcup_{0 \leqq t < t_2} N_t) \cup G$ is a neighborhood of A_{t_2} on which f has a unique holomorphic inverse. Thus we have proved $t_2 \in T$ and T is closed.

Now suppose $f(x) = f(y)$, $x \neq y$. Then there exist inverses g_1 and g_2 of f holomorphic in a connected neighborhood N of $A = \{(1 - t)f(x)\colon 0 \leqq t \leqq 1\}$ such that $g_1(f(x)) = x$ and $g_2(f(x)) = y$. Clearly, the set of points on which g_1 and g_2 agree is both open and closed (relative to N) so either $g_1 \equiv g_2$ or $g_1(u)$ is never equal to $g_2(u)$, $u \in N$.

But $0 \in N$ and $g_1(0) = g_2(0) = 0$ while also $f(x) \in N$ and $g_1(f(x)) = x \neq y = g_2(f(x))$. This contradiction completes the proof of the theorem.

REMARK. We do not know whether Theorem 2 remains true without assuming X is complete or without the boundedness condition on $[Df(x)]^{-1}$. The usefulness of this boundedness condition comes from the fact that for $r < 1$ this condition implies that the boundary of $f(B_r)$ (as a subset of Y rather than $f(B)$) is $f(||x|| = r)$.

EXAMPLE. Let $X = Y = \ell^\infty$ and define $f: B \to Y$ by $f(x) = (f_1, f_2, \cdots)$ where $f_1(x) = x_1$ and $f_j(x) = x_j(1 - x_1)$ if $j \geqq 2$. Then

$$[Df(x)]^{-1}(f(x)) = w(x) = (w_1, w_2, \cdots)$$

where $w_1 = x_1$ and $w_j = x_j/(1 - x_1)$ if $j \geqq 2$. Suppose $0 \neq x \in B$ and $x' \in T(x)$. If $|x_1| < ||x||$ then $x'(1, 0, 0, \cdots) = 0 = x'(i, 0, 0, \cdots)$ for otherwise we may choose α so that $||x|| = ||x + (\alpha, 0, 0, \cdots)||$ and $||x|| < x'(x) + x'(\alpha, 0, 0, \cdots) = x'(x + (\alpha, 0, 0, \cdots))$.

If $||x|| = |x_1|$, then $x'(x_1, 0, 0, \cdots) \geqq 0$ otherwise, choose $\alpha = -t\, x_1$ where t is small and positive so that $||x + (\alpha, 0, 0, \cdots)|| \leqq ||x||$ and $x'(x + (\alpha, 0, 0, \cdots)) > x'(x) = ||x||$.

Also, if $||x|| = |x_1|$ then $x'(ix_1, 0, 0, \cdots) = 0$ since $\sqrt{1 + t^2}\,||x|| \geqq x'((1 \pm it)x_1, x_2, \cdots) = x'(x) \pm tx'(ix_1, 0, 0, \cdots)$ so

$$\sqrt{\frac{1 + t^2 - 1}{|t|}}||x|| \geqq \pm\, x'(ix_1, 0, 0, \cdots)\,.$$

Similarly, $x'(i(0, x_2, x_3, \cdots)) = 0$ and $x'(0, x_2, x_3, x_4, \cdots) \geqq 0$. Therefore

$$\begin{aligned} x'(w(x)) &= x'(x_1, 0, 0, \cdots) + x'\Big(\frac{1}{1 - x_1}(0, x_2, x_3, \cdots)\Big) \\ &= x'(x_1, 0, 0, \cdots) + \operatorname{Re}\Big(\frac{1}{1 - x_1}x'(0, x_2, x_3, \cdots)\Big) \\ &> 0 \end{aligned}$$

so f is starlike. Note that as x tends to a boundary point of the form $(1, x_2, x_3, \cdots)$, $f(x)$ tends to the point $(1, 0, 0, 0, \cdots)$.

3. Convex maps of the unit ball.

We wish to obtain necessary and sufficient conditions that $f: B \to Y$ be a biholomorphic map of B onto a convex domain in Y. We begin with the following lemma.

