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A CLASS OF GENERALIZED FUNCTIONAL DIFFERENTIAL EQUATIONS

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In this paper, the equation $y' = Ay$ is solved, where A is a self-mapping of a certain set of functions. Also, a continuous dependence theorem is proven, and nth-order differential equations are considered.

1. Definitions. If p is a real number and $I = \{I_1, I_2, \cdots\}$ is a collection of intervals so that $p \in I_1$ and $I_n \subseteqq I_{n+1}$ for each positive integer n, then I is said to be a nest of intervals about p. Let $I_0 = \{p\}$ and $[a_n, b_n] = I_n$ for each nonnegative integer n. Let I^* denote the union of all the elements of I.

In general, B denotes a Banach space; and if D is a real number set, let $C[D, B]$ denote the set of continuous functions from D into B. Whenever D is an interval, $C[D, B]$ is considered a Banach space with supremum norm $|\cdot|$.

Let $C(I, B)$ denote the set of continuous functions whose domain is either I_0, I^*, or an element of I; and whose range is a subset of B.

Suppose A is a mapping from $C(I, B)$ into $C(I, B)$ so that

(i) domain f = domain Af, for all $f \in C(I, B)$,

(ii) $(Af)|_{I_k} = A(f|_{I_k})$, for all $f \in C(I, B)$ and $I_k \subseteqq$ domain f, for positive k, [Note: $f|_{I_k}$ is the restriction of f to I_k.] and

(iii) there is a function M from I^* into the nonnegative reals that is Lebesgue integrable on any interval contained in I, so that $\|Af(x) - Ag(x)\| \leqq M(x) \cdot |f - g|$, for all $f, g \in C[I_i, B]$ so that $f|_{I_{i-1}} = g|_{I_{i-1}}$ and $x \in I_i$, for each positive integer i.

Then, A is said to be an I-map with function M. Furthermore, if the phrase "$f|_{I_{i-1}} = g|_{I_{i-1}}$" is removed from part (iii) of the previous definition, A is said to be an I-map with strong function M.

2. Main results.

THEOREM A. *Suppose A is an I-map with function M; and* $\max\left\{\int_{a_i}^{a_{i-1}} M, \int_{b_{i-1}}^{b_i} M\right\} < 1$, *for all positive integers i. Then if $q \in B$, there is a unique $y \in C[I^*, B]$ so that $y' = Ay$ and $y(p) = q$.*

Proof. Let $\{(p, q)\} = y_0$. Then y_0 is the unique function in $C[I_0, B]$ so that $y_0(x) = q + \int_p^x Ay_0$ for all $x \in I_0$. Now, suppose n is a nonnegative integer so that y_n has been defined in $C[I_n, B]$ to be the unique function so that $y_n(x) = q + \int_p^x Ay_n$ for all $x \in I_n$. Then, $D =$

$\{f \in C[I_{n+1}, B]/f|_{I_n} = y_n\}$ is a complete metric space. If $f \in D$, let $Tf(x) = q + \int_p^x Af$, for all $x \in I_{n+1}$. Now if $x \in I_n$ and $f \in D$, then $T_f(x) = q + \int_p^x Af = q + \int_p^x (Af)|_{I_n} = q + \int_p^x A(f|_{I_n}) = q + \int_p^x Ay_n = y_n(x)$. Thus $(Tf)|_{I_n} = y_n$, and thus $Tf \in D$.

Suppose $f, g \in D$. Then,

$$\begin{aligned} |Tf - Tg| &= \max\{\|Tf(x) - Tg(x)\|/x \in I_{n+1}\} \\ &= \max\left\{\left\|\int_p^x (Af - Ag)\right\|\right\} \\ &\leqq \max\left\{\left|\int_p^x \|Af(s) - Ag(s)\|\,ds\right|\right\}. \end{aligned}$$

Note that $f|_{I_n} = g|_{I_n}$ and this implies that $A(f|_{I_n}) = A(g|_{I_n})$. Thus, $(Af)|_{I_n} = (Ag)|_{I_n}$; that is, $Af(s) = Ag(s)$ for all s in I_n. So

$$\begin{aligned} |Tf - Tg| &\leqq \max\left\{\sup\left\{\int_{b_n}^x \|Af(s) - Ag(s)\|\,ds/x \in [b_n, b_{n+1}]\right\},\right. \\ &\qquad \left.\sup\left\{\int_x^{a_n} \|Af(s) - Ag(s)\|\,ds/x \in [a_{n+1}, a_n]\right\}\right\} \\ &\leqq \max\left\{\sup\left\{\int_{b_n}^x M(s)\cdot|f - g|\,ds/x \in [b_n, b_{n+1}]\right\},\right. \\ &\qquad \left.\sup\left\{\int_x^{a_n} M(s)\cdot|f - g|\,ds/x \in [a_{n+1}, a_n]\right\}\right\} \\ &\leqq \max\left\{\int_{a_{n+1}}^{a_n} M, \int_{b_n}^{b_{n+1}} M\right\}\cdot|f - g|. \end{aligned}$$

