SLENDER RINGS AND MODULES

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An abelian group B is called slender if every homomorphism from $\prod_1^\infty Z$ into B is continuous for the discrete topology on B and the topology that $\prod_1^\infty Z$ has as a cartesian product if Z (the additive group of integers) is considered discrete. It can be seen that if B is slender, then every homomorphism from $\prod_1^\infty A_i$ into B is continuous for the same topologies, where A_i are arbitrary abelian groups. Slender groups were completely characterized by Nunke, and it follows from his characterization that all countable reduced torsion free groups are slender. Nunke's results depend on the fact that Z is slender (proved by Specker) and the use of some homological machinery. In this paper, slender modules are studied over an arbitrary ring. A ring R will be called slender if R is a slender R-module. In Theorem 1, a generalization of the Baire Category Theorem is used to show that a countable module over an arbitrary ring is slender if $\bigcap \mathfrak{m}B = 0$, where $\mathfrak{m}$ ranges over the set of ideals in R which are not zero divisors for B. It follows that countable torsion free reduced modules over any (countable) integral domain are slender. In Theorem 2, it is shown that a commutative ring R is slender if there exists an infinite set of maximal ideals with the property that the intersection of any infinite subset is 0.

In § 2, reflexive modules over a slender ring are studied, and it is shown that a projective module is reflexive if it is generated by a set with cardinality smaller than the first measurable cardinal. In § 3, Nunke's characterization is extended to modules over a slender Dedekind domain having only countably many ideals. The general approach in § 3 follows Nunke's, but all homological machinery is avoided by the use of topological techniques.

R will always be a ring with identity. The word ideal will mean a two sided ideal, the word module will mean a unitary left module.

If B is an R-module and $\mathfrak{m}$ an ideal in R, we say that $\mathfrak{m}$ is a zero divisor for B if $\mathfrak{m}x = 0$ for some nonzero $x \in B$. We say that B is torsion free if no nonzero ideal is a zero divisor for B. If $\mathscr{M}$ is a family of ideals in R, then by the $\mathscr{M}$-adic topology on B we mean the linear topology induced by the family of submodules $\mathfrak{m}B$ where $\mathfrak{m}$ is a finite intersection of ideals in $\mathscr{M}$. The word complete will only be used in reference to Hausdorff topologies.

1. Slender modules. Let $\{A_i \mid i \in I\}$ be a family of R modules. By the product topology on $\prod_I A_i$ we mean the topology this module has as a Cartesian product if each A_i is considered discrete. In the product topology, $\prod_I A_i$ is a complete topological module. If $a \in \prod_I A_i$, we will write $a(i)$ to denote the ith coordinate of a. If $J \subseteq I$, we will consider $\prod_J A_i$ as a submodule of $\prod_I A_i$ in the usual way.

If $\mathscr{M}$ is a family of ideals in R, we will also be interested in the topology on $\prod_I A_i$ which is the supremum of the product topology and the $\mathscr{M}$-adic topology. We will refer to this as the strong (or $\mathscr{M}$-strong) topology. It is a complete topology, and is metrizable if both $\mathscr{M}$ and I are countable. If $a \in \prod_I A_i$, then the sets of the form $a + \mathfrak{m} \prod_{i \notin J} A_i$ form a basis for the neighborhood system at a in the strong topology, where J is a finite subset of I and $\mathfrak{m}$ is a finite intersection of ideals in $\mathscr{M}$. Since these neighborhoods are closed in the product topology, the following generalization of the Baire Category Theorem is applicable.

LEMMA 1. *Let X be a complete metric space. Let $\mathscr{T}$ be a topology on X which is finer than the metric topology and such that $\mathscr{T}$ has a basis consisting of sets which are closed in the metric topology. Then X is not a countable union of sets which are nowhere dense with respect to $\mathscr{T}$.*

Proof. Let $G_1, G_2, \cdots$ be a countable family of sets which, with respect to $\mathscr{T}$, are dense and open. Then we can inductively find a sequence of points x_n and $\mathscr{T}$-neighborhoods U_n of x_n such that, for each n, $U_{n+1} \subseteq U_n \cap G_1 \cap \cdots \cap G_{n+1}$, and such that the U_n are closed in the metric topology and have diameter less than $1/n$. It follows that $\{x_n\}$ is a Cauchy sequence in the metric topology and hence converges metrically to a point x. Since the U_n are metrically closed, it follows that $x \in \cap G_n$. Thus any countable intersection of dense open sets is nonempty, and this is equivalent to the assertion of the lemma.

