PACIFIC JOURNAL OF MATHEMATICS
Vol. 139, No. 1, 1989

HIGHER DIMENSIONAL LINK OPERATIONS AND STABLE HOMOTOPY

ULRICH KOSCHORKE AND DALE ROLFSEN

The fundamental invariant, α, of link homotopy in higher dimensions takes values in the stable homotopy ring Π_*^S. Using the Eckmann-Pontrjagin-Thom correspondence between Π_*^S and Ω_*^{fr}, the framed bordism ring, we give new methods of calculating α and its nonstable version, A, and an extended definition to link maps of arbitrary dimensions. Also we show that the set of all links of two components (arbitrary dimensions) has a natural ring-like structure, compatible with homotopy. The geometric approach allows us to show these operations are compatible, via A, with the ring structure of the homotopy groups of spheres and of Π_*^S. Finally, this introduces a new bifiltration $\Pi_n^{p,q}$ of Π_*^S, which is of independent interest.

0. Introduction. Since Milnor [M] introduced the notion of link homotopy for classical links, there have been several papers ([S], [M-R], [F-R], [Ki], [Ko]) dealing with the same equivalence for links in higher dimensions. In particular, [M-R] discussed a link homotopy invariant α for two-component links: a p-sphere and q-sphere disjointly embedded in $\mathbf{R}^m$, euclidean m-space. For p and $q \leq m-2$, α is defined and takes values in the (stable) homotopy group $\pi_{p+q}(S^{m-1})$, a generalized linking number.

In this paper we will be concerned primarily with links of two components, that is, embeddings

$$L\colon S^p \amalg S^q \to \mathbf{R}^m, \quad \text{where } \amalg \text{ denotes disjoint union.}$$

Occasionally, it is more convenient to consider links in S^m, and if p and q are less than $m-1$, the theories are equivalent. We will also want to consider the more general situation of link *maps* and make no restriction on dimensions unless necessary. A map

$$f\colon S^p \amalg S^q \to \mathbf{R}^m$$

is a *link map* if $f(S^p)$ and $f(S^q)$ are disjoint (each component may self-intersect). Two links, or link maps, are said to be *link-homotopic* if they are homotopic through link maps.

The α-invariant was defined as follows in **[M-R]**. Suppose $L: S^p \amalg S^q \to \mathbf{R}^m$ is a link map, and let $\phi L: S^p \times S^q \to S^{m-1}$ be the map

$$\phi L(x,y) = (L(x) - L(y))/\|L(x) - L(y)\| \in S^{m-1}.$$

LEMMA 0.1 (SEE **[M-R]**). *Consider the map $\sigma: S^p \times S^q \to S^{p+q}$ which smashes the wedge $S^p \vee S^q$ to a point, and is elsewhere one-to-one. If $p, q \leq m-2$, this induces a bijection of homotopy sets*

$$\sigma^*: \pi_{p+q}(S^{m-1}) \to [S^p \times S^q; S^{m-1}].$$

This permits the following definition, for the dimensions assumed in the lemma:

$$\alpha(L) = (\sigma^*)^{-1}[\phi L], \quad \text{an element of } \pi_{p+q}(S^{m-1}).$$

It is clear that $\alpha(L)$ is invariant under link-homotopy of L, since link-homotopy of L induces an ordinary homotopy of ϕL.

Consider the sets

$$\mathbf{EL}^m_{p,q} = \{\text{link-homotopy classes of Euclidean } \textit{links } S^p \amalg S^q \to \mathbf{R}^m\},$$

and

$$\mathbf{ELM}^m_{p,q} = \{\text{link-homotopy classes of } \textit{link maps } S^p \amalg S^q \to \mathbf{R}^m\}.$$

Of course, $\mathbf{EL}^m_{p,q}$ is a subset (and a subgroup when they are groups) of $\mathbf{ELM}^m_{p,q}$. It is shown in **[M-R]** that $\mathbf{EL}^m_{p,q}$ is a group under connected sum, provided $p, q \leq m-3$, and then α is a homomorphism from $\mathbf{EL}^m_{p,q}$ to $\pi_{p+q}(S^{m-1})$, and in many cases is an isomorphism (see Proposition 4.1). The additivity of α will be extended below to link maps and arbitrary dimensions. ($\mathbf{ELM}^m_{p,q}$ is claimed to be a group by **[S]** when $p, q \leq m-3$, but the argument for the existence of inverses seems to be inconclusive.)