LEMMA 4. *If $f: B \to Y$ is a biholomorphic map of B onto a convex domain then $f(B_r)$ is convex for each r, $0 \leqq r \leqq 1$.*

Proof. Suppose $x_0, y \in B$, $||y|| \leqq ||x_0||$ and $x_0 \neq 0$. Then $tf(x_0) + (1-t)f(y) \in f(B)$ when $0 \leqq t \leqq 1$ and we wish to show $tf(x_0) + (1-t)f(y) \in f(B_r)$ when $||x_0|| < r$. Let ℓ be a (complex) continuous linear functional satisfying $\ell(x_0) = 1$, $||\ell|| \leqq 1/||x_0||$. Let $tf(x) + (1-t)f(\ell(x)y) = f(v(x, t))$. Then $v(0, t) = 0$ and $||v(x, t)|| < 1$ so $||v(x, t)|| \leqq ||x||$. Hence $f(v(x_0, t)) = tf(x_0) + (1-t)f(y)$ where $||v(x_0, t)|| \leqq ||x_0||$ and the lemma is proved.

DEFINITION 2. If $f: B \to Y$ is a biholomorphic map of B onto a convex domain, we say that f is convex.

If $X = C$, it is well known than a function f holomorphic in $|z| < 1$ satisfying $f'(0) \neq 0$ is convex if and only if $\mathrm{Re}\,[zf''(z)/f'(z) + 1] > 0$ in $|z| < 1$. The analogous condition for a general Banach space is necessary but not sufficient.

THEOREM 3. *If $f: B \to Y$ is convex then*

$$D^2f(x)(x, x) + Df(x)(x) = Df(x)(w(x)) \tag{3}$$

where $w \in \mathscr{P}(B)$.

Proof. Let $F(x, t) = 1/2(f(e^{i\sqrt{t}}x) + f(e^{-i\sqrt{t}}x))$ for $t \in I$. Then $F(x, t)$ satisfies the hypotheses of Lemma 2 with

$$\begin{aligned} F(x) &= \lim_{t \to 0^+} \frac{f(x) - \dfrac{1}{2}(f(e^{i\sqrt{t}}x) + f(e^{-i\sqrt{t}}x))}{t} \\ &= \lim_{t \to 0^+} -\frac{1}{2t}(Df(x)((2\cos\sqrt{t} - 2)x) \\ &\quad + D^2f(x)(i\sin\sqrt{t}\,x, i\sin\sqrt{t}\,x) + o(t)) \\ &= \frac{1}{2}(Df(x)(x) + D^2f(x)(x, x)) \,. \end{aligned}$$

Hence (3) follows with $w \in \mathscr{P}_0(B)$. Again, $Dw(o)(x) = x$ and by Lemma 3, $w \in \mathscr{P}(B)$.

The example $X = C^2$ with sup norm and $f(z_1, z_2) = (z_1 + z_2^2/2, z_2)$ shows the condition is not sufficient (f is not convex by Corollary 2 below).

Let $Q_0(B)$ be the class of functions $w: B \times B \to X$ which are holomorphic in each variable and which satisfy $w(x, x) = 0$ and $x'(w(x, y)) \geqq 0$ when $x' \in T(x)$ and $||y|| \leqq ||x||$. Let $Q(B)$ be the collection of all $w \in Q_0(B)$ which satisfy $x'(w(x, y)) > 0$ when $x' \in T(x)$ and $||y|| < ||x||$.

LEMMA 5. *If $w \in Q_0(B)$ and $|\alpha| < 1$ then $1/\alpha \cdot w(\alpha x, \alpha y) \in Q_0(B)$ (the* *limit value at* $\alpha = 0$ *is* $Dw(0, 0)(x, y))$. *Further, if* $x' \in T(x)$, $0 \neq x \in B$, *and* $\|y\| \leqq \|x\|$ *then* $x'(w(x, y)) = 0$ *if and only if* $x'(Dw(0, 0)(x, y)) = 0$.

Proof. See the proof of Lemma 3.

THEOREM 4. *Suppose* $f: B \to Y$ *is convex. Then* $f(x) - f(y) = Df(x)(w(x, y))$ *where* $w \in Q(B)$.

