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ON TWO CONGRUENCES FOR PRIMALITY

M. V. SUBBARAO

In this paper we consider the congruences

$$n\sigma(n) \equiv 2(\operatorname{mod} \varphi(n)) , \quad \varphi(n)t(n) + 2 \equiv 0 \,(\operatorname{mod} n) .$$

1. Introduction. Apart from the classical Wilson's theorem (that a positive integer $p > 1$ is a prime if and only if $(p-1)! + 1 \equiv 0 \,(\operatorname{mod} p)$) and its variants and corollaries, there is probably no other simple primality criterion in the literature in the form of a congruence. In this connection, we may recall Lehmer's congruence [1]:

$$n - 1 \equiv 0 \operatorname{mod} \phi(n) . \tag{1.1}$$

This is satisfied by every prime. We do not yet know if it has any composite n as a solution. In 1932, Lehmer [1] showed that if there exists a composite number n satisfying (1.1), then n must be odd and square free and have at least seven distinct prime factors. This result was improved in 1944 by Fr. Schuh [4] who showed that such a n must have at least eleven prime factors. In 1970, E. Lieuwens [2] corrected an error in the proof of Schuh.

In the congruences we shall consider,

$$n\sigma(n) \equiv 2(\operatorname{mod} \phi(n)) \tag{1.2}$$

and

$$\phi(n)t(n) + 2 \equiv 0(\operatorname{mod} n) , \tag{1.3}$$

where $\phi(n)$ is Euler's totient, and $t(n)$ and $\sigma(n)$ are respectively the number and sum of the divisors of n. Each of these is satisfied whenever n is a prime. It is a simple matter to solve (1.2) completely (Theorem 1). However, the problem of solving (1.3) for all composite integers n seems to be a deep one, and we offer only a partial solution.

2. THEOREM 1. *The only composite numbers n satisfying* (1.2) *are $n = 4$, 6, and* 22.

Proof. Let a solution of (1.2) be

$$n = 2^a p_1^{a_1} \cdots p_r^{a_r}$$

where $p_1, \cdots, p_r$ are the distinct odd prime divisors of n. If for some $i(1 \leqq i \leqq r)$, $a_i > 1$, then $p_i \mid \phi(n)$ and $p_i \mid n$, so that $p_i \mid 2$, an absurdity. Hence

$$a_1 = a_2 = \cdots = a_r = 1 .$$

An analogous argument shows that $a = 0, 1$ or 2. Hence $n = 2^a p_1 p_2 \cdots p_r$, where $a = 0, 1$ or 2. Next, when n is in this form, $2^r \mid \sigma(n)$ and $2^r \mid \phi(n)$, so that we should have $2^r \mid 2$, on using the congruence. Hence $r = 0$ or 1, and we get $n = 2, 4, p_1, 2p_1, 4p_1$ for the possible solutions of (1.2). However, $n = 4p_1$ is impossible, for otherwise $4 \mid \phi(n)$, and this would imply, on using the congruence, that $4 \mid 2$.

In the next place, if $n = 2p_1$, we have

$$6p_1(p_1 + 1) \equiv 2 \bmod (p_1 - 1) .$$

This shows that $(p_1 - 1) \mid 10$, and this gives $p_1 = 2, 3$, and 11. Hence all the possible composite solutions of (1.2) are $n = 4, 6$, and 22, and these are indeed solutions of the congruence.

3. The solution of congruence (1.3). Up to 100,000, the only composite solution of (1.3) is $n = 4$, and the question naturally arises if there is any composite solution > 4. While this is still open, we devote the rest of the paper to obtain some information about such a solution if it exists.

THEOREM 2. *Every composite solution $n > 4$ of the congruence* (1.3) *satisfies the following conditions*:

(A) *n is square-free.*

(B) *If p is an odd prime divisor of n, then there is no prime divisor of the form $px + 1$.*

(C) *Let K be defined by the relation*

$$\phi(n)t(n) + 2 = Kn . \tag{3.1}$$

Then K and n are of opposite parity and $4 \nmid K$.