1. The Hopf construction and the extended α-invariant $A(L)$. To fix notation, we recall that the *join* of spaces X and Y is defined as the quotient

$$X * Y \cong X \times Y \times [-1,1]/\sim$$

with identifications $(x,y,-1) \sim (x,y',-1)$ and $(x,y,1) \sim (x',y,1)$. Thus X and Y are naturally subsets of $X * Y$, as the -1 and $+1$ ends, respectively. Using the coordinate notation of the above product, one defines the join

$$f * f': X * X' \to Y * Y'$$

of two maps

$$f\colon X \to Y \quad \text{and} \quad f'\colon X' \to Y'$$

by the formula

$$(f * f')(x, y, t) = (f(x), f'(y), t) \in Y * Y'.$$

The notation of join coordinates also gives a convenient description of the *Hopf construction*. This associates to each map $\psi\colon S^p \times S^q \to S^{m-1}$ a well-defined map $G\psi\colon S^{p+q+1} \to S^m$ (a thorough discussion in a more general setting may be found in **[W]**).

The join of spheres is a sphere, and we choose fixed identifications

$$S^p * S^q \equiv S^{p+q+1}.$$

Also S^m may be identified with the suspension $S^m \equiv \Sigma S^{m-1}$ which we regard as $S^{m-1} \times [-1, 1]$, with the -1 and $+1$ ends smashed to points. Using coordinates of the join and suspension, define

$$G\psi\colon S^p * S^q \to \Sigma S^{m-1} \quad \text{by the equation}$$
$$G\psi(x, y, t) = (\psi(x, y), t).$$

Clearly G is compatible with homotopy and induces a map of homotopy sets

$$G\colon [S^p \times S^q; S^{m-1}] \to \pi_{p+q+1}(S^m).$$

We can now describe the Shapiro-Kervaire **[K]** invariant of two-component links (or link maps) of arbitrary dimensions. If $L\colon S^p \amalg S^q \to \mathbf{R}^m$ is a link-map, let $\phi L\colon S^p \times S^q \to S^{m-1}$ be the map defined in the previous section and define

$$A(L) = [G\phi L], \quad \text{an element of } \pi_{p+q+1}(S^m).$$

PROPOSITION 1.1. *$A(L)$ depends only on the link homotopy class of L. In the dimensions for which the α-invariant is defined, namely $p, q < m - 1$, we have*

$$A = \pm E\alpha$$

where $E\colon \pi_{p+q}(S^{m-1}) \to \pi_{p+q+1}(S^m)$ is the Freudenthal suspension, an isomorphism in these dimensions.

REMARK. We will shortly extend the definition of α. Then, in general, A will be a desuspension, rather than suspension, of α.

Proof. The first part is clear. Consider the diagram of homotopy sets:

$$\begin{tikzcd}[column sep=small] \pi_{p+q}(S^{m-1}) \arrow[rr, "\sigma^*"] \arrow[dr, "E"'] & & {[S^p \times S^q; S^{m-1}]} \arrow[dl, "G"] \\ & \pi_{p+q+1}(S^m) & \end{tikzcd}$$

This diagram is commutative, regardless of the dimensions. With $p, q < m-1$, the homotopy groups are stable, so the suspension map E is an isomorphism, and as noted in Lemma 0.1, σ^* is also an isomorphism. The result follows, with the sign ambiguity only because of differences in the literature regarding orientation conventions of the suspension and the Hopf construction.

2. Framed manifolds and the geometric α-invariants. We assume the reader is familiar with the construction of Eckmann-Pontrjagin-Thom ([E], [P], [T])

$$[M^n; S^k] \xrightarrow{P} \Omega^{fr}_{n-k}(M).$$

This is a bijection relating the homotopy classes of maps of a compact smooth manifold M^n into a sphere S^k with the framed bordism classes of framed $(n-k)$-submanifolds of M. Thus if $f\colon M^n \to S^k$ is a map (which we assume smooth) and $e \in S^k$ is a regular value of f, then $f^{-1}(e)$ is an $(n-k)$-dimensional submanifold of M^n with a framing (trivialization of its normal bundle) obtained by pulling back a framing of e in S^k. The framed bordism class of this framed submanifold is $P([f])$.

