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NONLINEAR FUNCTIONALS ON $C([0, 1] \times [0, 1])$

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Let M be a compact Hausdorff space. Let $\mathscr{C}(M)$ denote the Banach space of continuous functions f on M. We are interested in functionals Φ on $\mathscr{C}(M)$ with the following properties:

(i) $|\Phi(f)| \leqq \|f\|$ **for every** $f \in \mathscr{C}(M)$,

(ii) $\Phi(f+g) = \Phi(f) + \Phi(g)$ **whenever** $fg = 0$,

(iii) $\Phi(f+\alpha) = \Phi(f) + \alpha$ **for every** $f \in \mathscr{C}(M)$ **and every real number** α.

It was shown in [1] that any Φ which has properties (i), (ii), and (iii) is actually a continuous linear functional, in the particular case that $M = [0, 1]$. Thus in this case we can represent Φ by $\Phi(f) = \int f(x)\mu(dx)$ for some measure on M. It is the purpose of this paper to show that such a representation is not possible when $M = [0, 1] \times [0, 1]$, because there exist nonlinear functionals Φ which have properties (i), (ii), and (iii). We construct two classes of examples. The first class admits of a simple geometric interpretation. The examples in the second, and larger, class are defined less directly, using transfinite induction.

The general case, when M is an arbitrary compact Hausdorff space, can be carried to $M = [0, 1] \times [0, 1]$, in the following sense: Fix f and g in $\mathscr{C}(M)$. Let I_1 and I_2 be compact intervals containing $f(M)$ and $g(M)$ respectively. For any functional Φ on $\mathscr{C}(M)$, we can define Φ^* on $\mathscr{C}(I_1 \times I_2)$ by letting $\Phi^*(h) = \Phi(h(f, g))$ for each $h \in \mathscr{C}(I_1 \times I_2)$. Clearly if Φ satisfies (i), (ii), and (iii), then so does Φ^*, and a representation for Φ^* can be carried back to a representation for Φ on the algebra generated by f and g.

We prove in a forthcoming paper that conditions (i), (ii), and (iii) imply that Φ is linear provided that M is of (topological) dimension one.

2. Topological lemmas.

From now on, let M denote $[0, 1] \times [0, 1]$. Let f be a fixed function in $\mathscr{C}(M)$. We can define an equivalence relation on M as follows:

$x \sim y$ means that x and y are contained in some connected set upon which f is constant.

Let A_f be the collection of equivalence classes defined by this relation.

Then $A_f = \{l \mid l$ is a maximal connected component of $f^{-1}(\{\alpha\})$, $\alpha \in \boldsymbol{R}\}$.

The topology on M induces a topology on A_f as follows:

$B \subseteqq A_f$ is called open if $\bigcup_{l \in B} l$ is an open set of points in M.

We will call the elements of A_f the "level curves" of f.

Let $\theta_f\colon M \to A_f$ be the map that sends each point x into the equivalence class l that contains x.

Then θ_f is continuous.

Hence $A_f = \theta_f(M)$ is compact and connected.

We note that if E is an open (or closed) set in M which is a union of members of A_f then $\theta_f(E)$ is open (or closed) also.

LEMMA 1. *A_f is a Hausdorff space.*

Proof. Fix $l \in A_f$ and $x \in l$.

For each n, let F_n denote that maximal connected component of $\{z \mid f(x) - 1/n \leqq f(z) \leqq f(x) + 1/n\}$ which contains x.

Clearly F_n is closed, F_n is a union of members of A_f, and $l \subseteqq F_n$, for every n.

Hence $l \subseteqq \bigcap_{n=1}^{\infty} F_n$.

But a decreasing sequence of connected connected sets in a Hausdorff space has a connected intersection. Since f is constant on the connected set $\bigcap_{n=1}^{\infty} F_n$, therefore $l \supseteqq \bigcap_{n=1}^{\infty} F_n$, so $l = \bigcap_{n=1}^{\infty} F_n$.

Hence $\bigcap_{n=1}^{\infty} \theta_f(F_n) = \{l\}$.

For each n, let G_n denote that maximal connected component of $\{z \mid f(x) - 1/n < f(z) < f(x) + 1/n\}$ which contains x.

Clearly G_n is open, G_n is a union of members of A_f, and $l \subseteqq G_n$, for every n.

Hence $\theta_f(G_n)$ is an open set containing l, for each n.

Also $\bigcap_{n=1}^{\infty} \overline{\theta_f(G_n)} \subseteqq \bigcap_{n=1}^{\infty} \theta(F_n) = \{l\}$.

This proves Lemma 1.