(D) *If $n = m$ is a solution of* (1.3), *then $n = 2m$ is not a solution.*

Proof. For an odd prime p, if $p^2 \mid n$, then $p \mid \phi(n)$; hence on using (1.2), $p \mid 2$, which is absurd. Again if $4 \mid n$ and $n > 4$, a simple argument shows that (1.3) is impossible. This establishes result (A). The proofs of (B), (C), and (D) are equally easy.

LEMMA. *For a given r, the number of solutions n of* (2.11) *having r prime divisors is finite. In fact, if $p_1, p_2, \cdots, p_r$ are the prime divisors of n in increasing order of magnitude, and if*

$$Q_r = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{q_r}\right) \tag{3.2}$$

where q_r *is the* rth *prime in the sequence of primes* 2, 3, 5, $\cdots$ ($q_1 = 2$, $q_2 = 3$ etc.), *then*

$$2^r Q_r \leqq K \leqq 2^r\,, \tag{3.3}$$

$$p_1 < r\left(1 - \frac{K}{2^r}\right)^{-1}, \tag{3.4}$$

and for $i = 2, 3, \cdots, r$,

$$p_{i-1} < p_i < (r - i + 1)\left(1 - \frac{K}{2^r} - \frac{1}{p_1} - \cdots - \frac{1}{p_{i-1}}\right)^{-1}.$$

Proof. The relation (3.1) gives

$$\begin{aligned} K &= \frac{\phi(n)t(n)}{n} + \frac{2}{n} \\ &\leqq t(n) + \frac{2}{n}\,, \end{aligned}$$

for $n > 2$. Hence $K \leqq t(n)$. Since by Theorem 2, n is square free, $n = p_1, p_2, \cdots, p_r$, so that $t(n) = 2^r$. Hence $K \leqq 2^r$.

In the next place,

$$\begin{aligned} K &> 2^r \frac{\phi(n)}{n} \\ &= 2^r \prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right) \geqq 2^r Q_r\,. \end{aligned}$$

This completes the proof of (3.3). To prove (3.4), we note that

$$\begin{aligned} K &> 2^r \frac{\phi(n)}{n} = 2^r \prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right) \\ &> 2^r\left(1 - \frac{1}{p_1} - \cdots - \frac{1}{p_r}\right). \end{aligned}$$

Hence,

$$1 - \frac{K}{2^r} < \frac{1}{p_1} + \cdots + \frac{1}{p_r} < \frac{r}{p_r}\,,$$

and this gives

$$p_1 < r\left(1 - \frac{K}{2^r}\right)^{-1}.$$

Again, using

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_r} < \frac{1}{p_1} + \frac{r-1}{p_2}$$

and proceeding as before, we get

$$p_1 < p_2 < (r-1)\left(1 - \frac{K}{2^r} - \frac{1}{p_1}\right)^{-1}. \tag{3.5}$$

Continuing this process, we obtain

$$p_2 < p_3 < (r-2)\left(1 - \frac{K}{2^r} - \frac{1}{p_1} - \frac{1}{p_2}\right)^{-1}, \tag{3.6}$$

and finally,

$$p_{r-1} < p_r < \left(1 - \frac{K}{2^r} - \frac{1}{p_1} - \cdots - \frac{1}{p_{r-1}}\right)^{-1}. \tag{3.7}$$

This establishes (3.4).

For a given r, (3.3) shows that K can take only a finite number of values, and (3.4)–(3.7) show that $p_1, p_2, \cdots, p_r$ can take only a finite number of values. Thus for a given r, the congruence (1.3) has got only a finite number of solutions, since for a given r the upper and lower bounds for $K, p_1, p_2, \cdots, p_r$ are fixed by the relations (3.3) and (3.4). The actual solutions corresponding to any given r can be obtained after a finite number of trials. Following this method, we have obtained the following results. (The details of the numerous computations involved in the proofs of Theorems 3 and 4 below are available with the authors.)

