

CONSTRUCTIONS OF TWO-FOLD BRANCHED COVERING SPACES

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To Deane Montgomery

By equivariantly pasting together exteriors of links in S^3 that are invariant under several different involutions of S^3 , we construct closed orientable 3-manifolds that are two-fold branched covering spaces of S^3 in distinct ways, that is, with different branch sets. Sufficient conditions are given to guarantee when the constructed manifold M admits an induced involution, h , and when $M/h \cong S^3$. Using the theory of characteristic submanifolds for Haken manifolds with incompressible boundary components, we also prove that doubles, $D(K, \rho)$, of prime knots that are not strongly invertible are characterized by their two-fold branched covering spaces, when $\rho \neq 0$. If, however, K is strongly invertible, then the manifold branch covers distinct knots. Finally, the authors characterize the type of a prime knot by the double covers of the doubled knots, $D(K; \rho, \eta)$ and $D(K^*; \rho, \eta)$, of K and its mirror image K^* when ρ and η are fixed, with $\rho \neq 0$ and $\eta \in \{-2, 2\}$.

With each two-fold branched covering map, $p: M^3 \rightarrow N^3$, there is associated a PL involution, $h: M \rightarrow M$, that induces p . There can, however, be other PL involutions on M that are not equivalent to h , but nevertheless are covering involutions for two-fold branched covering maps of M (cf. [BGM]). Our purpose, in this paper, is to introduce ways of detecting such involutions and controlling their number. We begin with compact 3-manifolds with several obvious PL involutions.

An oriented link \tilde{L} in M is 2-symmetric, if $N^3 \cong S^3$ and if $h(\tilde{L}) = \tilde{L}$. In §1, we give examples of knots and links in S^3 that are 2-symmetric in two or more ways; for example, a trefoil knot is both strongly invertible and periodic (definitions in §1). In §2, we paste the exteriors, $E(\tilde{L})$ and $E(\tilde{L}')$, of 2-symmetric links, \tilde{L} and \tilde{L}' , together along a torus-boundary component of each exterior; Proposition 2.1 gives the pasting instructions f that must be followed in order for the involutions, h and h' , of $E(\tilde{L})$ and $E(\tilde{L}')$ to extend to an involution h_f of $E(\tilde{L}) \cup_f E(\tilde{L}')$. Theorems 2.2 and 2.3 allow us to conclude, under fairly relaxed conditions, that the orbit space of h_f is S^3 .

In the remainder of §2, we apply these results to our examples of 2-symmetric links to construct two-fold branched covering spaces of S^3 . (Recall that any such branched covering space can always be constructed by suitable surgery on some strongly invertible link in S^3 ; this idea was first introduced in [Mo₁].) The construction process shows clearly that many of the manifolds produced have at least two different Heegaard splittings. Some of our covering spaces are hyperbolic manifolds (obtained by surgery on the figure-eight knot) with different Heegaard splittings; W. Jaco has asked whether a hyperbolic manifold could have this property.

In §3, we take the opposite tack and construct 3-manifolds that are two-fold branched covering spaces of S^3 in exactly one way; that is, these manifolds characterize the type of the link in S^3 over which they branch. Theorem 3.7 characterizes many doubled-knot types in this way. If, however, K is a prime, strongly invertible knot, then the two-fold branched covering space of any double of K covers S^3 in at least two distinct ways (Proposition 3.8). Along the way, we classify the strong equivalence classes of involutions on Seifert fibered manifolds of the form $(O, o; 0|0; 2k/1)$ with fibered neighborhoods of two regular fibers removed, for k a nonzero integer (Theorem 3.6). In Corollary 3.10, we characterize the knot type of a prime knot by the topological type of the two-fold branched covering space of a certain doubled knot.

We wish to thank S. Bleiler, W. B. R. Lickorish, J. Przytycki and M. Sakuma for helpful comments. We work in the PL category. One can refer to [Ja] for most basic definitions and results.

1. Examples of 2-symmetric links. Let $p: M \rightarrow S^3$ be a two-fold branched covering map, let h be the covering involution, and let B be the branch set of p . An oriented link \tilde{L} in M is *2-symmetric* if $h(\tilde{L}) = \tilde{L}$. The image $L (= p\tilde{L})$ is a disjoint union of knots and arcs. The preimage of an arc is a *strongly invertible subknot* of \tilde{L} ; that of a knot in $L \cap B$, an *invariant subknot* of \tilde{L} ; that of a component K of L such that $\text{Lk}(B, K) \equiv 1 \pmod{2}$, a *periodic subknot* of \tilde{L} . The link \tilde{L} is *periodic*, if all components of \tilde{L} are periodic simultaneously. Finally, the preimage of a component K of L such that $\text{Lk}(B, K) \equiv 0 \pmod{2}$ is an *interchangeable two-component sublink* of \tilde{L} .

Let U be a relative regular neighborhood of L in (S^3, B) . In each torus component of ∂U , we take a meridian-longitude pair (m, l) of the corresponding knot. In each spherical component of ∂U , we take a pair (m, l) of arcs such that $m \cap B = \partial m$ and $l \cap B = \partial l$; moreover, $m \cap l$ is one point, and m together with an arc of $U \cap B$ bound a disk in U . In each component of the preimage of ∂U , take a *coordinate pair* (\tilde{m}, \tilde{l}) that

projects under p onto one of the above defined pairs (m, l) . These coordinate pairs are related to the canonical meridian-longitude pairs of the components of \tilde{L} as follows.

PROPOSITION 1.1. *Assume that $M = S^3$. Then the coordinate pair (\tilde{m}, \tilde{l}) of an invariant or periodic subknot \tilde{K} of \tilde{L} is also a meridian-longitude pair of \tilde{K} . If (\tilde{m}, \tilde{l}) belongs to a component \tilde{K}_1 of an interchangeable sublink $\tilde{K}_1 \cup \tilde{K}_2$ of \tilde{L} , then \tilde{m} is a meridian of \tilde{K}_1 , and any longitude of \tilde{K}_1 is homologous to $\tilde{l} + \text{Lk}(\tilde{K}_1, \tilde{K}_2)\tilde{m}$ in $p^{-1}(\partial U)$.*

Proof. Let A be an oriented, connected 2-manifold bounded by K and transverse to B . Then $\tilde{A} (= p^{-1}A)$ is an oriented 2-manifold and $\partial\tilde{A} = \tilde{K}$. Assume that l is contained in A . If $\tilde{K} (= p^{-1}K)$ is connected, then \tilde{l} is contained in \tilde{A} and is a longitude of \tilde{K} . If \tilde{K} is not connected, then the canonical longitude \tilde{l}' of \tilde{K}_1 is homologous to $\tilde{l} + \text{Int}(\tilde{A}, \tilde{l}')\tilde{m}$ in $p^{-1}(\partial U)$. The second part of the proposition follows, since

$$\text{Int}(\tilde{A}, \tilde{l}') = \text{Lk}(\tilde{K}_1, \tilde{l}') + \text{Lk}(\tilde{K}_2, \tilde{l}') = \text{Lk}(\tilde{K}_1, \tilde{K}_2). \quad \square$$

REMARK. When $M = S^3$, we shall take the arcs (m, l) of any spherical component of ∂U so that the preimages (\tilde{m}, \tilde{l}) form a meridian-longitude pair of the corresponding strongly invertible subknot of \tilde{L} .

We are interested in links in S^3 with more than one interpretation as a 2-symmetric link. We give some examples.

(a) *A strongly invertible link that is interchangeable in two different ways.*

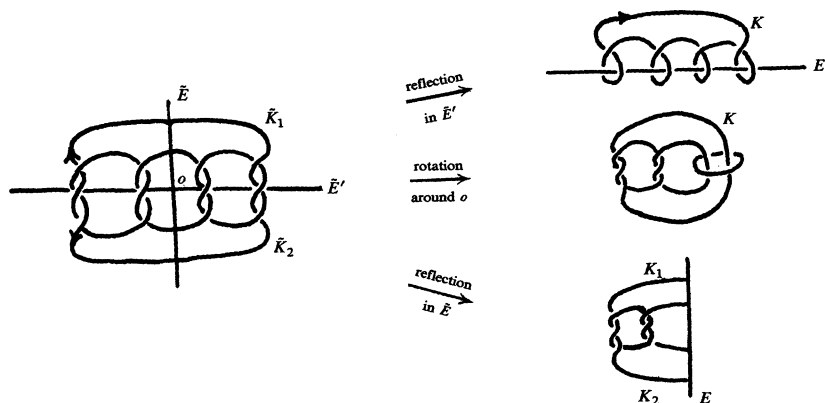


FIGURE 1

(b) *A periodic link that is interchangeable in two different ways.*

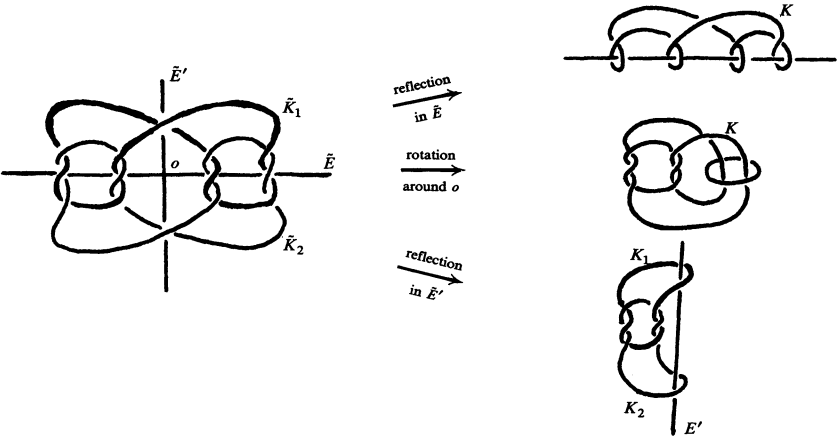


FIGURE 2

(c) *A knot that is periodic and strongly invertible.*

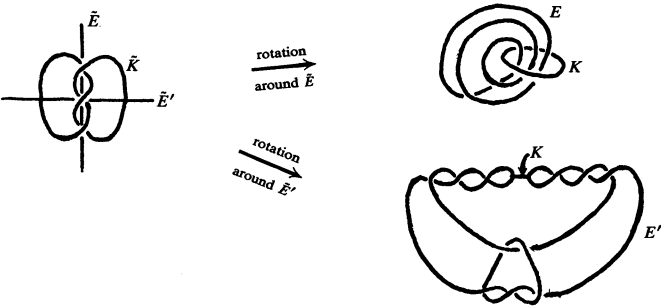


FIGURE 3

REMARK. In Figure 3 (and in the rest of the figures of this section), we depict the arc K so that its coordinates (m, l) lie in the plane of projection. We note that the two constructions of Figure 3 first appeared in [Mo₂].

(d) *An amphicheiral knot: periodic and, in two ways, strongly invertible.*

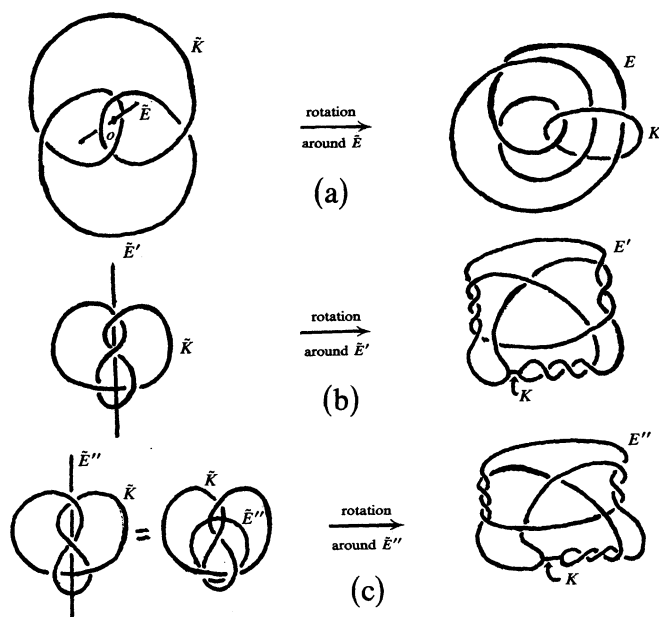


FIGURE 4

(e) $K \# K$: strongly invertible in two different ways.