COROLLARY 1. *If $\mathscr{M}$ is a family of ideals, then $\prod_1^\infty A_i$ is not a countable union of sets which are nowhere dense for the strong topology.*

We are now able to prove the two main theorems.

DEFINITION. If B is an R-module, we say that B is (left) *slender* if for every countable family of R-modules $\{A_i\}$ and every homomorphism ϕ from $\prod_1^\infty A_i$ into B, there is a natural number n such that $\phi(\prod_{i \geq n} A_i) = 0$. This amounts to saying that ϕ is continuous

for the product topology on $\prod_1^\infty A_i$ and the discrete topology on B. We say that R is a (left) *slender ring* if R is a slender R-module. (The concept of right slenderness can, of course, be similarly defined.)

THEOREM 1. *Let B be a countable R-module and $\mathscr{M}$ be the set of all ideals which are not zero divisors for B. If the $\mathscr{M}$-adic topology on B is Hausdorff, then B is slender.*

Proof. Let ϕ be a homomorphism from $\prod_1^\infty A_i$ into B. Then $\prod_1^\infty A_i$ is a countable union of sets $\phi^{-1}(x)$, for $x \in B$. Because ϕ is necessarily continuous in the $\mathscr{M}$-adic topology, each $\phi^{-1}(x)$ is closed in the $\mathscr{M}$-adic topology, and hence in the strong topology. By Corollary 1, some $\phi^{-1}(x)$ must have an interior in the strong topology. Thus there is a natural number n and an ideal $\mathfrak{m} \in \mathscr{M}$ such that $\phi(a + \mathfrak{m} \prod_{i \geq n} A_i) = \{x\}$ for some $a \in \prod A_i$. This is only possible if $\phi(\mathfrak{m} \prod_{i \geq n} A_i) = 0$. Since $\mathfrak{m}$ is not a zero divisor for B, this gives $\phi(\prod_{i \geq n} A_i) = 0$.

EXAMPLE 1. Let R be a countable commutative integral domain which is not a field. The R is a slender ring, and every countable reduced torsion free R-module B is slender. (Reduced means that no submodule is divisible.) In fact, in both cases $\mathscr{M}$ is the set of all nontrivial ideals in R. If $\mathfrak{m} \in \mathscr{M}$ and $0 \neq x \in R$, then $x \notin \mathfrak{m}x$ and $\mathfrak{m}x \in \mathscr{M}$, so R is $\mathscr{M}$-adically Hausdorff. If B is torsion free, then it is easily seen that $\bigcap_\mathfrak{m} \mathfrak{m}B$ is divisible and hence equal to 0 if B is reduced.

EXAMPLE 2. If R is a countable Dedekind domain, then the hypotheses of Theorem 1 hold if and only if B is countable torsion free and reduced. In fact, clearly it is necessary that B be reduced. Now if B is not torsion free, then there is a submodule A of B with $\mathfrak{a} = \text{Ann}\,(A) \neq 0$. Now every ideal in R is a product of maximal ideals. If $\mathfrak{m}$ is maximal and $\mathfrak{a} \subseteq \mathfrak{m}$, them $\mathfrak{m}$ is an associated prime for A, so $\mathfrak{m} \notin \mathscr{M}$. On the other hand, if $\mathfrak{m}$ does not contain $\mathfrak{a}$, then $\mathfrak{m} + \mathfrak{a} = R$ so that $\mathfrak{m}A = A$. Hence we have $A = \bigcap_\mathfrak{m} \mathfrak{m}A \subseteq \bigcap_\mathfrak{m} \mathfrak{m}B$.