Proof. Since $(1 - t)f(x) + tf(y) \in f(B)$ for all $x, y \in B$ when $0 \leqq t \leqq 1$, there exists $v(x, y, t): B \times B \times I \to B$ such that $f(v(x, y, t)) = (1 - t)f(x) + tf(y)$. By Lemma 4, $\|v(x, y, t)\| \leqq \max(\|x\|, \|y\|)$. Also,

$$f(v(x, y, t)) = f(x) + Df(x)(v(x, y, t) - x) + R(v(x, y, t), x)$$

so

$$\begin{aligned} f(x) - f(y) &= \lim_{t\to 0^+} \frac{f(x) - f(v(x, y, t))}{t} \\ &= \lim_{t\to 0^+} \left(Df(x)\left(\frac{x - v(x, y, t)}{t}\right) - \frac{1}{t} g(v(x, y, t), x)\right) \\ &= Df(x)(w(x, y)). \end{aligned}$$

Since $w(x, y) = \lim_{t\to 0^+} (x - v(x, y, t))/t$, it is clear that $w \in Q_0(B)$. Since $Dw(0, 0)(x, y) = x - y$, Lemma 5 implies that $w \in Q(B)$.

Again assuming a boundedness condition on $[Df(x)]^{-1}$ we can prove the converse.

THEOREM 5. *Suppose* $f: B \to Y$ *is holomorphic,* $Df(x)$ *has bounded inverse for each* $x \in B$ *and that for some* $w \in Q(B)$, $f(x) - f(y) = Df(x)(w(x, y))$. *Suppose further that for each* r, $0 < r < 1$, *there exists* $M(r) > 0$ *such that* $\|[Df(x)]^{-1}\| \leqq M(r)$ *when* $\|x\| \leqq r$. *Then* f *is convex.*

Proof. Since $w(x, 0) \in \mathscr{P}(B)$, $f(x) - f(0)$ is starlike. Let $v(x, y, t) = f^{-1}((1 - t)f(x) + tf(y))$ when $x, y \in B$ and $t \in [0, t_0]$ where t_0 is such that $(1 - t)f(x) + tf(y) \in f(B)$ when $0 \leqq t \leqq t_0$. Proceeding as in the proof of Theorem 2, we have

$$v(x, y, t) = x - tw(x, y) + o(t)$$

so $\|v(x, y, t)\|$ is decreasing (as a function of t) when $\|v(x, y, t)\| > \|y\|$. By methods similar to those used in the proof of Theorem 2, we conclude that the set of allowable values of $t_0 \in [0, 1]$ is nonempty, open and closed. Hence we may choose $t_0 = 1$ and it follows that $f(B)$ is convex.

THEOREM 6. *Let $f\colon B \to Y$ be convex, $0 \neq x \in B$ and let $x' \in T(x)$. Then the hyperplane $\{y \in Y\colon x'[(Df(x))^{-1}(y)] = x'[(Df(x))^{-1}(f(x))]\}$ is a supporting hyperplane for the convex set $f(B_{||x||})$.* If $y \neq 0$, $x'(y) = 0$ *and $||x + ty|| = ||x||$ for $0 < t < t_0$, then $x'[(Df(x))^{-1}(f(x + ty))] = x'[(Df(x))^{-1}(f(x))]$ for $0 < t < t_0$* (i.e., *$f(x + ty)$ lies in the supporting hyperplane described above).*

Proof. By Theorem 4, $||y|| < ||x||$ implies

$$0 < x'(w(x, y)) = x'[(Df(x))^{-1}(f(x) - f(y))]$$

so

$$x'[(Df(x))^{-1}(f(x))] > x'[(Df(x))^{-1}(f(y))] .$$

That is, all points in $f(B_{||x||})$ lie on the same side of the hyperplane and the first part of the theorem is proved.

Expanding $f(x + ty)$ in a power series about x we have

$$f(x + ty) = f(x) + tDf(x)(y) + \frac{1}{2}t^2D^2f(x)(y, y) + o(t^2)$$

so

$$\begin{aligned} w(x, x + ty) &= (Df(x))^{-1}(f(x) - f(x + ty)) \\ &= -ty - \frac{1}{2}t^2(Df(x))^{-1}(D^2f(x)(y, y)) + o(t^2) \end{aligned}$$

and

$$x'(w(x, x + ty)) = -\frac{1}{2}t^2x'(Df(x))^{-1}(D^2f(x)(y, y)) + o(t^2) .$$

By Lemma 5 and Theorem 4, we conclude $x'\{\lambda(Df(\lambda x))^{-1}(D^2f(\lambda x)(y, y))\} \leqq 0$ when $|\lambda| \leqq 1/||x||$ with equality when $\lambda = 0$. Since the function under consideration is a harmonic function of λ we conclude $x'((Df(x))^{-1}(D^2f(x)(y, y))) = 0$. Using induction on the higher order terms in the series we obtain the desired result, $x'(w(x, x + ty)) = 0$.