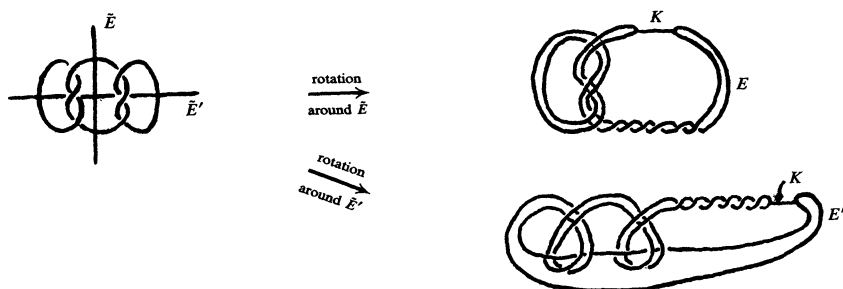


FIGURE 5

(f) A knot that is invariant, periodic, and strongly invertible.

First Example.

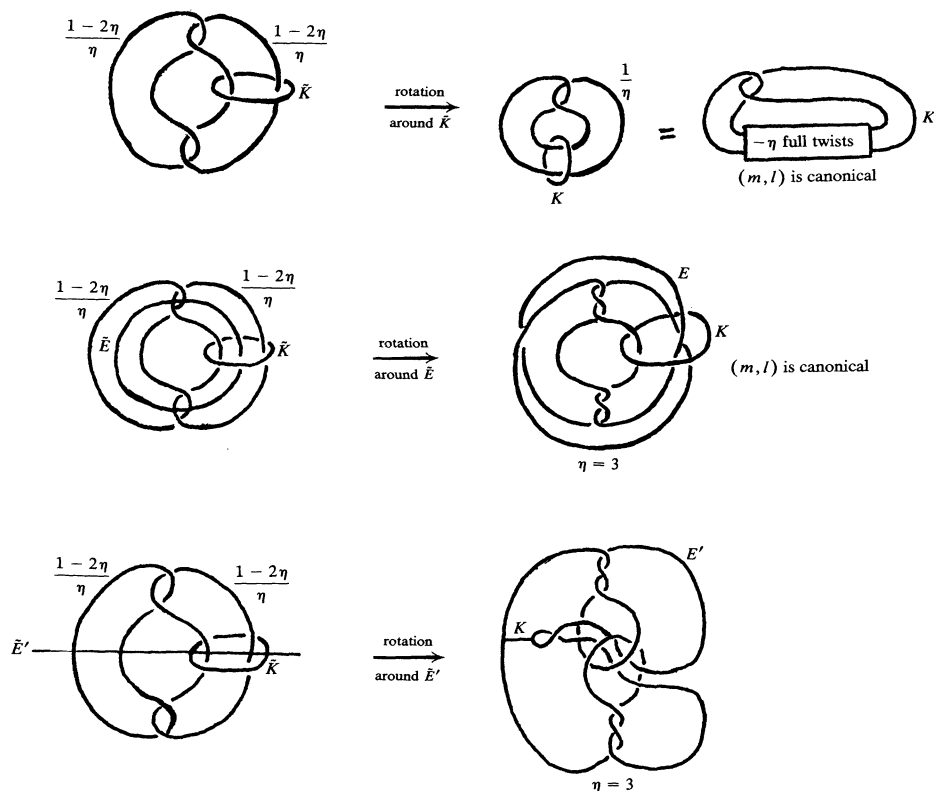


FIGURE 6

The knot \tilde{K} (see Figure 6) is contained in the manifold M obtained by surgery on a link N of two components. We take (\tilde{m}, \tilde{l}) as a meridian-longitude pair of \tilde{K} thought of as part of the link, $\tilde{K} \cup N \subset S^3$.

REMARK. For $\eta = 1$, the manifold M is the lens space $L(3, 1)$ depicted in Figure 7, which is a two-fold covering space of S^3 branched over the trefoil.

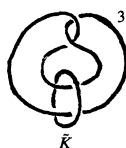


FIGURE 7

Second Example.

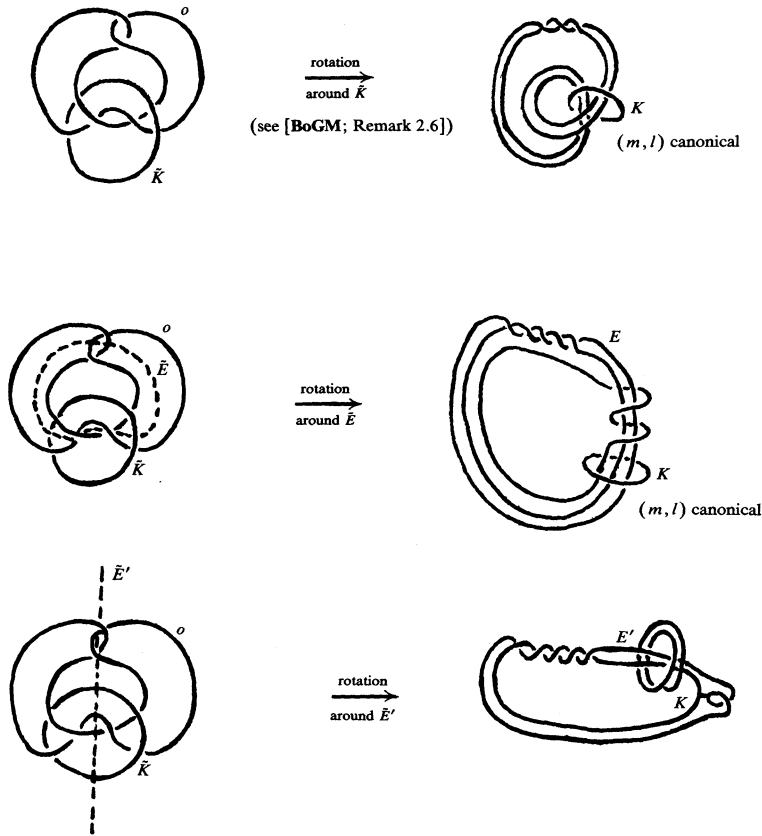


FIGURE 8

The knot \tilde{K} is contained in the manifold M obtained by surgery on the figure-eight knot N . We take (\tilde{m}, \tilde{l}) as a meridian-longitude pair of \tilde{K} thought of as part of the link $\tilde{K} \cup N$.

2. Constructions of two-fold branched coverings of S^3 . Our goal in this section is to construct two-fold branched covering spaces of S^3 by pasting together the exterior of links that are 2-symmetric in different ways. The following proposition gives the condition under which the resulting manifold admits an induced involution.

PROPOSITION 2.1. *Let $E(\tilde{L})$ and $E(\tilde{L}')$ denote the exteriors of 2-symmetric links \tilde{L} and \tilde{L}' in the manifolds M and M' , with involutions h and h' . Let \tilde{T} and \tilde{T}' be tori in $\partial E(\tilde{L})$ and $\partial E(\tilde{L}')$ with coordinates (\tilde{m}, \tilde{l}) and*

(\tilde{m}', \tilde{l}') . Form $E(\tilde{L}) \cup_{\tilde{f}} E(\tilde{L}')$, that is, the result of pasting $E(\tilde{L})$ and $E(\tilde{L}')$ together along \tilde{T} and \tilde{T}' according to a homeomorphism $\tilde{f}: \tilde{T}' \rightarrow \tilde{T}$, such that

$$(\tilde{m}, \tilde{l}) \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \sim (\tilde{f}\tilde{m}', \tilde{f}\tilde{l}')$$

on \tilde{T} ; the matrix, $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ is an integral matrix with determinant -1 . Then the involutions h and h' extend to an involution of $E(\tilde{L}) \cup_{\tilde{f}} E(\tilde{L}')$ if and only if one of the following holds:

- (i) both \tilde{T} and \tilde{T}' correspond to periodic knots and γ is even;
- (ii) \tilde{T} is periodic, \tilde{T}' is invariant, and α is even;
- (iii) \tilde{T} is invariant, \tilde{T}' is periodic, and δ is even;
- (iv) both \tilde{T} and \tilde{T}' are invariant, and β is even; or
- (v) both \tilde{T} and \tilde{T}' are strongly invertible. □

It is more difficult to decide whether the orbit space under such involutions (as in Proposition 2.1) is S^3 . The next two theorems give two interesting particular cases for which this happens.

THEOREM 2.2. *Let \tilde{L} be a 2-symmetric link in \tilde{M} such that the link part of L is trivial, that is, a collection of unlinked trivial knots. For each component \tilde{K} of \tilde{L} , let $N(\tilde{K})$ be a knot in a manifold. We assume that $N(\tilde{K})$ is 2-symmetric, except when \tilde{K} belongs to an interchangeable sublink of \tilde{L} . Identify the boundaries of $E(\tilde{L})$ and $E(N(\tilde{K}))$ by a homeomorphism, $f: \partial E(N(\tilde{K})) \rightarrow \partial U(\tilde{K})$, given homologically by*

$$(\tilde{m}, \tilde{l}) \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \sim (f\tilde{m}', f\tilde{l}')$$

on $\partial U(\tilde{K})$; the $\det \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = -1$ and we set $A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$; the set $U(\tilde{K})$ is a regular neighborhood of \tilde{K} in \tilde{M} , the pair (\tilde{m}, \tilde{l}) is a coordinate pair for \tilde{K} , and (\tilde{m}', \tilde{l}') is a coordinate pair for $N(\tilde{K})$. Assume also that

- (i) if both \tilde{K} and $N(\tilde{K})$ are periodic, then the orbit space of $E(N(\tilde{K}))$ is a solid torus and $A = \begin{pmatrix} \alpha & 2 \\ \beta & \delta \end{pmatrix}$;
- (ii) if \tilde{K} is periodic and $N(\tilde{K})$ is invariant, then $A = \begin{pmatrix} 2\alpha & 1 \\ \beta & \delta \end{pmatrix}$, if the orbit space of $E(N(\tilde{K}))$ is a solid torus, and $A = \begin{pmatrix} 0 & 1 \\ 1 & \delta \end{pmatrix}$, otherwise;
- (iii) if \tilde{K} is invariant and $N(\tilde{K})$ is periodic, then $A = \begin{pmatrix} \alpha & 1 \\ \beta & 2\delta \end{pmatrix}$, if the orbit space of $E(N(\tilde{K}))$ is a solid torus, and $A = \begin{pmatrix} 0 & 1 \\ 1 & 2\delta \end{pmatrix}$, otherwise;
- (iv) \tilde{K} and $N(\tilde{K})$ are not both invariant;
- (v) if \tilde{K} is strongly invertible, then so is $N(\tilde{K})$; and
- (vi) if \tilde{K} belongs to an interchangeable sublink $\tilde{K} \cup \tilde{K}'$ of \tilde{L} , then $N(\tilde{K}') = N(\tilde{K})$ and both matrices are equal to $\begin{pmatrix} \alpha & 1 \\ \beta & \delta \end{pmatrix}$, if $N(\tilde{K})$ is trivial in S^3 , and to $\begin{pmatrix} 0 & 1 \\ 1 & \delta \end{pmatrix}$, otherwise.

Then the resulting oriented closed 3-manifold M admits an involution whose orbit space is S^3 . \square

REMARK. The manifold M of Theorem 2.2(vi) might admit another involution arising from a second involution on \tilde{M} that interchanges $\tilde{K} \cup \tilde{K}'$. For example, one involution h on M might arise from an involution on \tilde{M} that preserves the orientation of each component of $\tilde{K} \cup \tilde{K}'$, and a second involution h' on M might arise from an involution that *reverses* the orientation of each component of $\tilde{K} \cup \tilde{K}'$. In the second case, we must also assume that $N(\tilde{K}) (= N(\tilde{K}'))$ is invertible, assuming that M itself arises from the first described involution on \tilde{M} . Generally, however, $M/h' \not\cong S^3$, as example $g(4)$ (following) shows, when $h' = h_c$ and $h = h_A$ in $g(4)$.