Since we are concerned with $[S^p \times S^q, S^{m-1}]$, consider the "natural" inclusion $j\colon S^p \times S^q \to S^{p+q+1}$ as the middle section of the join, or equivalently as the boundary of the tubular neighborhood $N(S^p)$ of an unknotted p-sphere in S^{p+q+1}, a framing in $S^p \times S^q$ becomes a framing in S^{p+q+1} by adjoining a vector normal to $j(S^p \times S^q)$ in S^{p+q+1} pointing outward from $N(S^p)$. As observed by Kervaire [K], this corresponds to the Hopf construction.

PROPOSITION 2.1. *The following diagram is commutative, up to sign depending only on the dimensions*:

$$\begin{tikzcd} \Omega^{fr}_{p+q-m+1}(S^p \times S^q) \arrow[r, "j^*"] & \Omega^{fr}_{p+q-m+1}(S^{p+q+1}) \\ {[S^p \times S^q; S^{m-1}]} \arrow[u, "P"] \arrow[r, "G"] & \pi_{p+q+1}(S^m). \arrow[u, "P"] \end{tikzcd}$$

Stabilization of the homotopy group corresponds to the inclusion

$$\Omega^{fr}_{p+q-m+1}(S^{p+q+1}) \xrightarrow{i} \Omega^{fr}_{p+q-m+1}$$

into the bordism group of all stably-framed manifolds, modulo stably-framed bordism.

We can now define the geometric α-invariant for a link map $f\colon S^p \amalg S^q \to \mathbf{R}^m$, $m \geq 2$. It is

$$\alpha(f) = ij * P\phi f \in \Omega^{fr}_{p+q-m+1}.$$

From the above discussion it is clear that $\alpha(f)$ corresponds to the stable-suspension of $A(f)$, hence is (up to sign) a generalization of the homotopy-theoretic α-invariant of **[M-R]**.

The situation is summarized in the following diagram, which commutes up to sign ($n = p + q - m + 1$).

$$\begin{tikzcd}
\mathbf{EL}^m_{p,q} \arrow[r, hook] \arrow[ddd, bend right=60, "A"'] & \mathbf{ELM}^m_{p,q} \arrow[dl, "\alpha"'] \arrow[dr, "\phi"] \arrow[ddddr, bend left=60, "\alpha"] & \\
\pi_{p+q}(S^{m-1}) \arrow[rr, "\sigma^*"] \arrow[dd, "E"] & & {[S^p \times S^q; S^{m-1}]} \arrow[d, "P", "\approx"'] \arrow[ddll, "G"'] \\
& & \Omega^{fr}_n(S^p \times S^q) \arrow[d, "j_*"] \\
\pi_{p+q+1}(S^m) \arrow[rr, "P", "\approx"'] \arrow[d, "E^\infty"] & & \Omega^{fr}_n(S^{p+q+1}) \arrow[d, "i"] \\
\pi^S_n \arrow[rr, "P", "\approx"'] & & \Omega^{fr}_n
\end{tikzcd}$$

PROPOSITION 2.2. *If $f\colon S^p \amalg S^q \to \mathbf{R}^m$ and $\tilde{\phi}\colon S^p \times S^q \to \mathbf{R}^m - \{0\}$ is the map $\tilde{\phi}(x, y) = f(x) - f(y)$, then $A(f)$ corresponds to the framed manifold $\tilde{\phi}^{-1}(\rho)$, where ρ is the ray in $\mathbf{R}^m$ through $e \in S^{m-1}$, a regular value of ϕ.*

Next we give a description of $\alpha(f)$ which is inspired by the old idea of measuring linking phenomena by intersecting with "Seifert surfaces". Note that we need to consider only smooth (or generic) link maps since small approximations don't change the link homotopy class.

PROPOSITION 2.3. *Given a smooth link map $f\colon S^p \amalg S^q \to \mathbf{R}^m$, let $f|S^p$ extend smoothly to a map $\tilde{f}\colon D^{p+1} \to \mathbf{R}^m$ such that*

$$\tilde{\phi}\colon D^{p+1} \times S^q \to \mathbf{R}^m, \quad \tilde{\phi}(x, y) := \tilde{f}(x) - f(y),$$

has $0 \in \mathbf{R}^m$ *as a regular value. Then the framed bordism class of* $\tilde{\phi}^{-1}(0)$ *is* $\alpha(f)$.