Hence T is a contraction map from the complete metric space D into D, and thus T has a unique fixed point y_{n+1}. So y_{n+1} is the unique function in $C[I_{n+1}, B]$ so that $y_{n+1}(x) = q + \int_p^x Ay_{n+1}$ for all x in I_{n+1}. So by induction y_k is defined for each positive integer k. Define $y(x) = y_m(x)$ whenever $x \in I_m \backslash I_{m-1}$. Then y is the desired function.

The following corollary (See [6].) shows that Theorem A guarantees the existence of solutions to some functional differential equations. Suppose g is a function from I^* to I^* so that $g(I_n) \subseteqq I_n$ for each positive integer n. Such a function is said to be an I-function. Let $A_k = \{x \in [a_k, a_{k-1}]/g(x) \notin I_{k-1}\}$ and let $B_k = \{x \in [b_{k-1}, b_k]/g(x) \notin I_{k-1}\}$, for each positive integer k. Also, suppose $\|F(x, y) - F(x, z)\| \leqq M(x)\cdot\|y - z\|$ for all $x \in I^*$, $y, z \in B$; and M is Lebesgue integrable on intervals.

COROLLARY. *If* $q \in B$, *and* $\max\left\{\int_{A_k} M, \int_{B_k} M\right\} < 1$, *for all* k; *then there is a unique* $y \in C[I^*, B]$ *so that* $y(p) = q$ *and* $y'(x) = F(x, y(g(x)))$ *for all* $x \in I^*$.

Proof. Let $(Af)(x) = F(x, f(g(x)))$. Then A is an I-map with function T, where

$$T(x) = \begin{cases} M(x) , & x \in A_n \cup B_n \\ 0 , & x \notin A_n \cup B_n \end{cases} , \quad \text{for } x \in I_n \backslash I_{n-1} .$$

The proof of the following is straightforward.

PROPOSITION. *Suppose I is a nest of intervals about p, and each of α and β is an I-function. Then*

(i) *Suppose P is of bounded variation on each interval contained in I^*, and let $Af(x) = \int_{\alpha(x)}^{\beta(x)} dF \cdot f$, for $f \in C(I, B)$ and $x \in$ domain f. Then A is an I-map with function M, where $M(x)$ is the variation of F over $[\alpha(x), \beta(x)] \backslash I_{k-1}$ where $x \in I_k \backslash I_{k-1}$.*

(ii) *Suppose K: $I^* \times I^*$ to the scalars which is continuous, and $Af(x) = \int_{\alpha(x)}^{\beta(x)} K(x, t) f(t) dt$, for $f \in C(I, B)$ and $x \in$ domain f. Then A is an I-map with function M, where $M(x) = \left| \int_{[\alpha(x), \beta(x)] \backslash I_{k-1}} |K(x, t)| dt \right|$ for $x \in I_k \backslash I_{k-1}$.*

It is easy to show that the set of I-maps, for a fixed nest of intervals I, is a near-ring under composition and addition. Thus, there are many types of differential equations that may be solved by combining I-maps of the types given in the corollary and the proposition.

3. Continuous dependence.

THEOREM B. *Suppose $A(z, \cdot)$ is an I-map with strong function M for each z in the topological space K, $q \in B$, and $M_k = \max \left\{ \int_{a_k}^{a_{k-1}} M, \int_{b_{k-1}}^{b_k} M \right\} < 1$, for all positive integers k. Let $y(z, \cdot)$ be the unique function, guaranteed by Theorem* A, *so that $y_2(z, \cdot) = A(z, y(z, \cdot))$ and $y(z, p) = q$. Then, there exists a sequence $\{L_i\}$ so that for $z, z_0 \in K$, $|y(z, \cdot) - y(z_0, \cdot)|_{I_i} \leqq L_i \cdot |A(z, y(z_0, \cdot)) - A(z_0, y(z_0, \cdot))|_{I_i}$, for each i. [In the previous inequality the norm is the supremum norm over I_i.]*

Indication of proof. Define $\{L_i\}$ as follows: Let $L_1 = \max(p - a_1, b_1 - p)/(1 - M_1)$. For $i \geqq 1$, let $L_{i+1} = \{L_i + \max(a_i - a_{i+1}, b_{i+1} - b_i)\}/(1 - M_{i+1})$.