EXAMPLES. We have chosen the following examples to illustrate the representation of 3-manifolds as two-fold branched cyclic covering spaces of S^3 in different ways. We shall use the following notation (Figure 9), in

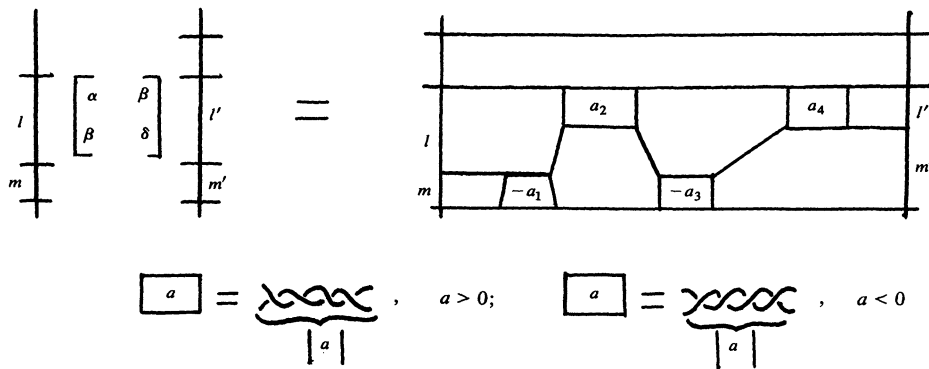


FIGURE 9

which $\det \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = +1$. Set

$$[a_1, \dots, a_k] = a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_k}},$$

and assume further that $\alpha/\beta = [a_1, \dots, a_{2n}]$ and $\gamma/\delta = [a_1, \dots, a_{2n-1}]$. Figure 9 corresponds to the case when

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} 43 & 30 \\ 10 & 7 \end{pmatrix}.$$

We also use the notation of Figure 10, with $\gamma/\delta = [a_1, \dots, a_{2n-1}]$ when $\alpha/\beta = [a_1, \dots, a_{2n}]$. In Figure 11, we depict the case $\alpha/\beta = 43/10$. We now give our examples.

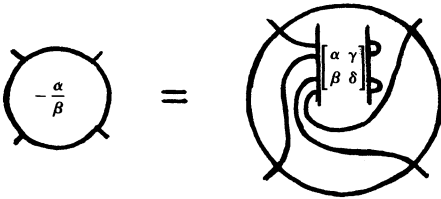


FIGURE 10

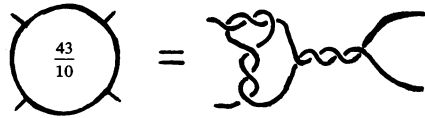


FIGURE 11

(a) Take \tilde{L} to be the link $\tilde{K}_1 \cup \tilde{K}_2$ of Figure 1, let $N(\tilde{K})$ be a trivial knot in S^3 , and take $A = \begin{pmatrix} 0 & 1 \\ 1 & \delta \end{pmatrix}$. The manifold M is a two-fold covering space of S^3 branched over the curves of Figure 12(a), because \tilde{L} is interchangeable, and over the curves of Figure 12(b), because \tilde{L} is strongly invertible.

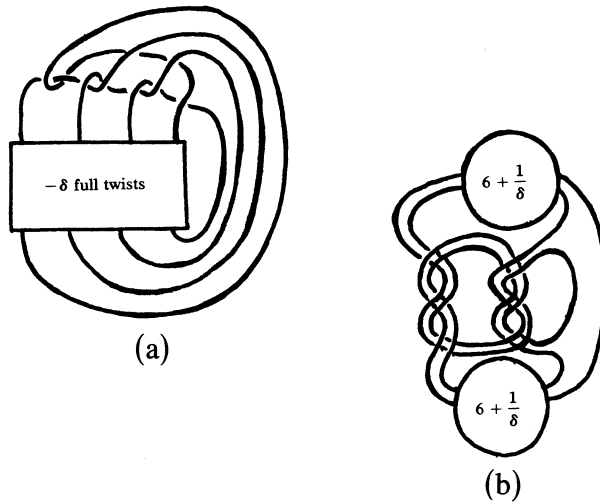


FIGURE 12

(b) Take \tilde{L} to be the (right-hand) trefoil knot of Figure 3, let $N(\tilde{K})$ be trivial in S^3 , and take $A = \begin{pmatrix} \alpha & 2 \\ \beta & \delta \end{pmatrix}$ (respectively, $A = \begin{pmatrix} 2\alpha & 1 \\ \beta & \delta \end{pmatrix}$). Then the manifold obtained by $2/\delta$ -surgery (δ odd) on the trefoil (respectively, $1/\delta$ -surgery) is a two-fold branched covering space of S^3 in the two ways depicted in Figure 13 (respectively, Figure 14) (cf. [G]).

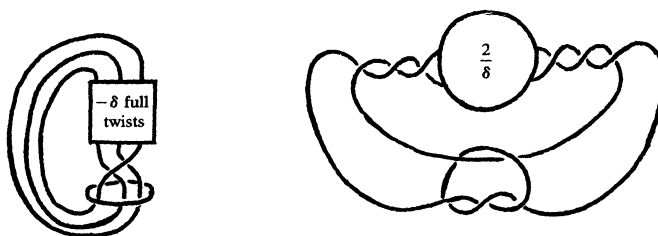


FIGURE 13

REMARK. The knot of Figure 14(a) is the $(3, 1 - 6\delta)$ -torus knot; the knot of Figure 14(b), for $\delta = -1$, is the $(-2, 3, 7)$ -pretzel knot (cf. [BGM]). The knots of Figures 14(a) and (b) are equivalent precisely when $\delta = 1$. By the same method used in [BGM], the homology 3-spheres obtained by surgery on the trefoil (except the Poincaré manifold) have at least two different Heegaard splittings of genus 2 (note that the knot of Figure 12(b) has 3 bridges).

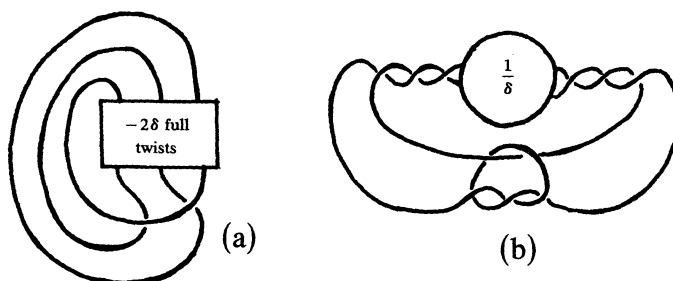


FIGURE 14

(c) Take \tilde{L} to be the figure-eight knot of Figure 4(a), let $N(\tilde{K})$ be trivial in S^3 , and take $A = \begin{pmatrix} \alpha & 2 \\ \beta & \delta \end{pmatrix}$ (respectively, $\begin{pmatrix} 2\alpha & 1 \\ \beta & \delta \end{pmatrix}$). The manifold obtained by $2/\delta$ -surgery (δ odd) (respectively, $1/\delta$ -surgery) on \tilde{L} is a two-fold branched covering space of S^3 in the way depicted in Figure 15 (Figure 16, respectively).

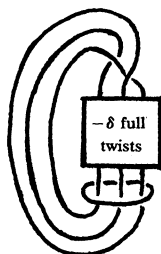


FIGURE 15

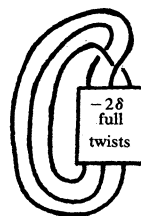


FIGURE 16

On the other hand, since the figure-eight knot is strongly invertible in two ways (Figures 4(b) and (c)), the manifold obtained by γ/δ -surgery on the figure-eight is a two-fold branched covering space of S^3 in two ways (Figure 17).

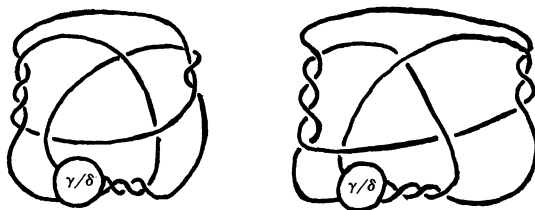


FIGURE 17

REMARKS. (1) For $\gamma/\delta = m$, the knots in Figure 17 are 3-bridge knots, and their Conway potential functions show that they are of different types, if $m \neq 0$ (S. A. Bleiler has also noted this [B1]). By the method used in [BGM], it follows that the manifold obtained by m -surgery ($m \neq 0$) on the figure-eight knot has at least two different Heegaard splittings. Since some of these manifolds are hyperbolic [Th], this answers the question of W. Jaco: Can there exist inequivalent, minimal, Heegaard splittings of a hyperbolic manifold?

Jozef Przytycki has pointed out that since the knots of Figure 17 are 3-braid knots, there exist infinitely many 3-manifolds with two different open-book decompositions of genus 1; the simplest such example is obtained by 5-surgery on the figure-eight knot.

(2) For $\gamma/\delta = -1$, the knots of Figure 17 take the form of those in Figure 18. (Note that the knot of Figure 16 (with $\delta = -1$) coincides with that of Figure 18(a).) The knot of Figure 18(b) is the left-handed $(3, 7)$ -torus knot. The manifold M is the Seifert fibered manifold $(O \circ; 0 | -1; 2, 3, 7)$ (see [M]). We also get M by a $+1$ -surgery on the left-handed trefoil, as Figure 14(b) shows (cf. [G], [BGM], and [Ta]).

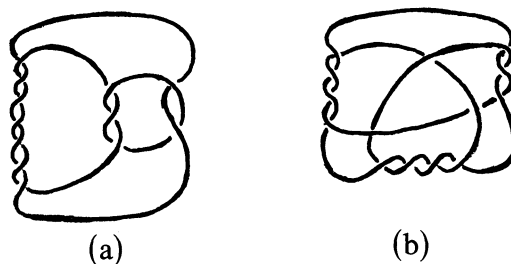


FIGURE 18

(3) Note that the manifold obtained by γ/δ -surgery on the figure-eight knot is the same as that obtained by $(-\gamma/\delta)$ -surgery.

(4) As observed in [Th], the manifolds obtained by ± 1 -, ± 2 - and ± 3 -surgery on the figure-eight knot are Seifert fibered manifolds. The one obtained by ± 4 -surgery is a graph manifold. We recognize these manifolds in Figure 19 [M].

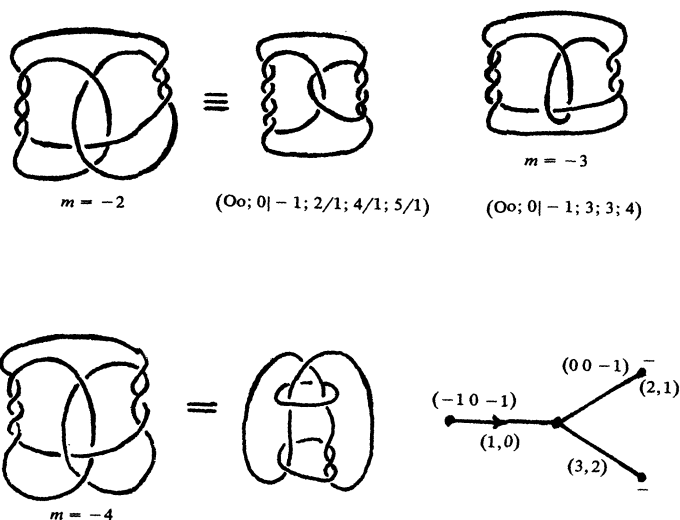


FIGURE 19

(5) For $\gamma/\delta = 1/\delta$ ($\delta \neq -1$), the manifold is the two-fold branched covering space of the knots in Figure 20, which are probably of different knot types. Because they have three local maxima, the corresponding manifold would have three different Heegaard splittings.