In particular, if $f|S^q$ *meets* $\tilde{f}(D^{p+1})$ *only where* $\tilde{f}$ *is an embedding, and transversely so, then* $\alpha(f)$ *can be represented by the framed "intersection" manifold* $(f|S^q)^{-1}(\tilde{f}(D^{p+1}))$.

Proof. If $e \in S^{m-1}$ is a regular value of ϕ then $\tilde{\phi}$ is transverse to the embedded ray $[0,\infty)\cdot e \subset \mathbf{R}^m$ both in a neighbourhood of $\tilde{\phi}^{-1}(0)$ and when restricted to the boundary $S^p \times S^q$ of its domain. After a slight perturbation, $\tilde{\phi}$ is everywhere transverse to this ray, and its inverse image gives the required framed bordism between $\phi^{-1}(e) = (\tilde{\phi}|S^p \times S^q)^{-1}$ (ray) and $\tilde{\phi}^{-1}(0)$.

Finally, a more direct approach leads to an interpretation of $\alpha(f)$ in terms of the "overcrossing locus". It has the advantage of taking place in the lower dimensional euclidean space $\mathbf{R}^{m-1}$.

Fix $e \in S^{m-1}$ and choose an isomorphism identifying $\mathbf{R}^{m-1}$ with the orthogonal complement of the line $\mathbf{R}e \cong \mathbf{R}$ in $\mathbf{R}^m$, so that $\mathbf{R}^m \cong \mathbf{R}^{m-1} \times \mathbf{R}$. Let $f = (\underline{f}, \overline{f})\colon S^p \amalg S^q \to \mathbf{R}^m$ be the corresponding decomposition for a link map f. After a small perturbation we may assume that f is smooth, and that $\underline{f}|S^p$ and $\underline{f}|S^q$ intersect transversely in $\mathbf{R}^{m-1}$, i.e.

$$\gamma\colon S^p \times S^q \to \mathbf{R}^{m-1}, \quad \gamma(x,y) := \underline{f}(x) - \underline{f}(y)$$

has $0 \in \mathbf{R}^{m-1}$ as a regular value. Then e and $-e$ are regular values of ϕ and $\alpha(f)$ can be represented by the "*overcrossing locus*"

$$N = \{(x,y) \in S^p \times S^q | \gamma(x,y) = 0,\ \overline{f}(x) > \overline{f}(y)\}$$

together with the framing

$$(1) \qquad g\colon TN + \underline{\mathbf{R}}^{m-1} \xrightarrow{\mathrm{Id}\oplus(T\gamma)^{-1}} TN + \nu = T(S^p \times S^q)|N.$$

Here $\underline{\mathbf{R}}^{m-1}$ denotes the trivial $\mathbf{R}^{m-1}$ bundle.

3. Link maps in $\mathbf{R}^4$. In this section we investigate how to compute $\alpha(f)$ in the special case $p = q = 2$, $m = 4$, and apply this to the example of **[F-R]**. This argument, by the first author of the present paper is an alternative to **[F-R]**, where different methods are used to establish that $\alpha(f) \neq 0$ for the Fenn-Rolfsen map $f\colon S^2 \amalg S^2 \to \mathbf{R}^4$. The reader may omit this section without loss of continuity.

Note that now $\alpha(f)$ lies in the group $\Omega_1^{fr} = \mathbf{Z}_2$ where the nontrivial element is represented by the invariantly framed (= nonstably parallelized) circle.

Given a link map

$$f = (\underline{f}, \overline{f})\colon S^2_{(1)} \amalg S^2_{(2)} \to \mathbf{R}^4 \cong \mathbf{R}^3 \times \mathbf{R}$$

we may assume that the projected maps $\underline{f}|S^2_{(1)}$ and $\underline{f}|S^2_{(2)}$ intersect transversely and only where they are both immersions (the singular points being isolated generically). Then each component of the 1-dimensional manifold N is an embedded circle $C \subset S^2_{(1)} \times S^2_{(2)}$ which projects into the immersed circles $\mathrm{pr}_i(C) \subset S^2_{(i)}$ under the obvious projections $\mathrm{pr}_i\colon S^2_{(1)} \times S^2_{(2)} \to S^2_{(i)}$, $i = 1, 2$. This suggests a more natural framing for

$$T(S^2_{(1)} \times S^2_{(2)})|C = \mathrm{pr}_i^*(TS^2_{(1)})|C + \mathrm{pr}_2^*(TS^2_{(2)})|_C\colon$$

each summand splits into a tangential and a normal direction along C. This framing differs from the one needed in formula (1) by d rotations as we go around C, where d is the sum of the normal degrees of the immersions $\mathrm{pr}_i|\colon C \to S^2_{(i)}$, $i = 1, 2$. By the Whitney-Graustein theorem d has the same parity as the added number of (generic) self-intersection points of these two immersions of C.