EXAMPLE 3. In general, B need not be torsion free in order to satisfy the hypothesis for Theorem 1. For instance, let $R = Z[x]$ and $B = R/2R$. Then B is not a torsion free R-module, but the hypothesis of Theorem 1 is easily seen to hold.

We cannot hope that Theorem 1 will remain valid if we simply require that B be countably generated. For instance, if R is a complete local ring, then R fails to be slender, as can be seen from

the proof of Theorem 6. However, there is some hope for certain uncountable rings, as is shown in Theorem 2.

THEOREM 2. *Let R be a commutative ring having an infinite set of maximal ideals with the property that the intersection of any infinite subset of $\mathscr{M}$ is 0. Then R is a slender ring.*

Proof. R must be an integral domain, since if $xy = 0$, then either x or y must belong to infinitely many ideals in $\mathscr{M}$ (since these are prime). There is no loss of generality in assuming that $\mathscr{M}$ is countable, so we may enumerate $\mathscr{M}$ as $\{\mathfrak{m}_{ij} \mid i, j = 1, 2, \cdots\}$. Now let $C_{mn} = \bigcup_{i=1}^{n}(R \backslash \mathfrak{m}_{mi})$ for every pair of natural numbers m, n. Let $C_0 = \{0\}$ and for $n \geqq 1$ let $C_n = \bigcap_m C_{mn}$. Then C_n is $\mathscr{M}$-adically closed and if $0 \neq x \in R$, then since x can belong to only finitely many $\mathfrak{m}_{ij}$, $x \in \bigcup_n C_n$. Now if $\mathfrak{a}$ is a nonzero ideal in R, then there are only finitely many $\mathfrak{m}_{ij}$ with $\mathfrak{a} \subseteq \mathfrak{m}_{ij}$. Thus for some m, none of the ideals $\mathfrak{m}_{mj}$ contain $\mathfrak{a}$. Thus for any fixed $b \in R$ and every n, we can find $x \equiv b \pmod{\mathfrak{a}}$ with $x \in \mathfrak{m}_{mi}$, $i = 1, \cdots, n$ by the Chinese remainder theorem. Hence $b + \mathfrak{a}$ is not contained in C_{mn} for any n, and so is not contained in any C_n.

Now let ϕ be a homomorphism from $\prod_1^\infty A_i$ into R. It now follows by Corollary 1 that some $\phi^{-1}(C_n)$ has an interior in the strong topology. This means that $\phi(a + \mathfrak{m} \prod_{i \geqq m} A_i) \subseteq C_n$ where $a \in \prod A_i$, m is a natural number, and $\mathfrak{m}$ is a finite intersection of ideals in $\mathscr{M}$. Since $\phi(\mathfrak{m} \prod_{i \geqq m} A_i)$ is an ideal in R, it follows from what we have said about the C_n that this can only happen if $\mathfrak{m}\phi(\prod_{i \geqq m} A_i) = 0$, and hence $\phi(\prod_{i \geqq m} A_i) = 0$, since R is an integral domain.

EXAMPLE 4. R is a Dedekind domain with infinitely many prime ideals.

EXAMPLE 5. $R = K[x_1, \cdots, x_s]$ where K is a field and the x_i are indeterminates. Let F be the algebraic closure of K and $V = F \otimes R$, regarded as an F-vector space. Let V_n be the subspace of V generated by those polynomials of degree at most n, let $r(n) = \dim V_n$, and let V_n^1 be the kernel of the restriction map $V^* \to V_n^*$, where $V^* = \text{Hom}(V, F)$. Each point $a \in F^s$ induces a linear functional $\phi(a) \in V^*$ by $\phi(a)(f) = f(a)$. Furthermore, suppose $W_1, \cdots, W_r$ are proper subspaces of V^* containing V_n^1. Choose $f_i \in V_n$ for each $1 \leqq i \leqq r$ such that f_i is annihilated by W_i. Now there is a point $a \in F^s$ such that $f_1 \cdots f_r$ does not vanish at a, hence we see that $\phi(F^s) \not\subseteq W_1 \cup \cdots \cup W_r$. This enables us to inductively choose a sequence of points $\{a_n\}$ as follows: choose a_n such that for all $i \leqq n$ and all $X \subseteq \{a_1, \cdots, a_{n-1}\}$, if $\langle V_i^1, \phi(X) \rangle \neq V^*$, then $\phi(a_n) \notin \langle V_i^1, \phi(X) \rangle$.