Let l be in A_f. Let x be in l. Let G be any open set in A_f containing l. Then $\theta_f^{-1}(G)$ is an open set containing x. Let H be that maximal connected component of $\theta_f^{-1}(G)$ which contains x. Then H is a union of members of A_f, because $\theta_f^{-1}(G)$ is. Hence $\theta_f(H)$ is an open, connected subset of G, containing l. This shows that A_f is locally connected.

LEMMA 2. *For any connected set C in A_f, $\theta_f^{-1}(C)$ is connected.*

Proof. Let F_1 and F_2 be closed sets in M, such that $F_1 \cup F_2 \supseteqq \theta_f^{-1}(C)$ and $F_1 \cap F_2 \cap \theta_f^{-1}(C) = \varnothing$.

Then any equivalence class l in C must lie entirely in F_1 or F_2 but not both, because l is connected.

Hence $\theta_f(F_1) \cap \theta_f(F_2) \cap C = \varnothing$.

Since $\theta_f(F_1) \cup \theta_f(F_2) \supseteqq C$ and C is connected, at least one of $\theta_f(F_1)$, $\theta_f(F_2)$ must be $\varnothing$. This proves Lemma 2.

DEFINITION 1. Let a and b be points in a topological space X. A set E in X is said to separate a and b if a and b do not lie in a connected component of $X - E$.

LEMMA 3. *Let E be a set in M which separates two points a and b. Then E contains a connected subset F which separates a and b.*

Proof. This is a special case of Theorem 1 in [2], §57 III, page 438.

LEMMA 4. *Let D be a set in A_f which separates two points l and k. Then D contains a connected set C which separates l and k.*

Proof. Choose $x \in l$ and $y \in k$.

Let $E = \theta_f^{-1}(D)$. Then E separates x and y in M, since θ_f is continuous.

Hence by Lemma 3, E contains a connected subset F which separates x and y.

Let $C = \theta_f(F)$.

Then C separates l and k by Lemma 2.

This proves Lemma 4.

DEFINITION 2. Let S be the unit circle in $\boldsymbol{R}^2$. A topological space which is homeomorphic to S is called a simple closed curve.

LEMMA 5. *A_f does not contain a simple closed curve.*

Proof. Let $\varphi: S \to A_f$ be continuous.

We will show that φ is not a homeomorphism.

Let g be the unique function on A_f such that $g \circ \theta_f = f$. Then g is clearly continuous. Furthermore, if C is a connected set in A_f upon which g is constant, we see by Lemma 2 that C must consist of one point.

Let $H = \varphi(S)$.

H is connected. If g is constant on H, then H is a one point set, and we are done. Thus we may assume that there exist points l and k in H such that $g(l) = \alpha < g(k) = \beta$.

Choose γ such that $\alpha < \gamma < \beta$.

Then clearly $g^{-1}(\{\gamma\})$ separates l and k.

Hence by Lemma 4 there must exist a connected set $C \subseteqq g^{-1}(\{\gamma\})$ such that C separates l and k. Thus l and k are separated by a single point. It is clear that this would not be possible if H were homeomorphic to S, so Lemma 5 is proved.

LEMMA 6. *Let K and L be two compact, connected subsets of A_f. Then $K \cap L$ is compact and connected.*

Proof. Follows from Lemma 5 and Theorem 1 in [2], §51 VI, page 300.

If we consider a continuous function f on a general topological space M, and form the space A_f of level curves of f, then Lemmas 5 and 6 no longer hold. For example, if M is the unit circle, we can find a function f such that A_f is homeomorphic to M.

3. Construction of functionals. As before, let M denote $[0,1] \times [0,1]$.

Let us suppose that for each $f \in \mathscr{C}(M)$ we have chosen a level curve $l_f \in A_f$. Then we can define a functional Φ as follows:

$$(1) \qquad \Phi(f) = f(x), \text{ any } x \in l_f .$$

We shall define the mapping $f \to l_f$ later in such a way that

$$(2) \qquad \forall f, g \in \mathscr{C}(M), l_f \cap l_g \neq \varnothing .$$

LEMMA 1. *If* (2) *holds, then Φ has properties* (i), (ii), *and* (iii) *of* §1.

Proof. (i) is clear.

For (ii), we note first that if $fg = 0$ then both f and g are constant on l_{f+g}. Since $l_{f+g} \cap l_f \neq \varnothing$, we must have $l_{f+g} \subseteqq l_f$. Similarly $l_{f+g} \subseteqq l_g$.

Let x be a point in l_{f+g}. Then $x \in l_f$ and $x \in l_g$. Hence $\Phi(f + g) = f(x) + g(x)$, $\Phi(f) = f(x)$, and $\Phi(g) = g(x)$. This proves (ii).

For (iii), we see similarly that $l_{f+c} = l_f$, and the proof follows.