If we use the above "more natural" framing in (1), the direction of TC in $T(S^2_{(1)} \times S^2_{(2)})|C \cong \underline{\mathbf{R}}^4$ remains constant, $\nu = \mathrm{pr}_1^*(TS^2_{(1)})|C \oplus \underline{\mathbf{R}}$, and $g|C$ defines an element

$$\rho(C) \in Z_2 \cong \pi_1(V_{3,2}) \cong \pi_1(\mathrm{SO}(3))$$

which can be represented by the following loop of 2-frames in $\mathbf{R}^3$: the immersion $\mathrm{pr}_1|\colon C \to S^2_{(1)}$ is accompanied by a 2-frame field tangential to $S^2_{(1)}$, given by the tangent vector along C and the normal vector; apply the tangent map of $\underline{f}\colon S^2_{(1)} \to \mathbf{R}^3$ to get the required map from the circle C into the Stiefel manifold $V_{3,2}$.

The correct framing (1) for C is the invariant one, and C contributes $+1$ towards $\alpha(f)$, if and only if $\rho(C) + d \in \mathbf{Z}_2$ is trivial. Thus we get

$$\alpha(f) = \sum_C \left(1 + \rho(C) + \sum_{i=1}^{2} \#\text{ double points of } \mathrm{pr}_i(C)\right) \tag{2}$$

where we sum over all circles in the overcrossing locus N.

Now we apply our two computing methods to the Fenn-Rolfsen link map f from $S^2_{(1)} \amalg S^2_{(2)}$ into the (x, y, z, w)-space $\mathbf{R}^4$. A slice-by-slice illustration is given in Figure 3.1: starting from the classical Whitehead link in the 3-space $w = 0$ we put two link homotopy trivialisations,

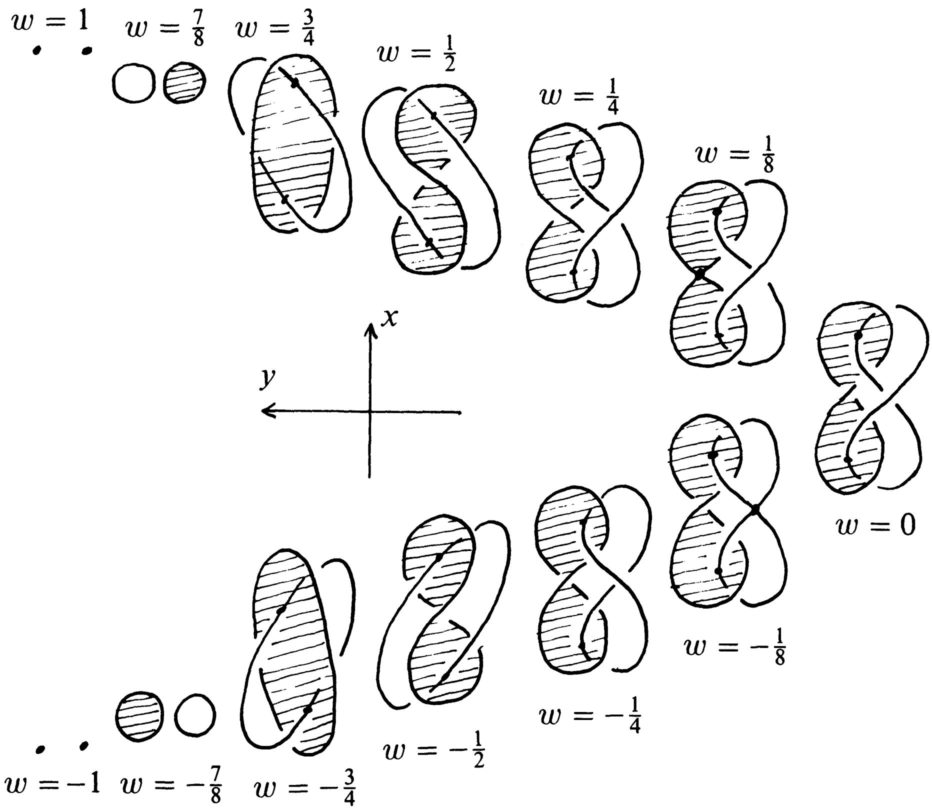

FIGURE 3.1

parametrized by the w-intervals $[-1,0]$ and $[0,1]$, together to obtain f. Each $f|S^2_{(i)}$, $i = 1, 2$, has just one double point (at $w = \pm\frac{1}{8}$).