THEOREM 3. *Any composite solution* $n > 4$ *of* (1.3) *must have at least* 4 *distinct odd prime factors.*

THEOREM 4. *For the congruence* (1.3) *we have the following*:

(3.8) *If* $K = 1$ *or* $3 \leqq K \leqq 14$, *there are no solutions.*
(3.9) *If* $K = 2$, *the only solutions are all the primes and* 4.
(3.10) *If* $K = 15$, *then* $r = 4$ *or* 5.
(3.11) *If* $17 \leqq K \leqq 29$, *then* $r = 5$.
(3.12) *If* $K = 30$ *or* 31, *then* $r = 5$ *or* 6.
(3.13) *If* $33 \leqq K \leqq 58$, *then* $r = 6$.
(3.14) *If* $59 \leqq K \leqq 63$, *then* $r = 6$ *or* 7.
(3.15) *If* $65 \leqq K \leqq 116$, *then* $r = 7$.
(3.16) *If* $117 \leqq K \leqq 127$, *then* $r = 7$ *or* 8.
(3.17) *If* $129 \leqq K \leqq 230$, *then* $r = 8$.
(3.18) *If* $231 \leqq K \leqq 255$, *then* $r = 8$ *or* 9.
(3.19) *If* $257 \leqq K \leqq 457$, *then* $r = 9$.
(3.20) *If* $458 \leqq K \leqq 551$, *then* $r = 9$ *or* 10.
(3.21) *If* $513 \leqq K \leqq 909$, *then* $r = 10$.
(3.22) *If* $910 \leqq K \leqq 1023$, *then* $r = 10$ *or* 11.

Proof. We illustrate the proof for the case when n is odd. Using the lemma, we have

$$\begin{aligned} 2^r \geqq K &> 2^r\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right) \\ &> 2^r\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{17}\right)\left(1-\frac{1}{23}\right)\cdots\left(1-\frac{1}{p_r}\right), \end{aligned}$$

on using part (B) of Theorem 2 and Theorem 3. Giving K successive integral values and examining the consistency of the resulting inequalities while keeping in view the restrictions of Theorem 2, we get the results of the theorem.

REMARK. Any solution n of (3.1) satisfies the relation

$$2^r < \frac{6480}{19019} K e^{\gamma} \log x (1 + \log^{-2} x)$$

where γ is Euler's constant, r is the number of distinct prime factors of n and $x = q_{r+5}$. To show this, we note that

$$\begin{aligned} 2^r &= t(n) < K\frac{n}{\phi(n)} \\ &< K\left(1-\frac{1}{3}\right)^{-1}\left(1-\frac{1}{5}\right)^{-1}\left(1-\frac{1}{17}\right)^{-1}\left(1-\frac{1}{23}\right)^{-1}\prod_{10\leqq i\leqq r+5}\left(1-\frac{1}{q_i}\right)^{-1}, \end{aligned}$$

on using Theorems 2 and 3. Hence

$$2^r < K\cdot\frac{1}{2}\cdot\frac{6}{7}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdot\frac{18}{19}\cdot Q_{r+5}^{-1}$$

where Q_{r+5} is defined as in (3.2). We now use the estimate given by Rosser and Schoenfeld [3, Theorem 8, Corollary 1] for Q_{r+5}^{-1}, namely $Q_{r+5}^{-1} < e^{\gamma} \log x(1 + \log^{-2} x)$, where $x = q_{r+5}$; and obtain the stated result.

In the next theorem, q_u denotes, as already noted, the uth prime in the sequence of primes $q_1 = 2$, $q_2 = 3, \cdots$.

THEOREM 5. *Let K and m be given and let q_u be the smallest prime factor of n which is a solution of the simultaneous equations*

$$\phi(n)t(n) + 2 = Kn \tag{3.8}$$

$$t(n) = mK\,. \tag{3.9}$$

Then n has a prime factor at least as large as

$$q_u^m + O(u^m \exp - \log^b u)$$

where b is any number $< 3/5$.