It can now be seen that no nonzero polynomial of degree n vanishes at any $r(n) + n$ of these points. Now for each a_n, we have the maximal ideal in R consisting of all polynomials vanishing at a_n. Let $\mathscr{M}$ be this set of maximal ideals.

EXAMPLE 6. A similar technique will work for $R = K[x_1, \cdots, x_s]/\mathfrak{p}$, where $\mathfrak{p}$ is a nonmaximal prime ideal.

The following theorem gives some properties of slender modules which are well known in abelian group theory [3, Theorems 47.2 and 47.4].

THEOREM 3. (1) *Submodules of slender modules are slender.*

(2) *A direct sum of slender modules is slender.*

(3) *B is slender if and only if every homomorphism ϕ from $\prod_1^\infty R$ into B is continuous for the product topology and discrete topology respectively.*

(4) *If B is slender and I is a set whose cardinality is smaller than the first measurable cardinal, then every homomorphism from $\prod_I A_i$ to B is continuous for the product topology on $\prod_I A_i$ topology on B.*

Proof. (1) Obvious.

(2) Let $\phi: \prod_1^\infty A_i \to \sum_K B_k$ where each B_k is slender. Let ϕ_k be the composition of ϕ with the projection onto B_k. Since B_k is slender, each ϕ_k is continuous for the product and discrete topologies. We must show that if $\{a_n\}$ is a sequence of elements in $\prod A_i$ which converges to 0 in the product topology, then $\phi(a_n)$ is eventually 0. Since there are in any case only countably many k for which $\phi_k(a_n) \neq 0$ for some n, we may suppose that we are dealing with $\sum_1^\infty B_k$. Now for any n, $\phi^{-1}(\sum_1^n B_k) = \bigcap_{k>n} \phi_k^{-1}(0)$ which is closed in the product topology by the continuity of each ϕ_k. Since $\prod A_i = \bigcup_{n=1}^\infty \phi^{-1}(\sum_1^n B_k)$, the Baire Category Theorem implies $\phi(\prod_{i \geq m} A_i) \subseteqq \sum_1^n B_k$ for some m and n. Since $\sum_1^n B_k$ is clearly slender, the result follows.

(3) Let $\phi: \prod_1^\infty A_i \to B$, and let $\{a_n\}$ be a sequence of elements in $\prod_1^\infty A_i$ converging to 0 in the product topology. For every natural n, let $e_n \in \prod_1^\infty R$ be defined by $e_n(i) = 1$ if $i = n$, $e_n(i) = 0$ otherwise. We can define a homomorphism $f: \prod_1^\infty R \to \prod_1^\infty A_i$ such that $f(e_n) = a_n$. Now if the composition $\phi f: \prod_1^\infty R \to B$ is continuous, then $\phi(a_n) = \phi f(e_n)$ is eventually 0.

(4) (Due to Los). Let $\phi: \prod_I A_i \to B$ where I has nonmeasurable cardinality and B is slender. It is clear that if there are infinitely many A_i with $\phi(A_i) \neq 0$, then we get a contradiction to

the fact that B is slender by taking a countably infinite subset of these and considering the appropriate restriction of ϕ. Hence we see that it suffices to show that if $\phi(\sum_I A_i) = 0$, then $\phi(\prod_I A_i) = 0$. Suppose this is not the case, and choose $a \in \prod_I A_i$ with $\phi(a) \neq 0$. Now for each subset X of I, let $a_X \in \prod_I A_i$ be defined by $a_X(i) = a(i)$ if $i \in X$, $a_X(i) = 0$ otherwise. We see that the family of sets X for which $\phi(a_X) = 0$ is a σ-algebra S which contains all the finite subsets of I. It also follows from the fact that B is slender that if we have a family of disjoint subsets X_i of I, we can have $\phi(a_{X_i}) \neq 0$ for at most finitely many X_i. From this we see that the quotient algebra P/S is finite, where P is the σ-algebra of all subsets of I. By choosing an atom of P/S, we can define a nontrivial measure on P which is countably additive and takes only 0 and 1 as values, and is 0 on all finite sets. This contradicts the fact that I has nonmeasurable cardinality.