To apply the intersection approach, let the circle describing $f(S^2_{(1)})$ at every w-level, $-1 < w < 1$, span a 2-disc as in Figure 3.1. This way we obtain an extension $\tilde{f}\colon D^3 \to \mathbf{R}^4$ of $f|S^2_{(1)}$. If we think of D^3 as the unit ball in (x, y, w)-space, then $\tilde{f}$ is roughly the identity whenever $x > \frac{1}{2}$, while for negative x there is a full rotation around the x-axis through (y, z)-space.

$\tilde{f}$ and $f|S^2_{(2)}$ intersect only where both maps are embeddings, along one circle C in the (x, w)-plane. Its inverse image in D^3 is a similar circle, and in $S^2_{(2)}$ it is a great circle. Thus

$$\tilde{C} := \tilde{\phi}^{-1}(0) = \{(r, s) \in D^3 \times S^2_{(2)} \,|\, \tilde{f}(r) = f(s)\}$$

is a circle in $D^3 \times S^2_{(2)}$ projecting onto both these inverse image circles.

In order to determine $\alpha(f)$ we need to study the framing

(3) $\quad l\colon \underline{\mathbf{R}}^5 = T\tilde{C} \oplus \underline{\mathbf{R}}^4 \xrightarrow{\mathrm{id}\oplus T\tilde{\phi}} T\tilde{C} \oplus \nu = T(D^3 \times S^2_{(2)})|\tilde{C} \to \underline{\mathbf{R}}^5.$

Here we use some identification $\tilde{T}C = \underline{\mathbf{R}}$, so that if the resulting homotopy class $[l] \in \pi_1(\mathrm{SO}(5)) = \mathbf{Z}_2$ is trivial then $\alpha(f) = 1$.

If the isomorphism k in (3) were given by the standard parallelization of D^3 and the tangent and normal vector field along (the projection of) $\tilde{C}$ in $S^2_{(2)}$, then after a homotopy $T\tilde{C}$ would correspond to a constant vector field on both sides in (3), and $[l]$ would correspond to

$$[T\tilde{f}] = [T\tilde{\phi}| : TD^3|\tilde{C} = \underline{\mathbf{R}}^3 \to \underline{\mathbf{R}}^4] \in \pi_1(V_{4,3}) = \mathbf{Z}_2.$$

Now $T\tilde{f}|C$ gives the identity in the x- and w-directions and, for $x > 0$, in the y-direction; but for $x < 0$ there is a full twist of the unit vector $(0, 1, 0)$ through the (y, z)-plane. This is cancelled by the fact that we have to pick another isomorphism k in (3), namely the one coming from a framing of all of $TS^2_{(2)}$ which differs from the one used above by one rotation. Thus $[l] = 0$ and $\alpha(f) = 1$.

Finally we apply the "overcrossing locus" method to the same example. Actually it will help our intuition to deform $f|S^2_{(2)}$ a little for $|w|$ close to 1 and to use the following picture for $|w| = \frac{3}{4}$ (see Figure 3.2). This link homotopy will not affect $\alpha(f)$.

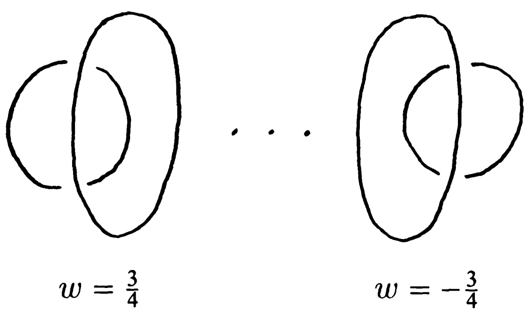

FIGURE 3.2

Let $e = (0, 0, 1, 0)$ be the unit vector in the positive z-direction (thought of as pointing towards the reader in Figures 3.1 and 3.2), and let $\mathbf{R}^3$ denote the complementary (x, y, w)-space. For $|w| < \frac{3}{4}$, each w-slice contains two overcrossing points; so here the overcrossing locus consists of two arcs A_+ and A_- parametrizable by w. The situation for $w \leq -\frac{3}{4}$ is illustrated in Figure 3.3 (for $w \geq \frac{3}{4}$, it is similar). Thus clearly the overcrossing locus N (and its image under $\underline{f}$) consists only of one circle without any self-intersections. By formula (2) $\alpha(f) = 1$ as soon as we have checked that $\rho(N) = 0$.