Proof. By Theorem 2, n is square free. Let it have r distinct prime divisors.

Then A. Walfisz [5, Satz 4, p. 187] has shown that if $\pi(x)$ denotes, as usual, the number of primes $\leqq x$, and

$$li\, x = \int_2^x \frac{dt}{\log t},$$

then

$$\pi(x) = li(x) + O(x\{\exp - A \log^{3/5} x(\log\log x)^{-1/5}\}),$$

where A is a positive constant. It follows that

$$\pi(x) = li(x) + O(x \exp - \log^a x)$$

for all $a < 3/5$. By using a standard argument, we can show that

$$\sum_{q \leqq x} \frac{1}{q} = \log\log x + c + O(\exp - \log^a x),$$

q varying over primes.

It follows that

$$\begin{aligned}\sum_{q \leqq x} - \log\left(1 - \frac{1}{q}\right) &= \sum_{q \leqq x} \frac{1}{q} + \sum_{q} \left\{-\log\left(1 - \frac{1}{q}\right) - \frac{1}{q}\right\} + O\left(\frac{1}{x}\right) \\ &= \log\log x + c + O(\exp - \log^a x)\end{aligned}$$

for all $a < 3/5$, where c is an absolute constant (not necessarily the same as the c used before).

Hence for any given h for which $h = O(x^m)$, we have

$$\begin{aligned}(3.10)\qquad &\sum_{x \leqq q \leqq x^m + h} - \log\left(1 - \frac{1}{q}\right) \\ &= \log\log(x^m + h) - \log\log x + O(\exp - \log^a x)\end{aligned}$$

for all $a < 3/5$. If we choose $h = x^m \exp(-\log^b x)$, where $b < a < 3/5$, we get

$$\begin{aligned}\sum_{x \leqq q \leqq x^m + h} - \log\left(1 - \frac{1}{q}\right) &= \log m + \frac{\exp - \log^b x}{m \log x} \\ &+ O\left\{\frac{\exp - 2\log^b x}{\log x} + O(\exp - \log^a x)\right\},\end{aligned}$$

and this is greater than $\log m$ for all sufficiently large x. Again, if we take $h = -x^m \exp(-\log^b x)$ where $b < a < 3/5$, then

$$\sum_{x \leqq q \leqq x^{m}+h} - \log\left(1 - \frac{1}{q}\right) = \log m - \frac{\exp(-\log^{b} x)}{m \log x} + O\left(\frac{\exp(-2\log^{b} x)}{\log x}\right) + O(\exp(-\log^{a} x)),$$

which is less than $\log m$ for all sufficiently large x. Hence, if $g(x)$ is the smallest number such that

$$\sum_{x \leqq q \leqq g(x)} - \log\left(1 - \frac{1}{q}\right) \geqq \log m,$$

then $g(x) = x^m + O(x^m \exp(-\log^a x))$ for all $a < 3/5$. Now going back to the relation

$$2^r \phi(n) + 2 = Kn .$$

This gives, with $m = 2^r/K$, the result

$$m + 2/\phi(n) = n/\phi(n) .$$

Taking q_u to be the smallest prime divisor of n, let the integer v be defined to be the smallest integer with the property

$$m < \prod_{i=u}^{v} \frac{q_i}{q_i - 1}$$

that is,

$$\sum_{q_u \leqq q \leqq q_v} - \log\left(1 - \frac{1}{q}\right) > \log m .$$

Then it follows that n must have a prime factor other than q_u and at least as large as q_v. The previous investigation shows that.

$$q_v = q_u^m + O(q_u^m \exp(-\log^a (q_u^m))),$$

that is,

$$q_v = q_u^m + O(u^m \exp(-\log^b u)) \text{ for any } b < a < 3/5 .$$

Hence, we have proved the theorem.

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