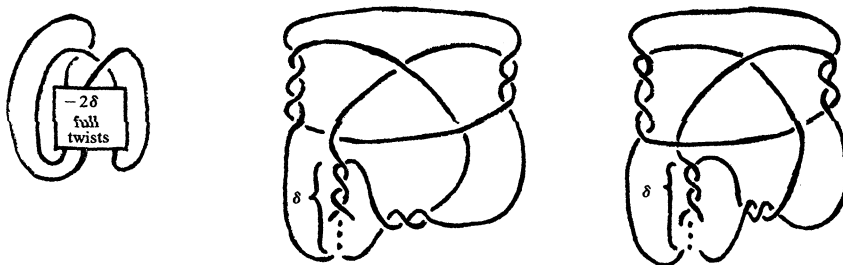


FIGURE 20

REMARK. That the knots in Figure 20 for Example c(5) are, in fact, of different type for $\delta \neq 0$ was kindly pointed out to us by M. Sakuma (see [FS]). The two authors of [FS] obtained this result independently. As noted on page 191 of [FS], part of the result was known to M. Takahashi [Ta₁].

(d) Take \tilde{L} to be the trefoil knot of Figure 3, take $N(\tilde{L})$ to be the knot \tilde{K} of Figure 6, and set $A = \begin{pmatrix} \alpha & 2 \\ \beta & \delta \end{pmatrix}$ (respectively, $A = \begin{pmatrix} 0 & 1 \\ 1 & \delta \end{pmatrix}$). The manifold M of Theorem 2.2 is a two-fold branched covering space of S^3 (Figures 21 and 22). Since both \tilde{L} and $N(\tilde{L})$ are strongly invertible, the manifold M is a two-fold branched covering space of S^3 in another form (Figure 23).

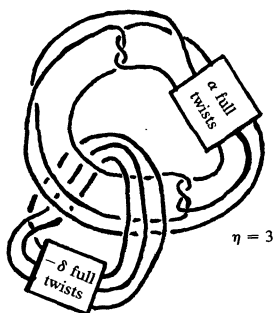


FIGURE 21

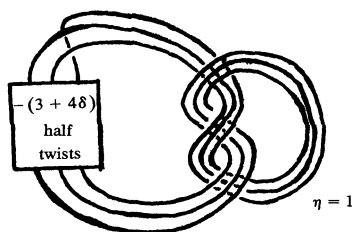


FIGURE 22

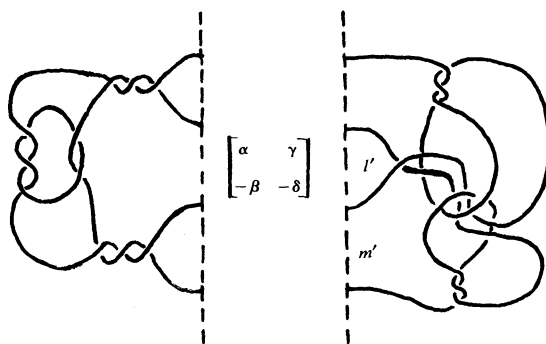


FIGURE 23

REMARK. For $\eta = 1$, the manifold M corresponding to $A = \begin{pmatrix} 0 & 1 \\ 1 & \delta \end{pmatrix}$ (Figure 22) is the result of 3-surgery on a double of the trefoil [BoGM].

(e) Take \tilde{L} to be the knot \tilde{K} of Figure 8, take $N(\tilde{L})$ to be the trivial knot in S^3 , and take $A = \begin{pmatrix} \alpha & 2 \\ \beta & \delta \end{pmatrix}$, $\begin{pmatrix} 2\alpha & 1 \\ \beta & \delta \end{pmatrix}$, or $\begin{pmatrix} \alpha & 1 \\ \beta & 2\delta \end{pmatrix}$. The corresponding manifolds M_A are two-fold branched covering spaces of S^3 , and the branch sets are depicted in Figure 24.

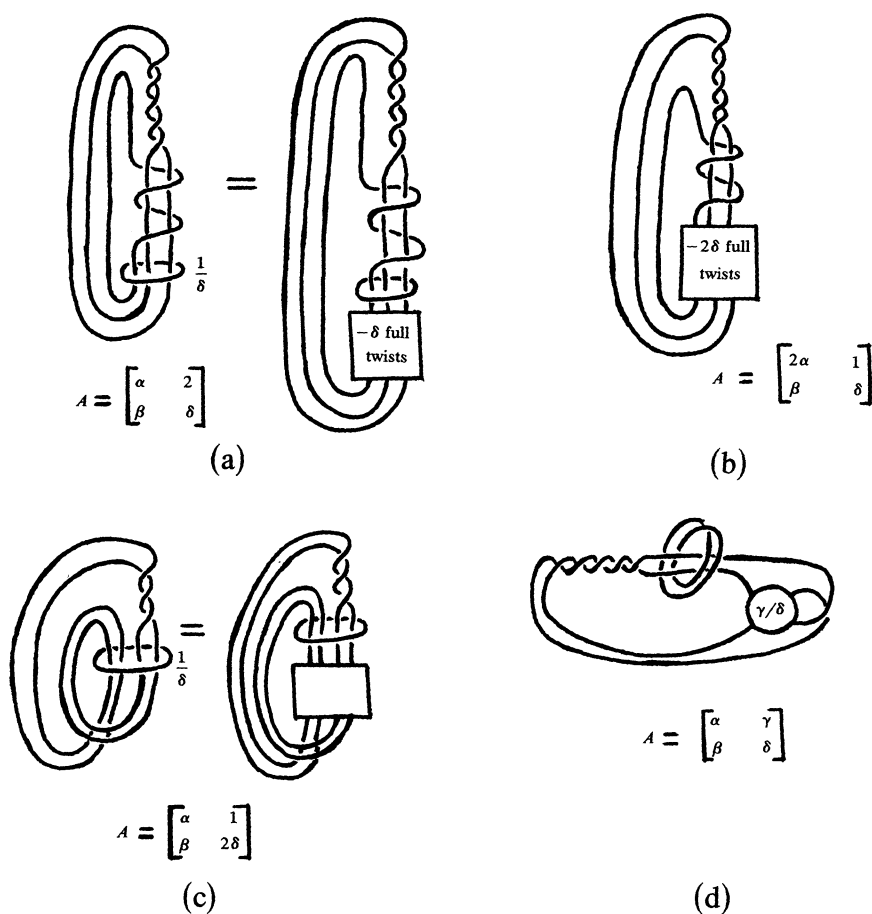


FIGURE 24

REMARKS.

(1) For $A = \begin{pmatrix} \alpha & 1 \\ \beta & 2 \end{pmatrix}$, the manifold M_A is the result of 0-surgery on the trefoil, and so M_A is a torus bundle over S^1 with periodic monodromy; in Seifert's notation, $M_A = (\text{Oo}; 0 | -1; 2, 3, 6)$ (cf, [M]). The branch set is shown in Figure 25(a). The branch set of Figure 25(b) comes from Figure 14. That these two branch sets (Figure 25) are equal was pointed out to us by Sakuma (see his paper [Sa]).



FIGURE 25

(2) For $A = \begin{pmatrix} \alpha & 1 \\ \beta & 0 \end{pmatrix}$, the manifold M_A is the result of 0-surgery on the figure-eight knot. The branch set is shown in Figure 26 in two ways (corresponding to the fact that M_A is symmetric).

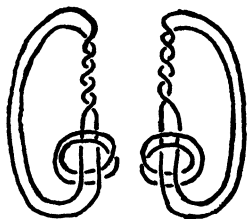


FIGURE 26

(f) Take \tilde{L} to be the knot \tilde{K} of Figure 8, let $N(\tilde{L})$ be the trefoil knot, and let A be $\begin{pmatrix} \alpha & 2 \\ \beta & \delta \end{pmatrix}$ or $\begin{pmatrix} \alpha & 1 \\ \beta & 2\delta \end{pmatrix}$. Then M_A is a two-fold branched covering space of S^3 and, in each case, is the result of 0-surgery on a double of the trefoil (cf. [BoGM]). The branch sets are shown in Figure 27(a) and (b).

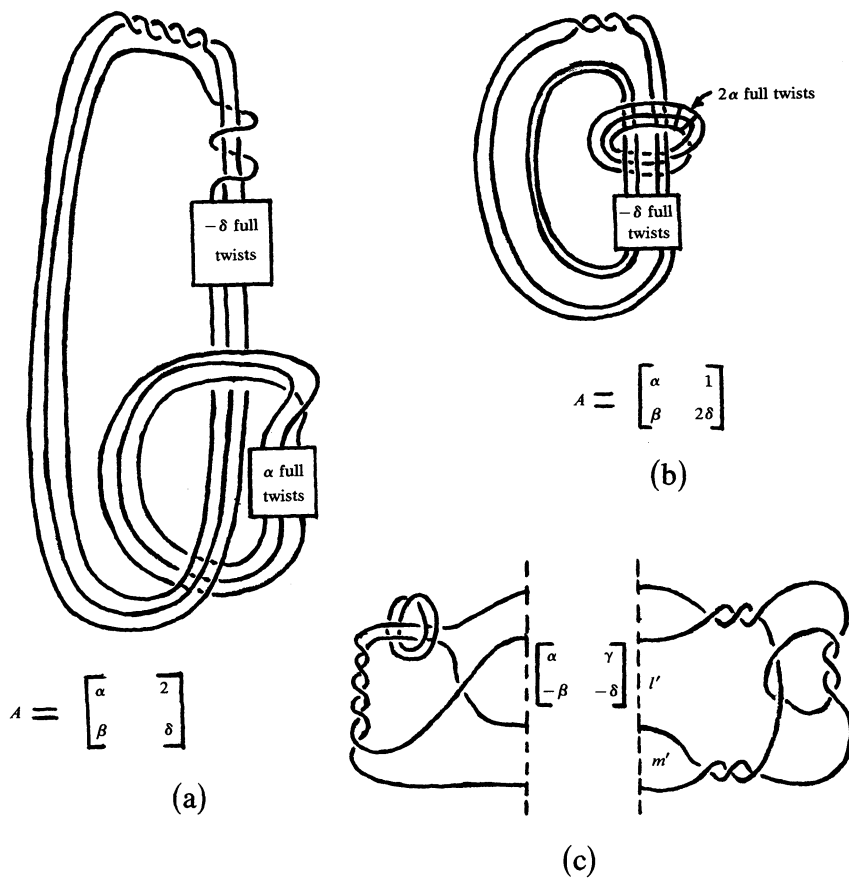


FIGURE 27

Since both \tilde{L} and $N(\tilde{L})$ are strongly invertible we get another two-fold branched covering space of S^3 , whose branch set is shown in Figure 27(c).