REMARKS. (1) The first measurable cardinal is at least as large as the first inaccessible cardinal [7].

(2) The nonmeasurability of $|I|$ is necessary, as is shown by the following example. Let I be a measurable cardinal and consider $\prod_I Z$. Let there be given a 0, 1-valued measure on I which is 0 on finite sets. For each $a \in \prod_I Z$, there is exactly one integer n such that the set $\{i \in I \mid a(i) = n\}$ has measure 1. Define $\phi(a) = n$. Then it is easily seen that ϕ is a homomorphism from $\prod_I Z$ to Z which is not continuous for the product and discrete topologies.

COROLLARY 2. *R is slender if and only if every projective R-module is slender. In particular, the slenderness of R can be regarded as a property of the category of R modules.*

COROLLARY 3. *If R is a subring of S such that S is a slender R-module, then S is a slender ring. In particular, this is true if R is a slender ring and S a projective R-module.*

Proof. This follows because any homomorphism of S-modules can be regarded as a homomorphism of R-modules.

COROLLARY 4. *If R is a slender ring, then so is the polynomial ring R[x], the ring of n dimensional matrices over R, and the group ring R[G], for any group G. In addition, if R is an algebra over a field K and F is an extension field of K, then $F \otimes_K R$ is slender.*

Proof. In each case, the larger ring is a free R-module, so this follows from Corollary 3.

2. **Reflexive modules.** If A is an A-module, we write $A^* = \text{Hom}\,(A, R)$. (A^* is a right R-module.) We say that A is *reflexive* if the natural map $\psi\colon A \to A^{**}$ given by $\psi(a)(f) = f(a)$ is an isomorphism.

THEOREM 4. *Let R be a ring and $\{M_i \mid i \in I\}$ a family of R-modules, with $\mid I \mid$ nonmeasurable. If R is slender, then $(\prod_I M_i)^* \approx \sum_I M_i$. If the M_i are reflexive, then $\prod_I M_i$ is reflexive if R is slender, and $\sum_I M_i$ is reflexive if R is right slender.*

Proof. The fact that $(\prod_I M_i)^* \approx \sum_I M_i^*$ when R is slender follows from Theorem 3, 4. Since we have $(\sum_I M_i)^* \approx \prod_I M_i^*$ in any case, the other assertions follow immediately.

COROLLARY 5. *If R is a right slender ring, then every projective module generated by a set with nonmeasurable cardinality is reflexive.*

Proof. Since summands of reflexive modules are reflexive, it suffices to prove the corollary for free modules, but this is immediate from the theorem and the fact that R is reflexive as an R-module.

COROLLARY 6. *Let R be a slender ring such that every non-finitely generated projective right R-module is free. Let $\mid I \mid$ be nonmeasurable and $\prod_I R = A \oplus B$. Then either A is a finitely generated projective module or $A \approx \prod_K R$ with $K \subsetneqq I$.*

Proof. Since summands of reflexive modules are reflexive, $A^{**} \approx A$ by Theorem 4. We have $(\prod_I R)^* \approx A^* \oplus B^*$, so A^* is projective since $(\prod_I R)^*$ is free. If A^* is finitely generated, then A^{**} is a finitely generated projective module. Otherwise, $A^* \approx \sum_K R^*$ and $A^{**} \approx \prod_K R$.

3. **Dedekind domains.** In this section, R will be a Dedekind domain and $\mathscr{M}$ will always be the set of nonzero ideals in R. When R is slender and $\mathscr{M}$ is countable, we will be able to completely characterize slender R-modules, as was done by Nunke [5] and [6], for the case $R = Z$. We will use some topological observations to avoid the homological machinery involved in Nunke's proof.