So let us consider along $f(N)$ a tangential vector field t and normal vector field n tangential to $\underline{f}(S^2_{(1)})$. Over each arc $A_\pm$ t points in the $\pm w$-direction; when w increases, n rotates by 180° in the counterclockwise sense in the (x, y)-plane (inspect Figures 3.1 and 3.2). This contributes one full rotation towards $\rho(N)$. One additional rotation comes from the behaviour of t and n along the two semicircles $|w| \geq \frac{3}{4}$. These two rotations cancel and we have $\rho(N) = 0$ and $\alpha(f) = 1$.

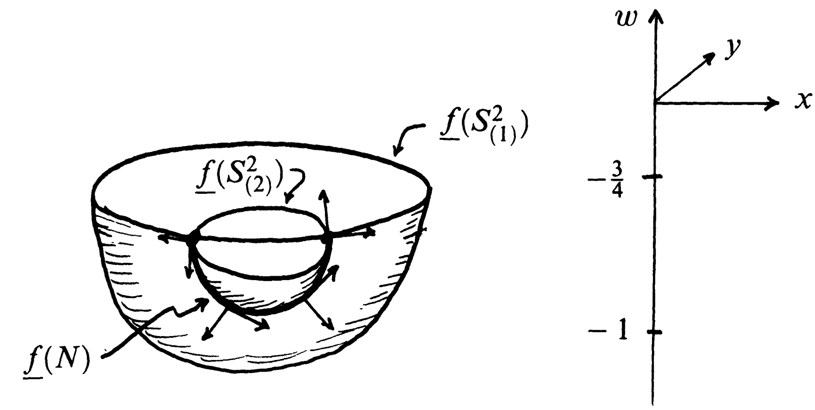

FIGURE 3.3

4. Operations on links and link maps. Let **EL** denote the set of link homotopy classes of links of 2-components in a euclidean space (regardless of dimensions) and let **ELM** similarly denote the set of link homotopy classes of 2-component link maps. As is well known, the set Π_* of all homotopy groups of spheres enjoys a number of algebraic operations, and is a ring with the usual addition, with an anticommutative multiplication induced by the join operation, as well as suspension homomorphisms, etc.

In this section we will examine analogous operations in **EL** and **ELM**: connected sum, suspensions, join of links, and precomposition. We will establish certain compatibility relations of these operations with their analogues in Π_* via A. In particular, roughly speaking, A would be a ring homomorphism if **EL** were a ring. The problem is that we don't know if the links of codimension less than three form a group under connected sum, and we know even less regarding link maps.

Here is a thumbnail description of the operations considered in more detail below. Connected sum is well known. It is required that the two links being added have the same number of components and the same dimensions of the corresponding components. To suspend a link in S^n, choose one component and suspend that into S^{n+1}, the suspension of S^n. To join two (ordered) links, in S^m and S^n, respectively, take the join of each ith component of one with the ith component of the other in the S^{m+n+1} which is the join of the two ambient spheres. Actually, to obtain formulas for A, we need links in euclidean space and so our official definitions of suspension and join are the corresponding constructions in $\mathbf{R}^m, \mathbf{R}^n$, etc. Finally, a composition link map is obtained from a link map $f\colon S^p \amalg S^q \to \mathbf{R}^m$, and arbitrary maps $g\colon S^{p'} \to S^p$ and $h\colon S^{q'} \to S^q$. The composite $F\colon S^{p'} \amalg S^{q'} \to \mathbf{R}^m$ is just the link map $F(x) = f(g(x))$ if $x \in S^{p'}, F(x) = f(h(x))$ if $x \in S^{q'}$. (Analogous definitions could be made for links of more than two components, of course.)