(g) The link $\tilde{K}_1 \cup \tilde{K}_2 (= \tilde{L} \subset S^3)$ of Figure 28 is interchangeable in two ways, strongly invertible, and periodic. Figures 28(a) and (c) show (respectively) the orbit spaces V and U of $E(\tilde{K}_1 \cup \tilde{K}_2)$ under involutions (rotations) of S^3 with fixed-point sets A and C ; the fixed-point set A is perpendicular to the plane of projection of $\tilde{K}_1 \cup \tilde{K}_2$. Figure 28(d) and the top portion of (b) show the images of $\tilde{K}_1 \cup \tilde{K}_2$ under covering maps induced by involutions (rotations) of S^3 with fixed-point set F (a circle in a plane perpendicular to the plane of projection of $\tilde{K}_1 \cup \tilde{K}_2$ and indicated by two crosses) and B , respectively. The bottom portion of Figure 28(b) shows the orbit space, $S^2 \times I$, of $E(\tilde{K}_1 \cup \tilde{K}_2)$ under an involution of S^3 with fixed-point set B . The fixed point sets A , B , and C are mutually

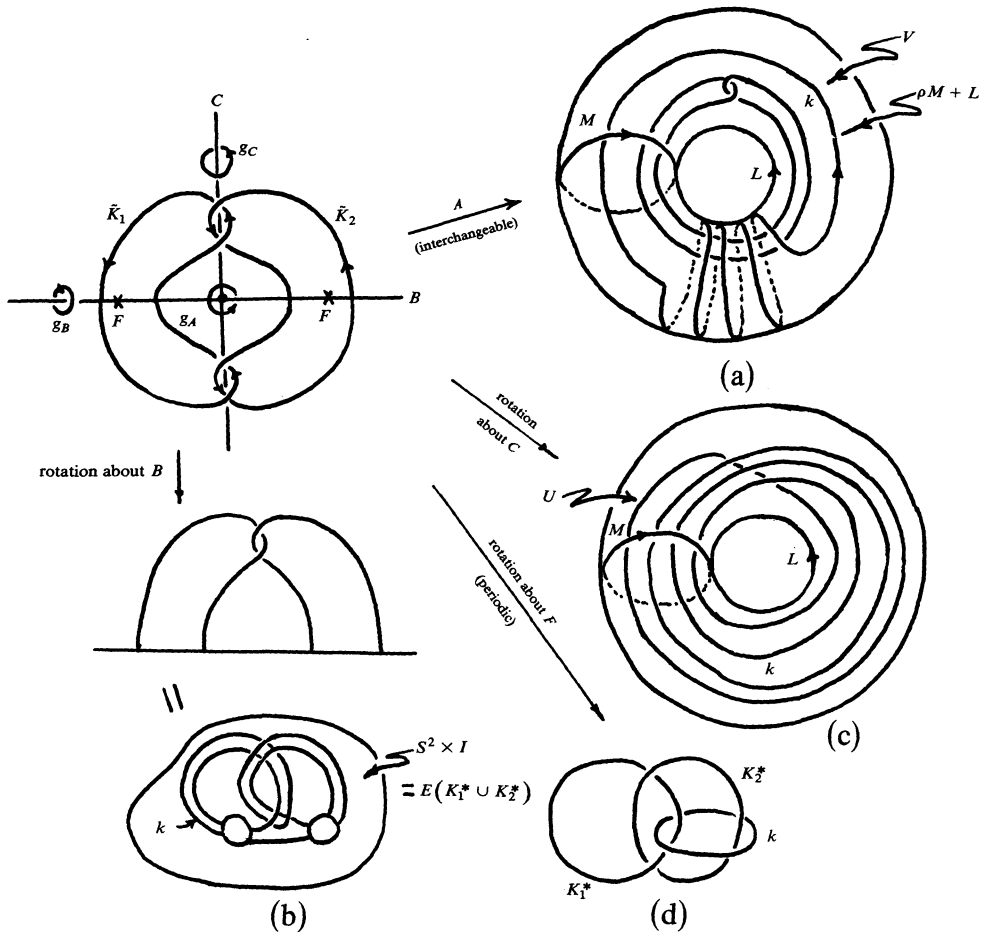


FIGURE 28

perpendicular in $R^3 \cup \{\infty\}$ ($= S^3$), and in all four cases k denotes the image of the fixed-point set under the particular covering projection. We shall denote the restrictions of the four involutions of S^3 to $E(\tilde{K}_1 \cup \tilde{K}_2)$ by g_A , g_B , g_C , and g_F ; note that $\text{Fix}(g_B)$ (the fixed-point set of g_B) is a collection of four arcs properly imbedded in $E(\tilde{K}_1 \cup \tilde{K}_2)$. Finally, we can choose tubular neighborhoods of \tilde{K}_1 and \tilde{K}_2 so that $E(\tilde{K}_1 \cup \tilde{K}_2)$ remains invariant under all four rotations of S^3 .

Set $\tilde{V} = E(\tilde{K}_1 \cup \tilde{K}_2)$, and let T_i be the boundary component of \tilde{V} corresponding to \tilde{K}_i ($i = 1, 2$); note that $g_A(T_i) = T_j$ ($i \neq j$). Let K be an oriented knot in S^3 , set $N(\tilde{K}_1) = N(\tilde{K}_2) = K$, and let (\tilde{m}', \tilde{l}') be the coordinate pair — in this case, a meridian-longitude pair — for K on $\partial E(K)$. Finally, let $E_1(K)$ and $E_2(K)$ be copies of $E(K)$ with corresponding copies $(\tilde{m}'_1, \tilde{l}'_1)$ and $(\tilde{m}'_2, \tilde{l}'_2)$ of (\tilde{m}', \tilde{l}') , and let ρ denote a fixed integer.

(1) By Theorem 2.2(vi) with pasting matrix $\begin{pmatrix} 0 & 1 \\ 1 & \rho \end{pmatrix}$, the involution g_A on \tilde{V} extends to an involution h_A on a two-fold branched covering space \tilde{D} of S^3 ; let $p: \tilde{D} \rightarrow S^3$ denote the covering map that h_A induces. The branch set pA is a doubled knot — the *double of K with twist ρ* . Set $D(K, \rho) = pA$, and write $\tilde{D} = E_1(K) \cup_{T_1} \tilde{V} \cup_{T_2} E_2(K)$.

REMARK. Note that (for fixed $\rho \neq 0$) the knot type of $D(K, \rho)$ depends on the *isotopy* type of K and that our class of doubled knots contains all doubled knots up to knot type. We only need the stricter doubled-knot notation, $D(K; \rho, \eta)$, for the statement of Corollary 3.10.

(2) Again, by Theorem 2.2(vi) with pasting matrix $\begin{pmatrix} 0 & 1 \\ 1 & \rho \end{pmatrix}$, the involution g_C on \tilde{V} extends to an involution on a two-fold branched covering space M of S^3 . The manifold M is the two-fold branched covering space of a $(4\rho + 1, 4)$ -cable about K (Figure 28(c)). We thank W. B. R. Lickorish for pointing out that in this case, it is necessary to change the orientation of \tilde{K}_1 (or \tilde{K}_2) in Figure 28 so that the *oriented* link $\tilde{K}_1 \cup \tilde{K}_2$ is interchangeable with respect to g_C . Thus, the coordinate pairs for \tilde{K}_1 (or \tilde{K}_2) (see Proposition 1.1) in (1) and (2) differ in the \tilde{l} -coordinate by $\pm 4\tilde{m}$, and so $M \cong \tilde{D}$ if and only if $M \cong S^3 \cong \tilde{D}$, which is clear from $g(4)$ (below).

(3) If K is strongly invertible, then the involution g_B extends to an involution h_B on \tilde{D} (by Theorem 2.2(v)) that induces a two-fold branched covering map $p': \tilde{D} \rightarrow S^3$. The $\text{Fix}(h_B)$ is a simple closed curve whose intersection with \tilde{V} is $\text{Fix}(g_B)$; abusing notation, we set $B = \text{Fix}(h_B)$. Hence, the branch set $p'(B)$ is a knot in S^3 (Figure 28(b)); set $D' = p'B$. By Proposition 3.8, the knot D' is not a doubled knot, when K is prime.

(4) If K is invertible, then g_C extends to an involution h_C on \tilde{D} . To obtain \tilde{D}/h_C , we attach the boundary of the solid torus U (Figure 28(c)),

to $\partial E(K)$ by a homeomorphism $\psi: \partial U \rightarrow \partial E(K)$ such that

$$(\psi M, \psi L) \sim (\tilde{m}', \tilde{l}') \begin{pmatrix} 1 + 4\rho & -\rho \\ -4 & 1 \end{pmatrix}.$$

If $\rho \neq 0$, then \tilde{D}/h_C is clearly not S^3 . If $\rho = 0$ and if we attach a 2-handle to $E(K)$ along the curve $\tilde{m}' - 4\tilde{l}'$ on $\partial E(K)$, then there is (at the present time) a possibility for obtaining a homotopy 3-cell (K might not have Property P).

(5) Suppose that g_F extends to an involution h_F on \tilde{D} with $\text{Fix}(h_F) = F$ (Figure 28). To see this involution, we consider the link $\tilde{K}_1 \cup \tilde{K}_2$ of Figure 28 as a $(2, 4)$ -torus link \tilde{L} on a standardly imbedded torus T in S^3 ; we have $E(\tilde{L}) = \tilde{V}$. The $\text{Fix}(h_F)$ is now the core of the solid torus bounded by T such that $|\text{Lk}(\tilde{K}_i, F)| = 1$ ($i = 1, 2$). The orbit space \tilde{V}/g_F ($\cong S^1 \times S^1 \times I$) is the exterior of the Hopf link $K_1^* \cup K_2^*$ of Figure 28(d). Moreover, if K is knotted, then each of $E_1(K)/h_F$ and $E_2(K)/h_F$ is a knot manifold (not a solid torus), if $\tilde{D}/h_F \cong S^3$, because the two-fold unbranched covering space of a solid torus is a solid torus. But then $E_2(K)/h_F$ would belong to the solid torus $S^3 - \text{Int}(E_1(K)/h_F)$ in a nontrivial way (not in a 3-cell in the solid torus). This is impossible [BM]; hence, if K is nontrivial and if g_F extends to \tilde{D} , then $\tilde{D}/h_F \not\cong S^3$.

(6) Since $\tilde{K}_1 \cup \tilde{K}_2$ is a $(2, 4)$ -torus link, there is one final obvious involution g_f of $E(\tilde{K}_1 \cup \tilde{K}_2)$. This is a free involution that interchanges the boundary components of $E(\tilde{K}_1 \cup \tilde{K}_2)$. We will show that later (Theorem 3.6) that the five involutions g_A, g_B, g_C, g_F , and g_f on \tilde{V} are the only possible ones, up to strong equivalence (defined in the next section). One can easily see, by considering orbit spaces and fixed-point sets, that no two of these involutions are strongly equivalent.

REMARK. Example (g) is important, because it forms the core of the next section.

In the statement of Theorem 2.3, we shall use the concept of *surgery instruction* χ . Given a link in S^3 , the function χ associates to each component of the link a rational number and determines, in the usual way, a manifold obtained by Dehn surgery on the link.

THEOREM 2.3. Let \tilde{L} be a 2-symmetric link in \tilde{M} such that the link part of L admits a surgery instruction χ that produces S^3 . Assume also that

(i) if $\tilde{K}_1 \cup \tilde{K}_2$ is an interchangeable sublink of \tilde{L} , then both $N(\tilde{K}_1)$ and $N(\tilde{K}_2)$ are trivial in S^3 , the matrix $A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$, and $\chi(p\tilde{K}_1) = \gamma/\delta$;

- (ii) if \tilde{K} and $N(\tilde{K})$ are both periodic, then the orbit space of $E(N(\tilde{K}))$ is a solid torus, the matrix $A = \begin{pmatrix} \alpha & 2\gamma \\ \beta & \delta \end{pmatrix}$, and $\chi(p\tilde{K}) = \gamma/\delta$;
- (iii) if \tilde{K} is periodic and $N(\tilde{K})$ is invariant, then the orbit space of $E(N(\tilde{K}))$ is a solid torus, $A = \begin{pmatrix} 2\alpha & \gamma \\ \beta & \delta \end{pmatrix}$, and $\chi(p\tilde{K}) = \gamma/2\delta$;
- (iv) if \tilde{K} is invariant and $N(\tilde{K})$ is periodic, then the orbit space of $E(N(\tilde{K}))$ is a solid torus, $A = \begin{pmatrix} \alpha & \gamma \\ \beta & 2\delta \end{pmatrix}$, and $\chi(p\tilde{K}) = \gamma/\delta$;
- (v) if \tilde{K} and $N(\tilde{K})$ are both invariant, then the orbit space of $E(N(\tilde{K}))$ is a solid torus, $A = \begin{pmatrix} \alpha & \gamma \\ 2\beta & \delta \end{pmatrix}$, and $\chi(p\tilde{K}) = 2\gamma/\delta$; and
- (vi) if K is strongly invertible, then so is $N(\tilde{K})$. Then, the resulting oriented, closed 3-manifold M admits an involution whose orbit space is S^3 . \square

3. Double covers of doubled knots. So far, we have constructed a variety of 3-manifolds with nonunique representations as two-fold branched covering spaces of S^3 . The uniqueness question, however, is also of interest. In fact, in [BGM; §1, p. 98], the authors ask for a cataloguing of those knots (and links) in S^3 whose two-fold branched covering spaces characterize their type, and some basic cases are already known to be on the list. Results of F. Waldhausen [W₂], J. L. Tollefson [Tol₁], and C. Hodgson [Ho] show that S^3 , and $S^2 \times S^1$, and all lens spaces have unique representations as two-fold branched covering spaces of S^3 , and J. H. Rubinstein [Ru] has shown that certain Seifert fibered 3-manifolds also have this property. In this section, we extend this list to include the two-fold branched covering spaces of all the doubles (with nontrivial twisting) of any prime knot that is not strongly invertible. (Note that this is a larger class of knots than that covered in the theorem of our abstract [MW]; we do not require that the exterior of the prime knot that we are doubling contain no essential annuli.)