Throughout this section, we will write $P = \prod_1^\infty R_i$, with $R_i = R$. We will write $U_n = \prod_{i \geq n} R_i$ (considered as a submodule of P), so that the U_n form a neighborhood basis at 0 for the product topology on P. The supremum of the $\mathscr{M}$-adic and product topologies on P will again be called the strong topology. We recall that P is complete in the strong topology.

THEOREM 5.   *Let R be a Dedekind domain and $P = \prod_1^\infty R_i$, $R_i = R$. If A is a submodule of P which is closed in the product topology, then $A \approx \prod N_i$, where each N_i is a projective module. In particular, if rank A is finite, then A is projective. If R is slender and A has infinite rank, then $A \approx P$.*

Proof. For each n, we let $V_n = A \cap U_n$ and we let π_n be the projection of P onto R_n. Then the restriction of π_n to V_n has kernel V_{n+1} and $\pi_n(V_n)$ is projective (as a submodule of R_n). Hence we have $V_n = N_n \oplus V_{n+1}$ where N_n is projective, i.e., we get $A = N_1 \oplus N_2 \oplus \cdots \oplus N_n \oplus V_{n+1}$. It is readily seen that this gives us an isomorphism from A to $\prod N_i$. Now if there are only finitely many nonzero N_i, then A is finite rank projective. Otherwise, if R is slender, we have $A \approx (\prod N_i)^{**} \approx (\sum N_i^*)^* \approx (\sum R)^* \approx \prod R$, because nonfinitely generated projective modules are free.

COROLLARY 7.   *If R is slender and A is a countably generated module, then A^* is reflexive and A^{**} is projective.*

Proof. A^* is submodule of $\prod_K R$ which is closed in the product topology, where K is a set of generators for A. The result now follows from Theorems 4 and 5.

We now proceed to a sequence of lemmas leading directly to the characterization of slender modules.

LEMMA 2.   *Let A be an R-module with a metrizable topology. If A is complete and B a closed submodule, then the quotient topology on A/B is complete.*

Proof. [2, §3, Prop. 4].

COROLLARY 8.   *If $\mathscr{M}$ is countable and $\{A_i \mid i = 1, 2, \cdots\}$ is any countable family of R modules, then the reduced part of $\prod A_i / \sum A_i$ is $\mathscr{M}$-adically complete.*

Proof. $\prod A_i$ is complete in the strong topology. The $\mathscr{M}$-adic closure K of $\sum A_i$ in A_i consists of those x such that, for every $\mathfrak{m} \in \mathscr{M}$, $x(i) \in \mathfrak{m}A_i$ for sufficiently large i. It is evident that this is also the closure of $\sum A_i$ in the strong topology and that $K/\sum A_i$ is divisible. Hence $\prod A_i/K$ is the reduced part of $\prod A_i/\sum A_i$. Also, because K is dense in $\prod A_i$ in the product topology, the topology induced on $\prod A_i/K$ by the strong topology on $\prod A_i$ is the same as the $\mathscr{M}$-adic topology on this module. Hence the result follows from Lemma 2.

LEMMA 3. *If $\mathcal{M}$ is countable and A is a submodule of P such that P/A is torsion free and $(P/A)^* = 0$, then the reduced part of P/A is $\mathcal{M}$-adically complete.*

Proof. Because P/A is torsion free, the reduced part of P/A is isomorphic to P/K, where K is the $\mathcal{M}$-adic closure of A in P. So without loss of generality, we may assume that A is $\mathcal{M}$-adically closed. Now $P/(A + U_n)$ is a finitely generated quotient of P/A. If this is not a torsion module, then it has a projective summand and it follows that $(P/A)^* \neq 0$. Hence for each n there is a nonzero ideal $\mathfrak{m}_n$ such that $\mathfrak{m}_n P \subseteq A + U_n$. This means that the $\mathcal{M}$-adic topology on P/A is also the quotient topology that P/A inherits when P is given the strong topology. Thus the result follows from Lemma 2.