Note that if x is not an extreme point of the closed ball of radius $||x||$, $x \neq 0$, then y and x' as given in the theorem do exist.

We now apply Theorem 6 to two particular spaces which show that the requirement that a holomorphic map of the unit ball be convex is very restrictive at least in some spaces. We asume that (S, R, μ) is a measure space with μ a positive measure defined on the σ-ring R of subsets of S. We assume S has two disjoint subsets of finite positive measure.

THEOREM 7. *Let $X = L(\mu)$ (the space of complex valued integrable* *functions on S). If $f\colon B \to Y$ is convex then $f(x) - f(0)$ is linear.*

Proof. We may assume $Y = X$, $f(0) = 0$ and $Df(0) = I$ (otherwise, replace f by $[Df(0)]^{-1}(f(x) - f(0)))$. Since $f(x) = x + 1/2 \cdot D^2f(0)(x, x) + \cdots$, it is sufficient to show $D^nf(0) = 0$, $n = 2, 3, \cdots$. The proof is by induction. Let $x = \alpha x_E + \beta x_F$ where E and F are disjoint sets of finite positive measure, x_E and x_F are characteristic functions for E and F respectively, $|\alpha|\mu(E) + |\beta|\mu(F) < 1$ (so $x \in B$) and $\alpha\beta \neq 0$. Let x' be the real continuous linear functional on X given by

$$x'(u) = \mathrm{Re}\Big[|\alpha|/\alpha\int_E u\,d\mu + |\beta|/\beta\int_F u\,d\mu\Big]$$

and let

$$y = -\alpha/(|\alpha|\mu(E))(1 - ir)x_E + \beta/(|\beta|\mu(F))(1 + is)x_F$$

where r and s are real.

Then $x'(y) = 0$ and

$$\begin{aligned}||x + ty|| &= |\alpha|\,(\mu(E) - t/|\alpha|)\sqrt{1 + (rt/(|\alpha|\mu(E) - t))^2} \\ &\quad + |\beta|\,(\mu(F) + t/|\beta|\sqrt{1 + (st/(|\beta|\mu(F) + t))^2} \\ &= ||x|| + |\alpha|\mu(E)r^2t^2/(2(|\alpha|\mu(E) - t)^2) \\ &\quad + |\beta|\mu(F)s^2t^2/(2(|\beta|\mu(F) + t)^2) + o(t^2) \geqq ||x||\end{aligned}$$

if $|t|$ is sufficiently small.

Therefore, using Theorem 4 and the definition of Q we conclude $x'(w(x + ty, x)) \geqq 0$ if $|t|$ is sufficiently small. Since $x'(y) = 0$, it follows as in the proof of Theorem 6 that $x'(Dt(x))^{-1}(D^2f(x)(y, y)) = 0$.

Letting $L_x(\cdot, \cdot) = (Df(x))^{-1}(D^2f(x)(\cdot, \cdot))$ we have

$$\begin{aligned}\mathrm{Re}\Big[\int_E &(\alpha(1 - ir)^2/(|\alpha|\mu^2(E))L_x(x_E, x_E) \\ &- 2\beta(1 - ir)(1 + is)/(|\beta|\mu(E)\mu(F))\cdot \\ &L_x(x_E, x_F) + |\alpha|\beta^2(1 + is)^2/(\alpha|\beta|^2\mu^2(F))L_x(x_F, x_F))d\mu \\ &+ \int_F (\alpha^2|\beta|(1 - ir)^2/(|\alpha|^2\beta\mu^2(E))L_x(x_E, x_E) \\ &- 2\alpha(1 - ir)(1 + is)/(|\alpha|\mu(E)\mu(F))\cdot \\ &L_x(x_E, x_F) + \beta(1 + is)^2/(|\beta|\mu^2(F))L_x(x_F, x_F))d\mu\Big] = 0\,.\end{aligned}$$

In (4), if we replace α and β by $\lambda\alpha$ and $\lambda\beta$ and multiply through by $|\lambda|$ we obtain an analytic function of λ for $|\lambda| < R$ where $R > 1$ which has 0 real part and is 0 at $\lambda = 0$. Therefore the quantity in brackets is 0. Since α, β, r, and s are variable, we conclude that each of the six terms above is 0. Letting α and β tend to 0, we conclude that if A and B are disjoint measurable sets of finite measure then

$$\int_A D^2f(0)(x_A, x_A)d\mu = \int_B D^2f(0)(x_A, x_A)d\mu = \int_A D^2f(0)(x_A, x_B)d\mu = 0 .$$

If A and $E \cup F$ are disjoint and A has finite positive measure, we may replace the functional x' in the argument above by x_1' where

$$x_1'(u) = x(u) + \mathrm{Re}\int_A \gamma u d\mu$$

where $|\gamma| \leqq 1, \gamma$ complex. This leads to the conclusion, $\int_A D^2f(0)(x_E, x_F)d\mu = 0$.