Let D be a fixed closed, connected set in M. Let z be a fixed point in M. For any fixed f in $\mathscr{C}(M)$, let $\theta_f(D) = C$, where θ is the map defined in §2. We then have that C is a closed, connected set in A_f.

Let φ_1: $[0, 1] \to M$ and φ_2: $[0, 1] \to M$ be any two continuous maps such that $\varphi_1(0) = \varphi_2(0) = z$, $\varphi_1(1) \in D$, $\varphi_2(1) \in D$.

Let

$$t_1 = \inf \{t \mid \theta_f(\varphi_1(t)) \in C\} ,$$
$$t_2 = \inf \{t \mid \theta_f(\varphi_2(t)) \in C\} .$$

LEMMA 2. $\theta_f(\varphi_1(t_1)) = \theta_f(\varphi_2(t_2))$.

Proof. Let $L_1 = \theta_f(\varphi_1([0, t_1]))$. Let $L_2 = \theta_f(\varphi_2([0, t_2]))$.

Then L_1 and L_2 are closed, connected sets in A_f.

$L_1 \cap C = \theta_f(\varphi_1(t_1))$, by the definition of t_1. Similarly $L_2 \cap C = \theta_f(\varphi_2(t_2))$.

Thus $L_1 \cup C$ and $L_2 \cup C$ are connected sets.

By Lemma 6 of §2, $(L_1 \cup C) \cap (L_2 \cup C)$ is connected. That is, $(L_1 \cap L_2) \cup C$ is connected.

Hence $L_1 \cap L_2 \cap C \neq \emptyset$.

Hence $\{\theta_f(\varphi_1(t_1))\} \cap \{\theta_f(\varphi_2(t_2))\} \neq \emptyset$.

This proves Lemma 2.

DEFINITION 1. For each $f \in \mathscr{C}(M)$ we will define l_f to be the unique element $\theta_f(\varphi_1(t_1))$ described above.

Intuitively, one may regard $\theta_f(D)$ as being a collection of hairs covering D. Suppose that one releases a bug from z and allows it to crawl to D. The first hair that it reaches is called l_f. Lemma 2 shows that this definition does not depend on the path of the bug.

Let U_f denote the maximal connected component of z in $M - l_f$. If $z \in l_f$ let $U_f = \emptyset$. Let V_f denote the union of the other components of $M - l_f$.

LEMMA 3. $U_f \cap D = \emptyset$.

Proof. If $z \in l_f$, the result is trivial. Otherwise, suppose there exists a point $y \in D \cap U_f$. Since U_f is open and connected, we can find φ: $[0, 1] \rightarrow U_f$ such that φ is continuous, $\varphi(0) = z$, and $\varphi(1) = y$.

Let $t_0 = \inf \{t \mid \theta_f(\varphi(t)) \in C\}$.

Since $\varphi(t_0) \in U_f$, clearly $\theta_f(\varphi(t_0)) \neq l_f$. This contradicts Lemma 2, so our assumption that there exists a point $y \in D \cap U_f$ must be false. This proves Lemma 3.

LEMMA 4. *Let f and g be in $\mathscr{C}(M)$.*

Then $l_f \cap l_g \neq \emptyset$.

Proof. Suppose $l_f \cap l_g = \emptyset$. Since l_f contains points in D, l_f is not completely contained in U_g. Hence $l_f \cap U_g = \emptyset$, or in other words $l_f \subseteqq V_g$, since l_f is connected. Similarly $l_g \subseteqq V_f$. Hence $[V_f \cup V_g] \cup [U_f \cap U_g] = M$. Since M is connected, and $V_f \cup V_g \neq \emptyset$, we must have $U_f \cap U_g = \emptyset$. Hence z is not in both U_f and U_g. Suppose $z \notin U_f$. Then $z \in l_f$. Hence $z \notin U_g$. Hence $z \in l_g$. This contradicts our assumption $l_f \cap l_g = \emptyset$, so Lemma 4 is proved.

EXAMPLE 1. Let l_f be chosen as in Definition 1. Let Φ be defined by equation (1). It follows from Lemma 4 and Lemma 1 that Φ satisfies (i), (ii), and (iii) of §1.

THEOREM 1. *Suppose $z \notin D$, and D contains more than one point. Then Φ is nonlinear.*

Proof. It is easy to see that two continuous maps $\varphi_1: [0, 1] \to M$ and $\varphi_2: [0, 1] \to M$ can be found such that $\varphi_1(0) = \varphi_2(0) = z$, $\varphi_1(1) \in D$, $\varphi_2(1) \in D$, $\varphi_1(1) \neq \varphi_2(1)$, $\varphi_1(t) \notin D$ for $t < 1$, $\varphi_2(t) \notin D$ for $t < 1$.