EXAMPLE. Let g be an I-function and let $N > 0$. Then let K be the metric space of all I-functions that are pointwise never more that N from g. Define $A(h, y) = y(h|_{\text{dom}\, y})$ and $d(h_1, h_2) = \sup\{|h_1(x) - h_2(x)| / x \in I^*\}$; d is the metric.

4. *N*th order equations.

THEOREM C. *Suppose A is an I-map with function M, n is a positive integer, and $q_0, q_1, \cdots, q_{n-1} \in B$. Let*

$$N_k = \max\left\{\int_{a_k}^{a_{k-1}}\int_{s_1}^{a_{k-1}}\cdots\int_{s_{n-1}}^{a_{k-1}} M(s_n)ds_n\cdots ds_1\,,\right.$$
$$\left.\int_{b_{k-1}}^{b_k}\int_{b_{k-1}}^{s_1}\cdots\int_{b_{k-1}}^{s_{n-1}} M(s_n)ds_n\cdots ds_1\right\}.$$

Then, if $N_k < 1$, for all positive integers k, there is a unique $y \in C[I^*, B]$ so that $y^{(n)} = Ay$ and $y(p) = q_0, \cdots, y^{(n-1)}(p) = q_{n-1}$.

Indication of proof. Use induction, Theorem A, and the following lemma.

LEMMA. *Suppose H is an I-map with function S, and $q \in B$, then define $Kf(x) = q + \int_p^x Hf$, for all $f \in C(I, B)$ and $x \in$ domain f. Then K is an I-map with function T, where $T(x) = \int_x^{a_{k-1}} S$, whenever $x \in (a_k, a_{k-1}]$; and $T(x) = \int_{b_{k-1}}^x S$, whenever $x \in [b_{k-1}, b_k)$.*

The proof of Theorem D is straightforward and Theorem E is a special case of Theorem D. Both of these theorems are imitations of standard theorems of ordinary differential equations.

THEOREM D. (*A generalized system of equations theorem.*) *Suppose B_i is a Banach space with norm $||\cdot||_i$, for each positive integer i between 1 and n. Let $B' = \{(x_1, x_2, \cdots, x_n)/x_i \in B_i\}$. Also, let $||(x_1, \cdots, x_n)|| = \max\{||x_i||_i/1 \leqq i \leqq n\}$, for all elements of B'.* [*Then B' is a Banach space.*] *Furthermore, suppose H_i: $C(I, B')$* to *$C(I, B_i)$ for $1 \leqq i \leqq n$ so that*

(1) *if $f \in C(I, B')$, domain f = domain $H_i f$,*

(2) *if $f \in C(I, B')$, and $I_k \subseteqq$ domain f, $k > 0$, then $(H_i f)|_{I_k} = H_i(f|_{I_k})$, and*

(3) *there is M_i: I^* to the reals which is Lebesgue integrable on intervals so that if $f, g \in C[I_k, B']$, $f|_{I_{k-1}} = g|I_{k-1}$, and $x \in I_k$, then $||H_i f(x) - H_i g(x)|| \leqq M_i(x)\cdot|f - g|$. Now, define A: $C(I, B')$ to $C(I, B')$ so that $Af = (H_1 f, H_2 f, \cdots, H_n f)$, for all $f \in C(I, B')$.*

Then A is an I-map with function $\max\{M_i/1 \leqq i \leqq n\}$.

THEOREM E. *Suppose B' is as in Theorem* D, *with $B = B_i$, for all i. Also, suppose $H = H_n$ and $M = M_n$, where H_n and M_n are as in Theorem* D. *Suppose $q_0, \cdots, q_{n-1} \in B$ and*

$$\max\left\{\int_{a_k}^{a_{k-1}} \max\{1, M\}, \int_{b_{k-1}}^{b_k} \max\{1, M\}\right\} < 1, \text{ for all } k > 0 .$$

Then, there is a unique $y \in C[I^*, B]$ *so that*

$$y^{(n)} = H((y, y^{(1)}, \cdots, y^{(n-1)})) \textit{ and } y^{(i)} = q_i, \textit{ for } 0 \leqq i \leqq n - \iota .$$

EXAMPLE. Suppose each g_i is an I-function, then for appropriate functions F_i, Theorem E guarantees the existence of a solution to

$$y^{(n)}(x) = \sum_{k=1}^{n} F_k(x, y^{(n-k)}(g_k(x))), \text{ for all } x \in I^* .$$

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