Hence we may now conclude that for simple functions y we have $D^2f(0)(y, y) = 0$. This implies $D^2f(0)(y, y) = 0$ for all $y \in L(\mu)$. Similarly, $D^kf(0)(y, y, \cdots, y) = 0$ for $k = 3, 4, \cdots$.

The theorem now follows from the power series for f about 0.

Choosing S to be the positive integers and μ counting measure we obtain the following corollary.

COROLLARY 1. *If $X = \ell^1$ and $f: B \rightarrow Y$ is convex then $f(x) - f(0)$ is linear.*

THEOREM 8. *Let $X = L^\infty(\mu)$ (the space of essentially bounded, complex valued, measurable functions on S) and suppose $f: B \rightarrow Y$ is convex. If $x, y \in B$ where $x(s) = y(s)$ for all $s \in E \subset S$ and $\mu(E) > 0$ then $(Df(0))^{-1}(f(x))(s) = (Df(0))^{-1}f(y)(s)$ for almost all $s \in E$.*

Proof. We may assume $Y = X$, $f(0) = 0$ and $Df(0) = I$ (otherwise, replace f by $(Df(0))^{-1}(f(x) - f(0)))$. We will show that if $\mu(E) > 0$, then for almost all $s \in E$ we have $f(x)(s) = f(xx_E)(s)$ (where x_E is the characteristic function of E). We accomplish this by showing, $(Df(x))^{-1}D^nf(x)(y, \cdots, y)(s) = 0$ a.e. on E when $y(s) = 0$ on E $(n \geqq 2)$ and $D^2f(x)(x_E, y) = 0$ when $y = 0$ on E. An induction argument will then show $D^nf(x)(x_E, y, y, \cdots, y) = \lim z \rightarrow 0\,[D^{n-1}f(x + zy)(x_E, y, y, \cdots, y) - D^{n-1}f(x)(x_E, y, y, \cdots, y)]/z = 0$ for $n \geqq 3$ when $y = 0$ on E and using the fact that $D^nf(x)$ is n-linear and symmetric (writing $y = yx_E + y(1 - x_E))$ we have

$$\begin{aligned} \mathrm{D}^nf(x)(y, y, \cdots, y) &= D^nf(x)(yx_E, yx_E, \cdots, yx_E) \\ &\quad + D^nf(x)(y(1 - x_E), y(1 - x_E), \cdots, y(1 - x_E)) \end{aligned}$$

for all $n \geqq 1$ when E is measurable. Hence for $s \in E$ we have

$$D^nf(0)(y, y, \cdots, y)(s) = D^nf(0)(yx_E, yx_E, \cdots, yx_E)(s) \quad \text{a.e. on } E .$$

Therefore,

$$
\begin{aligned}
f(x)(s) &= \left[\sum_{k=1}^{\infty} \frac{1}{k!} D^k f(0)(x, x, \cdots, x)\right](s) \\
&= \left[\sum_{k=1}^{\infty} \frac{1}{k!} D^k f(0)(x, x, \cdots, x)(s)\right] \\
&\sum_{k=1}^{\infty} \frac{1}{k!} D^k f(0)(xx_E, xx_E, \cdots, xx_E)(s) \\
&= f(xx_E)(s) \text{ for almost all } s \in E \,.
\end{aligned}
$$

Let E be a set of positive measure, let $0 < \varepsilon < r < 1$ and

$$
(7) \qquad x = \alpha x_E + (1 - x_E)u,\ u \in X
$$

where $|\alpha| = r$ and $\|u\| \leqq r - \varepsilon$.