Choose $f, g \in \mathscr{C}(M)$ such that $f = 0$ on $\varphi_2([0, 1])$, $g = 0$ on $\varphi_1([0, 1])$, and $f + g \geqq 1$ on D.

Then $\Phi(f) = 0$, $\Phi(g) = 0$, but $\Phi(f + g) \geqq 1$.

We will now describe a more general way of defining the map $f \to l_f$ so that equation (2) is satisfied.

LEMMA 5. *Let f be in $\mathscr{C}(M)$. Let H be a collection of closed, connected sets in A_f. Suppose for every F_1 and F_2 in H that $F_1 \cap F_2$ is nonempty. Then $\bigcap_{F \in H} F$ is nonempty.*

Proof. First, assume H has three elements, F_1, F_2, and F_3. We will show that $F_1 \cap F_2 \cap F_3 \neq \varnothing$.

Since $F_1 \cap F_2 \neq \varnothing$, therefore $F_1 \cup F_2$ is connected. Similarly $F_1 \cup F_3$ is connected.

By Lemma 6 of §2, $(F_1 \cup F_2) \cap (F_1 \cup F_3)$ is connected. That is, $F_1 \cup [F_2 \cap F_3]$ is connected. Hence $F_1 \cap F_2 \cap F_3 \neq \varnothing$.

Now assume that Lemma 5 has been proved when H has n elements. Suppose H has $n + 1$ elements, $F_1, F_2, \cdots, F_{n+1}$.

Let $K_i = F_i \cap F_{n+1}$, $i = 1, \cdots, n$.

By Lemma 6 of §2, the K_i are closed and connected.

By Lemma 5 with $n = 3$, for every i and j we have $K_i \cap K_j \neq \varnothing$.

Hence by our inductive assumption $K_1 \cap K_2 \cap \cdots \cap K_n \neq \varnothing$. But $K_1 \cap \cdots \cap K_n = F_1 \cap \cdots \cap F_{n+1}$.

Thus we have proved Lemma 5 for the case that H has $n + 1$ elements.

Hence by induction Lemma 5 is true for any finite collection H.

This implies that any arbitrary H has the finite intersection property. Lemma 5 follows by the compactness of A_f.

LEMMA 6. *Let Γ be a map whose domain is a certain subset S of $\mathscr{C}(M)$, such that $\Gamma(f) \in A_f$ for each $f \in S$, and such that for each f and g in S, $\Gamma(f) \cap \Gamma(g) \neq \varnothing$. Let h be in $\mathscr{C}(M)$, h not in S. Then we can define $\Gamma(h) \in A_h$ in such a way that for every $f \in S$, $\Gamma(f) \cap \Gamma(h) \neq \varnothing$.*

Proof. Let $H = \{\theta_h(\Gamma(f)), f \in S\}$.

Each set $\theta_h(\Gamma(f))$ is a closed, connected subset of A_h. For every f and g in S,

$$\theta_h(\Gamma(f)) \cap \theta_h(\Gamma(g)) \supseteqq \theta_h(\Gamma(f) \cap \Gamma(g)) \neq \varnothing .$$

By Lemma 5,

$$\bigcap_{f \in S} \theta_h(\Gamma(f)) \neq \varnothing .$$

Choose any $l \in \bigcap_{f \in S} \theta_h(\Gamma(f))$, and call it $\Gamma(h)$.

For each $f \in S$, $l \in \theta_h(\Gamma(f))$, so $l \cap \Gamma(f) \neq \varnothing$.

This proves Lemma 6.

EXAMPLE 2. Using Lemma 6 and Zorn's lemma, we can start with any map Γ of the sort described in Lemma 6, and extend it to all of $\mathscr{C}(M)$ in such a way that for any f and g in $\mathscr{C}(M)$, $\Gamma(f) \cap \Gamma(g) \neq \varnothing$. Let l_f be defined to be $\Gamma(f)$ for each $f \in \mathscr{C}(M)$. Let Φ be defined as before, using equation (1). Once again by Lemma 1, Φ has properties (i), (ii), and (iii).

We could take our original domain S for Γ to consist of the three functions x, y, and $x + y$ where x and y are the usual coordinates on M. Let $\Gamma(x) =$ the line joining (0, 0) and (0, 1). Let $\Gamma(y) =$ the line joining (0, 0) and (1, 0). Let $\Gamma(x + y) =$ the line joining (0, 1) and (1, 0) and (1, 0). Clearly $\Phi(x) = \Phi(y) = 0$, but $\Phi(x + y) = 1$, so Φ is nonlinear.

We note that all the functionals constructed are monotone and continuous. This may be verified directly without too much difficulty.

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