Much of the notation that we shall use comes from Example (g) of the previous section and will not generally be referenced or explained in the present section. We begin with several lemmas leading to our Theorems 3.6 and 3.7.

Autohomeomorphism h and g of a manifold M are *strongly equivalent*, if there exists a homeomorphism $f: M \rightarrow M$ such that f is isotopic to the identity and such that $h = fgf^{-1}$. To prove our main result (Theorem 3.7) of this section, we need to classify the strong equivalence classes of involutions on \tilde{V} . For this we need several lemmas. Note first that \tilde{V} is a Seifert-fibered solid torus of type $(2, 1)$ [Ja; p. 84] with a fibered neighborhood of a regular fiber removed (\tilde{V} is a bundle over S^1 with a disk-with-two-holes as the fiber). Hence, \tilde{V} has one exceptional fiber ($= \text{Fix}(g_F) = F$), and the orbit manifold V_0 of \tilde{V} is an annulus.

LEMMA 3.1. *The only closed incompressible surfaces in the two-component link exterior \tilde{V} are boundary-parallel tori.*

Proof The space \tilde{V} is the exterior of a $(2, 4)$ -torus link (the link $\tilde{K}_1 \cup \tilde{K}_2$ of Figure 28), which is a 2-bridge link, and it is well known that the exterior of a 2-bridge link contains no closed nonperipheral incompressible surfaces (see, for example, [GL; Corollary 1.2]. \square

REMARK. Note that \tilde{V} is the exterior of a well-defined infinite class of links [BuM], but we shall be interested only in $\tilde{K}_1 \cup \tilde{K}_2$.

A surface in a 3-manifold is *essential*, if it is properly imbedded, incompressible, boundary incompressible, and not boundary parallel. A surface in a Seifert fibered manifold is *fiber complete*, if the surface is a union of (Seifert) fibers; this is a variation of the term “saturated” [Ja; p. 87]. By a “surface,” we shall always mean a “connected surface.”

LEMMA 3.2. *The manifold \tilde{V} contains exactly three distinct ambient-isotopy classes of essential annuli, each containing a fiber-complete representative.*

Proof. To obtain representatives of the three classes, we lift three, suitable, properly imbedded arcs (missing the exceptional point) in the orbit manifold V_0 to \tilde{V} . The boundary of one of these annuli is in T_1 , the boundary of another is in T_2 , and the third annulus has a boundary component in each of T_1 and T_2 . Obviously, these are the only fiber-complete essential annuli in \tilde{V} (up to isotopy). But, in \tilde{V} , any essential annulus is isotopic to one that is either fiber complete or transverse to all the (Seifert) fibers [Wa₁]. The restriction of the projection map $\tilde{V} \rightarrow V_0$ to an annulus transverse to all the fibers would, however, yield a covering with exactly one branch point of an annulus by an annulus. Since an Euler-characteristic argument shows that this is impossible, our conclusion follows. \square

REMARK. Bleiler has also observed Lemmas 3.1 and 3.2 [Bl].

LEMMA 3.3. *Let Q be a (fixed) annulus from the unique isotopy class of essential annuli in \tilde{V} (Lemma 3.2) with one boundary component in each component of $\partial\tilde{V}$, and let h be a PL involution on \tilde{V} . Then there exists a homeomorphism, $\sigma: \tilde{V} \rightarrow \tilde{V}$, such that σ is isotopic to the identity and $\sigma h \sigma^{-1}(Q) = Q$.*

Proof. By [Tol₂], we can assume that h preserves the Seifert-fiber structure on \tilde{V} . Hence, h induces an involution h_0 on the orbit manifold V_0 ; if $e: \tilde{V} \rightarrow V_0$ is the projection map, then $eh = h_0e$, and the exceptional point v_0 in V_0 (corresponding to the exceptional fiber in \tilde{V}) is a fixed point of h_0 .

Now considering V_0 as $S^1 \times I$, we can find a product fibering of V_0 and an involution g of S^1 such that, relative to this fibering, $h_0(x, t) = (g(x), k(t))$, in which $k(t) = t$ or $1 - t$, for $(x, t) \in S^1 \times I$; this is just a two-dimensional analogue of Theorem 1.3 of [Ki]. Then there exists a unique point $x_0 \in S^1$ such that $\{x_0\} \times I$ contains the exceptional point v_0 , and so $h_0(\{x_0\} \times I) = \{x_0\} \times I$. Splitting V_0 along $\{x_0\} \times I$, we obtain a disk D_0 , which must contain a fixed point (x_1, t_1) of h_0 . Assuming, as we can, that D_0 is the complement in V_0 of a small h_0 -invariant neighborhood of $\{x_0\} \times I$, it follows that $h_0(\{x_1\} \times I) = \{x_1\} \times I \neq \{x_0\} \times I$.

Let $Q' = e^{-1}(\{x_1\} \times I)$. Then Q' is isotopic to Q in \tilde{V} and $h(Q') = Q'$. To complete the proof, let σ be any homeomorphism on \tilde{V} such that σ is isotopic to the identity and $\sigma Q' = Q$. Then $\sigma h \sigma^{-1}(Q) = Q$. \square

LEMMA 3.4. *Let W be a Seifert-fibered solid torus and let h be a PL involution on W preserving the fibration. Then there exists a meridional disk D of W that is transverse to the fibration and to the $\text{Fix}(h)$, and for which either $h(D) \cap D = \emptyset$ or $h(D) = D$.*

Proof. The proof is the same as that of Lemma 3 in [KT₂, p. 267] except that admissible disks (the collection Σ in the proof of [KT₂; Lemma 3]) are transverse to the fibration, and the h -general position isotopies and the isotopies of α -operations and β -operations [KT₁, pp. 223, 224, 226] are to be taken as fiber preserving (cf. proof of Corollary 4.3 of [Tol₃; p. 334]). (Note that $\Sigma \neq \emptyset$, because we can easily find a meridional disk transverse to the fibering and then put it into h -general position by a fiber-preserving isotopy.) \square

REMARKS. The conclusion of Lemma 3.4 also holds when W is a solid Klein bottle, but Lemma 3.4 cannot be extended to any other Seifert fibered manifold, because a solid torus and a solid Klein bottle are the only Seifert fibered manifolds with a compressible boundary component.

LEMMA 3.5. *Let $\{A_1, \dots, A_n\}$ be a nonempty collection of disjoint fiber-complete annuli on the boundary of a Seifert-fibered solid torus W of type (p, q) ($(p, q) = 1$ and $0 < q \leq p/2$), and let h_1 and h_2 be PL*

involutions on W such that $h_1|_{A_i} = h_2|_{A_i}$ and $h_j(A_i) \in \{A_1, \dots, A_n\}$ ($i = 1, \dots, n$; $j = 1, 2$). Then both h_1 and h_2 are OP (orientation preserving), and there exists a homeomorphism $f: W \rightarrow W$ isotopic to the identity $\text{rel}(A_1 \cup \dots \cup A_n)$ for which $h_2 = fh_1f^{-1}$.

Proof. Let τ be a boundary component of A_1 , and let (μ, λ) be a nonoriented meridian-longitude pair for W such that μ meets each (Seifert) fiber of ∂W exactly p times and λ meets each such fiber exactly q times. Orient λ . Then, for suitable orientations of μ and τ , we have $\tau \sim q\mu + p\lambda$ on ∂W . If h_j ($j = 1, 2$) were OR (orientation reversing), then either $h_j(\tau) \sim q\mu - p\lambda$ or $h_j(\tau) \sim -q\mu + p\lambda$ (on ∂W). But $h_j(\tau) \sim \pm(q\mu + p\lambda)$ (because $h_j(A_1) \in \{A_1, \dots, A_n\}$), and neither $q\mu + p\lambda$ nor $-(q\mu + p\lambda)$ is homologous to either $q\mu - p\lambda$ or $-q\mu + p\lambda$ (because $p, q \neq 0$). Hence, both h_1 and h_2 are OP.

Choose a (p, q) -fibering on ∂W that contains the curves in $\{\partial A_1, \dots, \partial A_n\}$ as (regular) fibers and for which $h_1|_{\partial W}$ is fiber preserving. To do this, cut ∂W along each of the curves in $\{\partial A_1, \dots, \partial A_n\}$. The closure of each component is an annulus and there are $2n$ of them; let A_0 be one of them. Now either $h_1(A_0) \cap A_0 = \emptyset$ or $h_1(A_0) = A_0$, and $\partial A_0 \subset \partial A_1 \cup \dots \cup \partial A_n$. If $h_1(A_0) \cap A_0 = \emptyset$, then any (Seifert) fibering of A_0 that includes ∂A_0 induces a fibering of $h_1(A_0)$ so that $h_1|_{(A_0 \cup h_1(A_0))}$ is fiber preserving. If $h_1(A_0) = A_0$, then (as pointed out in the proof of Lemma 3.3), we can regard $h_1|_{A_0}$ as a product map $\varphi \times \psi: S^1 \times I \rightarrow S^1 \times I$ such that $\varphi^2 = 1$ and $\psi(t) = t$ or $1 - t$, for each $t \in I$. Hence, we can take $\{S^1 \times \{t\} \mid t \in I\}$ as an h_1 -invariant fibering of A_0 when $h_1(A_0) = A_0$. Continuing through the remaining annuli of our cut-open version of ∂W , just as for A_0 , we can choose the fibering of ∂W to be h_1 -invariant.

Now extend this fibering of ∂W to a (p, q) -fibering \mathcal{F}' of W . By [Tol₂; Lemma 6, p. 526], there exists a fiber structure \mathcal{F} of W such that h_1 preserves the fibers of \mathcal{F} and such that \mathcal{F}' and \mathcal{F} agree on ∂W .

By Lemma 3.4, there exists a meridional disk D_1 of W that is transverse to both \mathcal{F} and $\text{Fix}(h_1)$, and for which either $h_1(D_1) \cap D_1 = \emptyset$ or $h_1(D_1) = D_1$. We now apply the proof of Lemma 3 of [KT₂] (this is just the equivariant-disk theorem referred to in the proof of Lemma 3.4 (above)) to find a second meridional disk D_2 of W such that $D_1 \cap (A_1 \cup \dots \cup A_n) = D_2 \cap (A_1 \cup \dots \cup A_n)$ (which is a collection of spanning arcs $\{a_1, \dots, a_{pn}\}$ of A_1, \dots, A_n), such that D_2 is transverse to $\text{Fix}(h_2)$ and such that either $h_2(D_2) \cap D_2 = \emptyset$ or $h_2(D_2) = D_2$. The set Σ of admissible disks in the proof of Lemma 3 (of [KT₂]) is taken to be

the set of all meridional disks of W that intersect $A_1 \cup \cdots \cup A_n$ in $a_1 \cup \cdots \cup a_{pn}$ (only) and that are either h_2 -invariant and in general position with respect to $\text{Fix}(h_2)$ or are in h_2 -general position; the proof of Lemma 3 (of [KT₂]) then goes through with isotopies of W that leave $A_1 \cup \cdots \cup A_n$ pointwise fixed.