Let $y \in B$ satisfy

$$
(8) \qquad y(s) = 0 \text{ if } s \in E \,.
$$

Then $\|x + \beta y\| = \|x\| = r$ for $|\beta|$ sufficiently small. Let x' be the real continuous linear functional on X given by

$$
x'(v) = \operatorname{Re} \int_E \left(\frac{\bar{\alpha}}{r\mu(E)}\right) v \, d\mu \,.
$$

Then $x' \in T(x)$ and $x'(\beta y) = 0$ so Theorem 6 applies. Since arg β is arbitrary we may therefore conclude

$$
(9) \qquad (Df(x))^{-1} D^2 f(x)(y, y)(s) = 0 \quad \text{a.e. on } E \,.
$$

But $(Df(x))^{-1}D^2 f(x)(y, y)$ is an analytic function of α for $|\alpha| < 1$ (E, u, and y remain fixed) which satisfies (9) when $|\alpha| > \|u\|$. Hence (9) holds when $|\alpha| < 1$ and $\|u\| < 1$. For arbitrary $x \in B$, we may approximate xx_E on E by simple functions to see that (9) holds for all $x \in B$ if $y(s) = 0$ on E. An induction argument shows that (9) holds when $D^2 f(x)(y, y)$ is replaced by $D^n f(x)(y, y, \cdots, y)$ where $n \geqq 2$ and $y = 0$ on E.

We now wish to show $D^2 f(x)(x_E, y) = 0$ for all $x \in B$ and $y \in X$ satisfying $y(s) = 0$ if $s \in E$.

Let x and y be given by (7) and (8) and set $h = \beta y + t\alpha x_E$ where $|\beta| = t^{1/2}$. Then for t sufficiently small, $\|x + h\| \sqrt{1 + t^2}\, \|x\|$ and

$$
\begin{aligned}
w\left(x, \frac{1}{\sqrt{1 + t^2}}(x + h)\right) &= (Df(x))^{-1}\left(f(x) - f\left(\frac{1}{\sqrt{1 + t^2}}\right)(x + h)\right) \\
&= - v - (Df(x))^{-1}(D^2 f(x)(v, v)) - \cdots
\end{aligned}
$$

where

$$
v = \left(\frac{1}{\sqrt{1 + t^2}} - 1\right)x + \frac{1}{\sqrt{1 + t^2}} \cdot h \,.
$$

With x' as above, applying Theorem 4 we obtain

$$\begin{aligned} 0 &\leqq x'\left(w\left(x, \frac{1}{\sqrt{1+t^2}}(x+h)\right)\right) \\ &= \operatorname{Re}\left(\frac{-irt\beta}{\mu(E)}\int_E (Df(x))^{-1}D^2f(x)(x_E, y)d\mu\right) \\ &\quad + o(t^2). \end{aligned}$$

Since $|\beta| = t^{1/2}$ and arg β is arbitrary, we conclude

$$\int_E (Df(x))^{-1}D^2f(x)(x_E, y)d\mu = 0 \text{ when } y = 0 \text{ on } E. \tag{10}$$

Let $F \subset S$, F measurable, and $\mu(F) > 0$. Equation (10) implies

$$\int_{F\cap E}(Df(x))^{-1}D^2f(x)(x_E, y)d\mu = 0 .$$

Also,

$$\begin{aligned} \int_{F-E} L_x(x_E, y)d\mu = \int_{F-E}\Big[&L_x(yx_F, x_E) \\ &+ \frac{1}{2}L_x(x_E + y(1-x_F), x_E + y(1-x_F)) - \frac{1}{2}L_x(x_E, x_E) \\ &- \frac{1}{2}L_x(y(1-x_F), y(1-x_F))\Big]d\mu = 0 \end{aligned}$$

for the first term is 0 by (10) applied to $F - E$ and the other 3 terms are 0 by (9) applied to $F - E$. Hence we conclude $(Df(x))^{-1}D^2f(x)(x_E, y)(s) = 0$ a.e. on S so $D^2f(x)(x_E, y) = 0$.

Again taking S to be the positive integers with μ the counting measure we obtain the following corollary.

COROLLARY 2. *If* $X = \ell^\infty$ *and* $f: B \to Y$ *is convex then* $f(x) - f(0) = Df(0)(g(x))$ *where* $g(x) = (g_1(x_1), g_2(x_2), \cdots)$ *and* $g_k(x_k) = x_k + a_{2k}x_k^2 + \cdots$ *maps* $|x_k| < 1$ *onto a convex domain.*

4. Discussion. The techniques used in this paper may also be used to extend the concepts of close-to-convexity and spirallikeness to Banach spaces. There are at least two different ways of extending the analytic condition for close-to-convexity to Banach spaces and these apparently lead to different classes of functions. Results concerning these concepts are as yet incomplete and will be discussed in a later paper.

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