REMARK. Corollary 8 and Lemma 3 fail if $\mathcal{M}$ is uncountable. In fact, let A be the $\mathcal{M}$-adic closure in P of $\sum_1^\infty R$. If R is slender, then clearly $(P/A)^* = 0$. Now choose $a \in P \setminus A$, and let $\mathfrak{m}_0$ be a maximal ideal. For each set of ideals $\mathfrak{m}_1^{k_0}, \cdots, \mathfrak{m}_n^{k_n}$, where $\mathfrak{m}_1, \cdots, \mathfrak{m}_n$ are maximal ideals distinct from $\mathfrak{m}_0$, consider the set of x such that $x \equiv a \pmod{\mathfrak{m}_0^{k_0}}$ and $x \in \mathfrak{m}_i^{k_i}$, $i = 1, \cdots, n$. These sets form a $\mathcal{M}$-adically Cauchy filter in P/A, and hence, if P/A is complete, converge to a point x such that $x \neq 0$, and $x \in \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ distinct from $\mathfrak{m}_0$. Then if y is a representative in P for x, we have $y \notin A$, and every maximal ideal $\mathfrak{m} \neq \mathfrak{m}_0$, almost all the coordinates of y belong to $\mathfrak{m}$. If $\mathcal{M}$ is uncountable, this implies that almost all the coordinates of y must be 0, which is a contradiction.

LEMMA 4. *Let R be slender and $A \subseteq P$, and let A'' be the set of $x \in P$ such that $f(x) = 0$ for every $f \in P^*$ for which $f(A) = 0$. Then A'' is a summand of P and $(A''/A)^* = 0$.*

Proof. We follow Nunke's proof. The fact that $(A''/A)^* = 0$ follows easily once we know that A'' is a summand. Let B be the image of the natural map $P^* \to A^*$. By Theorem 4, P^* is countably generated, hence B is countably generated. Now $B \subseteq A^* \subseteq \prod_A R$. Just as for abelian groups [3, Theorem 47.1], it is readily shown that these facts imply that B is projective. Thus the sequence

$$0 \longrightarrow (P/A)^* \longrightarrow P^* \longrightarrow B \longrightarrow 0$$

is split exact. Dualizing, we get a split exact sequence

$$0 \longrightarrow B^* \longrightarrow P \longrightarrow (P/A)^{**} \longrightarrow 0 .$$

But it is easily seen that the image of B^* in P is exactly A''.

LEMMA 5. *Let R be slender and $\mathcal{M}$ countable. If a torsion free module B is a homomorphic image of P, then the reduced part of B is the direct sum of a $\mathcal{M}$-adically complete module and a module which is either finite rank projective or isomorphic to P.*

Proof. Let A be the kernel of $\phi: P \to B$. We have $P = A'' \oplus C$ (Lemma 4) where C is either finite rank projective or isomorphic to P (Corollary 6). Hence $B \approx (A''/A) \oplus C$. Since $(A''/A)^* = 0$, the result follows from Lemma 3.

THEOREM 6. *Let R be a slender Dedekind domain such that the set $\mathcal{M}$ of nonzero ideals in R is countable. Then an R-module B is slender if and only if B is reduced, torsion free, and contains no subgroup which is isomorphic to P or $\mathcal{M}$-adically complete.*

Proof. Sufficiency. By Theorem 3, it suffices to show that if $\phi: P \to B$, then ϕ is continuous for the product and discrete topologies. From Lemma 5 and the hypothesis, the image of ϕ must be finite rank projective and hence, by Corollary 2, slender, so the result follows immediately.

Necessity. If B is not torsion free, then we have a submodule of the form $R/\mathfrak{m}$ with $\mathfrak{m}$ maximal, and if B is not reduced, then it contains a submodule isomorphic to the quotient field Q of R, and neither of these is slender. Clearly B is not slender if it contains a submodule isomorphic to P. Finally, suppose B has an $\mathcal{M}$-adically complete submodule, and let $\{b_n\}$ be a sequence of elements in this submodule which $\mathcal{M}$-adically converges to 0. Then we define $\phi: P \to B$ by $\phi(x) = \sum_1^\infty x(i)b_i$, and ϕ fails to be continuous.

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