Since ∂D_j meets each A_i in p (≥ 2) arcs, the collection of curves in $\{\partial D_j, h_j(\partial D_j), \partial A_1, \dots, \partial A_n\}$ ($= \Gamma_j$; $j = 1, 2$) induces an h_j -invariant cell decomposition \mathcal{C}_j of ∂W ; note that either $h_1(D_1) \cap D_1 = \emptyset$ and $h_2(D_2) \cap D_2 = \emptyset$ or $h_1(D_1) = D_1$ and $h_2(D_2) = D_2$, because $h_1|_{A_i} = h_2|_{A_i}$ for each i . The 0-cells of \mathcal{C}_j are just the points in

$$[\partial D_j \cup h_j(\partial D_u)] \cap [\partial A_1 \cup \cdots \cup \partial A_n].$$

The 0-cells divide the curves of Γ_j into arcs — the 1-cells of \mathcal{C}_j .

Since $\partial D_1 \cap \partial D_2 \cap (A_1 \cup \cdots \cup A_n)$ is the collection $\{a_1, \dots, a_{pn}\}$ of pn disjoint arcs (1-cells of both \mathcal{C}_1 and \mathcal{C}_2) and since $h_1|_{A_i} = h_2|_{A_i}$ ($i = 1, \dots, n$), there is a homeomorphism $f_1: W \rightarrow W$ that is isotopic to the identity $\text{rel}(A_1 \cup \cdots \cup A_n)$ and that takes D_1 onto D_2 and $h_1(D_1)$ onto $h_2(D_2)$. Note that $f_1(\mathcal{C}_1) = \mathcal{C}_2$ and that $f_1 h_1 f_1^{-1}(\mathcal{C}_2) = \mathcal{C}_2$; we shall denote $f_1 h_1 f_1^{-1}$ by h_1 and say (following Hartley [Ha]) that we have *varied* h_1 .

Obviously, h_1 and h_2 correspond on the 0-cells of \mathcal{C}_2 . It is also obvious that, if d is either a 1-cell or 2-cell of \mathcal{C}_2 , then $h_1(d) = h_2(d)$; moreover, $h_1(D_2) = h_2(D_2)$. Consequently, by Hartley's trick (Lemma 2.1 and technique *A* of [Ha]), we can vary $h_1|(D_2 \cup h_2(D_2) \cup \partial W)$ so that h_1 and h_2 correspond on $D_2 \cup h_2(D_2) \cup \partial W$ (cf. [Ha, Proof of Lemma 2.3, p. 178]). Finally, if we split W along the disk(s), D_2 and $h_2(D_2)$, and apply Hartley's trick to the resulting 3-cell(s), we can vary h_1 so that $h_1 = h_2$ on W . Since all these variations of h_1 can be chosen to leave $A_1 \cup \cdots \cup A_n$ pointwise fixed, the proof is complete. \square

Let S be a properly imbedded, two-sided surface in a 3-manifold M and let $\Sigma_S(M)$ denote M split along S . There is a natural projection $\gamma: \Sigma_S(M) \rightarrow M$, and $\gamma^{-1}(S)$ is the union of two copies of S each of which is mapped by γ homeomorphically onto S . An involution g on M with $g(S) = S$ lifts to an involution \bar{g} of $\Sigma_S(M)$ such that $\gamma \bar{g} = g \gamma$, because S is two-sided in M .

THEOREM 3.6. *Let k be any (fixed) nonzero integer and let V_k denote the Seifert fibered manifold $(O, o: 0|0; 2k/1)$ with fibered neighborhoods of two regular fibers removed. There exist exactly five (nontrivial) strong*

equivalence classes of PL involutions on V_k , and every PL involution on V_k is orientation preserving. The fixed-point sets of these five classes are as follows: (1) \emptyset , for one class; (2) S^1 , for three classes; and (3) four properly imbedded arcs, for one class.

Proof. The Seifert manifold V_k is the exterior (in S^3) of $(2, 4|k|)$ -torus link (meeting each meridian (on some torus) twice and each longitude $4|k|$ times) and can also be obtained from $(S^1 \times I) \times S^1$ by $(2k/1)$ -surgery on a regular fiber in the (product) Seifert fibration. Note that the conclusions of Lemmas 3.1, 3.2 and 3.3 hold for each V_k just as for V_{-1} ($= \tilde{V}$); the proofs are the same.

Let g_1 and g_2 be any two PL involutions on V_k , and let A_0 be an annulus from the unique isotopy class of essential annuli in V_k with one boundary component in each boundary component of V_k (Lemma 3.2). By Lemma 3.3, we can assume that $g_i(A_0) = A_0$ without changing the strong-equivalence class of g_i ($i = 1, 2$). Now suppose that $g_1|_{A_0}$ and $g_2|_{A_0}$ are strongly equivalent. We will show that g_1 and g_2 are themselves strongly equivalent.

Because $g_1|_{A_0}$ and $g_2|_{A_0}$ are strongly equivalent and because A_0 is bicollared in V_k by a product neighborhood meeting each component of ∂V_k in an annulus, we can vary g_1 so that $g_1|_{A_0} = g_2|_{A_0}$; we assume that this has been done. The space $\Sigma_{A_0}(V_k)$ ($= V_k$ split along A_0) is a solid torus W ; let $\gamma: W \rightarrow V_k$ denote the natural projection. Then $\gamma^{-1}(A_0)$ is a pair of disjoint annuli $\{A_1, A_2\}$ on the ∂W each of whose median is a $(2|k|, 1)$ -torus knot. If \bar{g}_1 and \bar{g}_2 are the lifts to W of g_1 and g_2 , then $\bar{g}_j(A_i) \in \{A_1, A_2\}$ ($i = 1, 2$; $j = 1, 2$). By the first paragraph in the proof of Lemma 3.5, it follows that each of \bar{g}_1 and \bar{g}_2 is OP; consequently, each of g_1 and g_2 is also OP. Thus, by considering a small 3-cell Q in V_k that meets A_0 in a disk in ∂Q , one can easily show that both g_1 and g_2 preserve the sides of A_0 or both reverse them. But then, $\bar{g}_1(A_i) = \bar{g}_2(A_i)$ ($i = 1, 2$), and so $\bar{g}_1|_{A_i} = \bar{g}_2|_{A_i}$, because $g_1|_{A_0} = g_2|_{A_0}$. It follows immediately from Lemma 3.5 that g_1 and g_2 are strongly equivalent. Hence, if $g_1|_{A_0}$ and $g_2|_{A_0}$ are strongly equivalent, then so are g_1 and g_2 .

As Kim noted in [Ki; p. 382] and as we noted earlier, any involution on $S^1 \times I$ is strongly equivalent to one of the form $\varphi \times \psi$, with $\varphi^2 = \text{id}$ and $\psi(t) = t$ or $1 - t$, for each $t \in I$. Therefore, there are exactly five (nontrivial) strong-equivalence classes of involutions on $S^1 \times I$ and, hence, at most five such classes of involutions on V_k .

As we have seen, however, there are at least five strong-equivalence classes of involutions (represented by g_A , g_B , g_C , g_F and g_f) on \tilde{V} . As

with the $(2, 4)$ -torus link $\tilde{K}_1 \cup \tilde{K}_2$ (whose exterior is \tilde{V}), it is not difficult to see that a $(2, 4|k|)$ -torus link (whose exterior is V_k) is interchangeable in two ways, strongly invertible, periodic, and freely periodic (by an involution in S^3). Therefore, there are at least five strong-equivalence classes of involutions, each composed of OP involutions on V_k . \square

THEOREM 3.7. *If K is a prime knot that is not strongly invertible, then the two-fold branched cyclic covering space \tilde{D} of a double $D(K, \rho)$ of K characterizes the knot type of $D(K, \rho)$, if $\rho \neq 0$ (or if K has property P).*

Proof. We have $\tilde{D} = E_1(K) \cup_{T_1} \tilde{V} \cup_{T_2} E_2(K)$. Let h_A be an OP involution such that $\text{Fix}(h_A) = A$ (see Figure 28) and such that the induced covering map $p: \tilde{D} \rightarrow S^3$ maps A onto $D(K, \rho)$. Let h' be a second OP involution on \tilde{D} with $\text{Fix}(h')$ a simple closed curve A' . Moreover, suppose that h' induces a covering map $p': \tilde{D} \rightarrow S^3$ with $p'(A') = D'$. We shall show that $D(K, \rho)$ and D' belong to the same knot type, provided that the prime knot K is not strongly invertible and that $\rho \neq 0$ (which is a technical condition to avoid the possibility that K does not have property P).

Let Σ denote a characteristic submanifold (called a characteristic Seifert pair in [Ja; p. 172]) for \tilde{D} . By [JS; Lemma V.3.3, p. 152], the inclusion map $i: \tilde{V} \rightarrow \tilde{D}$ is nondegenerate [JS; p. 55]. By [Ja; Theorem IX.17, p. 174], the inclusion map i is homotopic to a map $g: \tilde{V} \rightarrow \tilde{D}$ such that $g(\tilde{V}) \subseteq \Sigma$, and so we can assume (after an ambient isotopy) that $\tilde{V} \subseteq \Sigma$ [JS; Squeezing Theorem, p. 139]. Since all incompressible tori in \tilde{V} are boundary parallel (that is, \tilde{V} is *simple*), we can assume that, in fact, \tilde{V} is either a component of Σ or contained in a nonsimple component σ of Σ . Suppose that the latter holds. Then $\sigma = W_1 \cup_{T_1} \tilde{V} \cup_{T_2} W_2$ for which at least one of W_1 and W_2 is not \emptyset .

By [JS; Corollary V.5.1, p. 164], \tilde{D} contains a unique (up to ambient isotopy), mutually disjoint, minimal family of incompressible tori T (a *characteristic family*) such that each component of $\Sigma_T(\tilde{D})$ is either a Seifert fibered manifold or a simple manifold. According to the splitting theorem [JS; p. 157], we can assume that $T \subseteq \partial\Sigma$; we can evidently also assume that $\partial\sigma \subseteq T$ [JS; Proposition V.4.4, p. 156].

Since σ is not simple, it must be Seifert fibered [JS; Corollary V.5.1 p. 164]. We assume that $W_1 (\subset \sigma)$ is not \emptyset . The torus T_1 is incompressible in σ (since K is nontrivial), and so $\pi_1(\sigma)$ contains an infinite cyclic normal subgroup [JS; Lemma II.4.2(i), p. 23] N . Evidently, $\pi_1(\sigma)$ is a *nontrivial* free product with amalgamation (on $\pi_1(T_1)$), since K is a

nontrivial knot and $W_1 \cong S^1 \times S^1 \times I$. Now suppose that there is no Seifert fibration of σ in which T_1 is saturated (that is, fiber complete). Then $N \not\subset \pi_1(T_1)$ [Ja; VI.25, p. 102], and so $\pi_1(T_1)$ has index two in each of $\pi_1(W_1)$ and $\pi_1(\tilde{V} \cup W_2)$ [GH; Lemma 1, p. 305]. By the corollary of [Ze₁, p. 1116] (which is a corollary to the theorem of [Ze₂, p. 366]), it follows that $\pi_1(\tilde{V} \cup W_2) \approx Z \times Z$, which is not true, whether $W_2 = \emptyset$ or not. Hence, there is a Seifert fibration of σ in which T_1 is fiber complete, and so W_1 is Seifert fibered manifold.

By [Ja; Lemma IX.22, p. 188], W_1 is either a cable space, a torus-knot space, or a composing space (an S^1 -bundle over a disk with n holes). (It follows that $W_2 \cong W_1$, because $E_1(K) \cong E_2(K)$, the family T is minimal, and the fiber structure on h_*W_1 is compatible with that of $W_1 \cup \tilde{V}$.) There is an essential fiber-complete annulus A_1 in W_1 with $\partial A_1 \subset T_1$, and by the construction of \tilde{D} , $p(\partial A_1)$ is a pair of meridians of $V (= p\tilde{V})$ on ∂V (cf. Figure 28(a)). But then pA_1 is an essential annulus in $E(K)$ whose boundary components are meridians of K , and so W_1 (as well as W_2) is a composing space; that is, K is a composite knot, contrary to hypothesis. Therefore, $\sigma = \tilde{V}$; that is, \tilde{V} is a component of Σ .

Since $\partial\sigma \subseteq T$, it follows that $\{T_1, T_2\} \subseteq T$. Now there exists a characteristic family of tori T' for \tilde{D} such that $h'T' = T'$ [MS; Proof of Theorem 3.6]. By [JS; Corollary V.5.1, p. 164], there is a homeomorphism $f: \tilde{D} \rightarrow \tilde{D}$ such that f is isotopic to the identity and such that $f(T') = T$. We shall denote $fh'f^{-1}$ by h' ; hence, $h'T = T$.

We claim that $h'(T_1 \cup T_2) = T_1 \cup T_2$. If not, then for some i and j in $\{1, 2\}$, we have $h'T_i \not\subset \partial\tilde{V}$ and $h'T_i \subset E_j(K)$. Hence, $h'\tilde{V} \subset E_j(K)$, for otherwise, $h'\tilde{V}$ contains a characteristic torus (either T_1 or T_2) in its interior that is not boundary parallel (because of the minimality of T), which contradicts Lemma 3.1. But if $h'\tilde{V} \subset E_j(K)$, then K must be a companion of itself, because \tilde{V} is a cable space. Thus, since no knot can be a companion of itself [Sch], we have $h'(T_1 \cup T_2) = T_1 \cup T_2$, and so $h'\tilde{V} = \tilde{V}$.

Next we claim that $A' (= \text{Fix}(h')) \subset \text{Int } \tilde{V}$. If $A' \subset \text{Int } E_i(K)$, then $h'|_{\tilde{V}}$ is a free involution. By Theorem 3.6, there is exactly one strong equivalence class of *free* involutions on \tilde{V} , and each involution of this class interchanges the boundary components of \tilde{V} . Hence, $h'E_i(K) = E_j(K)$ ($i \neq j$). This is a contradiction, however, because $\text{Fix}(h') \subset E_i(K)$. On the other hand, if $A' \cap (T_1 \cup T_2) \neq \emptyset$, then $h'T_i = T_i$, for at least one $i \in \{1, 2\}$, and $A' \cap T_i$ contains exactly four points, because h' is OP. But then K is strongly invertible, contradicting our hypothesis. Thus $A' \subset \text{Int } \tilde{V}$.

Therefore, by Theorem 3.6, there exists a homeomorphism $\varphi: \tilde{V} \rightarrow \tilde{V}$ such that φ is isotopic to the identity (on \tilde{V}) and such that $\varphi(h'|_{\tilde{V}})\varphi^{-1}$ is either g_A , g_C or g_F . Since φ is isotopic to the identity, we can extend φ to \tilde{D} . Thus, $(\varphi h' \varphi^{-1})|_{\tilde{V}}$ is either g_A , g_C , or g_F ; we denote $\varphi h' \varphi^{-1}$ by h' . But, as we pointed out in the constructions of h_C and h_F , if $h'|_{\tilde{V}}$ is either g_C or g_F , then $\tilde{D}/h \not\cong S^3$, because $\rho \neq 0$ or K has property P . Hence, $h'|_{\tilde{V}} = g_A (= h_A|_{\tilde{V}})$.

To see that D' and $D(K, \rho)$ belong to the same knot type, define $\alpha: \tilde{D} \rightarrow \tilde{D}$ by $\alpha|_{(E_1(K) \cup \tilde{V})} = \text{id}$ and $\alpha|_{E_2(K)} = (h_A h')|_{E_2(K)}$ (this is part of Hartley's trick [Ha]). Clearly, α is a well-defined homeomorphism and $h_A = \alpha h' \alpha^{-1}$. Hence, there exists a homeomorphism $\bar{\alpha}: (S^3, D') \rightarrow (S^3, D(K, \rho))$, defined by $\bar{\alpha} p' = p \alpha$. \square

Recall that, if K is strongly invertible, then there is an involution h_B on \tilde{D} that induces a two-fold covering $p': \tilde{D} \rightarrow S^3$ branched over a knot D' ; also, $\text{Fix}(h_B) = B$.

PROPOSITION 3.8. *If K is a nontrivial, strongly invertible, prime knot, then $D(K, \rho)$ and D' belong to distinct knot types; in fact, D' is not a doubled knot (Figure 28(b)).*

Proof. Since \tilde{D} is irreducible, D' is a prime knot. Since K is nontrivial and \tilde{V} is boundary irreducible, the torus $T_1 (= \partial E_1(K))$ is incompressible in \tilde{D} . Now T_1 is invariant under h_B , meets $\text{Fix}(h_B) (= B)$ in exactly four points, and splits \tilde{D} into irreducible, boundary-irreducible parts. It follows from a result of W. B. R. Lickorish, as Bleiler pointed out in [Bl], that D' is the join of two prime tangles.

On the other hand, Bleiler has shown that any double of a *prime* knot is *not* the join of two prime tangles [Bl; Theorem 3.1], and so $D(K, \rho)$ and D' belong to distinct knot types. Moreover, if D' is a doubled knot (say, $D' = D(K', \rho')$), then K' is composite [Bl; Theorem 3.1].

Thus, assuming that $D' = D(K', \rho')$ (with K' composite), we can write

$$\tilde{D} = E_1(K') \cup_{T'_1} \tilde{V}' \cup_{T'_2} E_2(K')$$

with respect to K' ; as usual we have

$$\tilde{D} = E_1(K) \cup_{T_1} \tilde{V} \cup_{T_2} E_2(K).$$

Let Σ be a characteristic submanifold for \tilde{D} . After ambient isotopies of \tilde{D} , we can assume (as shown in the Proof of Theorem 3.7) that \tilde{V} is a component σ of Σ (because K is prime) and that \tilde{V}' belongs to a

component σ' such that

$$\sigma' = W_1 \cup_{T_1'} \tilde{V}' \cup_{T_2'} W_2,$$

in which W_i is a composing space ($i = 1, 2$). Since the number of components of $\partial\sigma'$ is $2n$ (≥ 4) (here n is the number of prime factors of the composite knot K') and the number of components of $\partial\sigma$ is 2, we have $\sigma \not\cong \sigma'$; hence, $\sigma \cap \sigma' = \emptyset$. Therefore, $\sigma' \subset E_i(K)$, for $i = 1$ or $i = 2$. Consequently, σ' is a torus-knot space, a cable space, or a composing space [Ja; Lemma IX.22, p. 188]. But σ' is obviously neither a torus-knot space nor a cable space. It is not a composing space either, because it has one exceptional fiber and a composing space has none. Thus, we have a contradiction to our assumption that D' is a doubled knot. \square

COROLLARY 3.9. *If K is a prime knot and if $\rho \neq 0$, then the two-fold branched covering space \tilde{D} of the double $D(K, \rho)$ of K characterizes $D(K, \rho)$ among all doubled knots.*

Proof. We begin as we did in the proof of Theorem 3.7. We have $\tilde{D} = E_1(K) \cup_{T_1} \tilde{V} \cup_{T_2} E_2(K)$. Let h_A be an OP involution on \tilde{D} such that $\text{Fix}(h_A) = A$ (Figure 28) and such that the induced covering $p: \tilde{D} \rightarrow S^3$ maps A onto $D(K, \rho)$. Let h' be a second OP involution on \tilde{D} with $\text{Fix}(h')$ a simple closed curve A' . Moreover, suppose that h' induces a covering $p': \tilde{D} \rightarrow S^3$ with $p'(A') = D(K', \rho')$. If K is not strongly invertible, then $D(K, \rho)$ and $D(K', \rho')$ belong to the same knot type (Theorem 3.7).

So suppose that K is strongly invertible. Then, by an argument analogous to that in the proof of Theorem 3.7, we can vary h' so that $h'(\tilde{V}) = \tilde{V}$ and $h'|_{\tilde{V}}$ is either g_A or g_B . The proof of Proposition 3.8 shows that, on the one hand, $D(K', \rho')$ is the join of two prime tangles and K' is composite, if $h'|_{\tilde{V}} = g_B$, but that, on the other hand, a contradiction occurs, because K is prime. Hence, $h'|_{\tilde{V}} = g_A$, and the last paragraph of the proof of Theorem 3.7 shows that $D(K', \rho')$ and $D(K, \rho)$ belong to the same knot type. \square

Unoriented knots K_1 and K_2 in S^3 belong to the same isotopy type, if there exists an OP homeomorphism $(S^3, K_1) \rightarrow (S^3, K_2)$. Let $\{K\}$ denote the isotopy type of a knot K ; let K^* denote the mirror image of K . With each doubled knot, we associate a pair of integers (ρ, η) such that the twisting number ρ is arbitrary and the self-intersection number η belongs to $\{-2, 2\}$. The triple $(\{K\}, \rho, \eta)$ characterizes $\{D(K; \rho, \eta)\}$, the isotopy

type of the (ρ, η) -double of K ; moreover, $\{D(K; \rho, \eta)\}$ completely determines the triple $(\{K\}, \rho, \eta)$ except in two cases when K is trivial [Sch; p. 234] (cf. [Wh₁; p. 261]). Note that the mirror image of $D(K; \rho, \eta)$ is $D(K^*, -\rho, -\eta)$. Our last result follows from Theorem 2.1 of [Wh₁] and from Corollary 3.9.

COROLLARY 3.10. *Let K and K' be knots in S^3 , and suppose that K is prime. Let ρ and η be fixed integers; $\rho \neq 0$, but otherwise arbitrary; $\eta = \pm 2$. Finally, let \tilde{D} , \tilde{D}' and \tilde{D}'_0 denote the two-fold branched covering spaces of $D(K; \rho, \eta)$, $D(K'; \rho, \eta)$, and $D(K'^*; \rho, \eta)$, respectively. Then K and K' belong to the same knot type if and only if either $\tilde{D} \cong \tilde{D}'$ or $\tilde{D} \cong \tilde{D}'_0$.*

Proof. Necessity. Suppose that K and K' belong to the same knot type. Then, by [Wh₁; Theorem 2.1, p. 263], either $E(D(K; \rho, \eta)) \cong E(D(K'; \rho, \eta))$ or $E(D(K; \rho, \eta)) \cong E(D(K'^*; \rho, \eta))$. Hence, either $D(K; \rho, \eta)$ and $D(K'; \rho, \eta)$ or $D(K; \rho, \eta)$ and $D(K'^*; \rho, \eta)$ belong to the same knot type ([G], [BM]). But then, either $\tilde{D} \cong \tilde{D}'$ or $\tilde{D} \cong \tilde{D}'_0$, as required.

Sufficiency. Suppose that $\tilde{D} \cong \tilde{D}'$ or $\tilde{D} \cong \tilde{D}'_0$. By Corollary 3.9, either $D(K; \rho, \eta)$ and $D(K'; \rho, \eta)$ or $D(K; \rho, \eta)$ and $D(K'^*; \rho, \eta)$ belong to the same knot type. Hence either $\{D(K; \rho, \eta)\} = \{D(K'; \rho, \eta)\}$, $\{D(K; \rho, \eta)\} = \{D(K'^*; \rho, \eta)\}$, $\{D(K; \rho, \eta)\} = \{D(K'^*; -\rho, -\eta)\}$, or $\{D(K; \rho, \eta)\} = \{D(K'; -\rho, -\eta)\}$. But neither of the last two possibilities can hold, because K is nontrivial and $\eta = \pm 2$ (cf. the proof of Theorem 2.1 of [Wh₁]). Therefore, either $\{K\} = \{K'\}$ or $\{K\} = \{K'^*\}$ (see the paragraph preceding the statement of this corollary). \square

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