Connected sum. To form a connected sum $L\#L'$ of the two links $L, L'\colon S^p \amalg S^q \to \mathbf{R}^m$, one takes representatives separated by an $(m-1)$-dimensional hyperplane $\mathbf{R}^{m-1}$ and connects $L(S^p)$ with $L'(S^p)$ and $L(S^q)$ with $L'(S^q)$ by tubes, disjoint from the other components and from each other. Up to homotopy, this is the same as requiring that each component of L meet $\mathbf{R}^{m-1}$ in a single point, and likewise L', so they form a disjoint union of wedges. Then $L\#L'$, as a map, has domain $S^p \amalg S^q$, and on each component factors through the standard "folding map" of homotopy theory, and sends the 'northern' and 'southern' hemispheres of each sphere, respectively, into the 'upper' and 'lower' half-spaces of $\mathbf{R}^n$, separated by $\mathbf{R}^{n-1}$. We do not know, in general, whether this is independent of the choices, even up to link homotopy, except in codimension ≥ 3 (see below).

EXAMPLE 4.1. Classical links. For classical links $S^1 \amalg S^1 \to \mathbf{R}^3$ connected sum is *not* well-defined in the usual sense. For example, if L is the Hopf link (two simply-linked circles, as in fibres of the Hopf fibration of S^3), one realization of $L\#L$ is the unlink, while another (twisting the tube, or band) realization of the same sum is the Whitehead link. Of course these two versions of $L\#L$ are link-homotopic and it turns out that, modulo link homotopy, connected sum *is* well-defined for classical two-component links. Since each link map is homotopic with an embedding, $\mathbf{EL}^3_{1,1} = \mathbf{ELM}^3_{1,1}$ and they form groups under connected sum. In fact an elementary argument (or see [M]) shows that classical 2-component links are classified, up to homotopy, by their linking number, and α is an isomorphism of this group onto the infinite cyclic group Π_0^S. On the other hand, connected sum is *not* well-defined for the three-component classical case, even up to homotopy. The following picture shows two 3-component links L and L' and two realizations of $L\#L'$. One sum gives the unlink, while the other yields the Borromean link, which (by [M]) is *not* homotopic with the unlink.

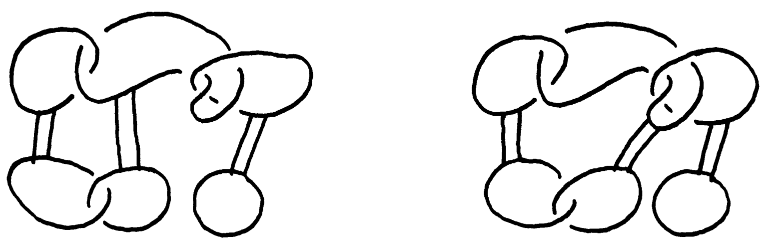

FIGURE 4.1

We now review what is known for higher dimensions regarding connected sum.

PROPOSITION 4.1 (P. Scott [S]). *Connected sum is well-defined (up to homotopy) for links (or link maps) of spheres in $\mathbf{R}^m$ of codimension at least three. That is, if L and L': $S^p \amalg S^q \to \mathbf{R}^m$ are links (link maps), then all links (link maps) arising as $L\#L'$ are link-homotopic, provided $m - p \geq 3$ and $m - q \geq 3$. Moreover, if L' is homotopic with L'', then $L\#L'$ is homotopic with $L\#L''$.*

The α-invariant was defined earlier, $\alpha\colon \mathbf{EL}_{p,q}^m \to \pi_{p+q}(S^{m-1})$.

PROPOSITION 4.2 (Massey and Rolfsen [M-R]). *Suppose $m - p \geq 3$ and $m - q \geq 3$. Then α is a group homomorphism. Moreover, α is*

(i) *injective (i.e. a faithful invariant) if $2p + 2q \leq 3m - 6$;*

(ii) *an isomorphism if $p + 2q \leq 2m - 3$ and $2p + q \leq 2m - 3$.*

COROLLARY. *The same is true of A: $\mathbf{EL}_{p,q}^m \to \pi_{p+q+1}(S^m)$ and for the geometric α-invariant with values in $\Omega_{p+q+1-m}^{fr}$.*

For the codimension 2 case the following results hold (see [M-R] for proofs).

PROPOSITION 4.3 (Due to J. Levine). *If $1 < p \leq q = m - 2$ then $\alpha(L)$ is zero for any link of $S^p \amalg S^q$ in $\mathbf{R}^m$.*

We note that this does not hold for link maps. §3 gives an example of a link map of $S^2 \amalg S^2$ in $\mathbf{R}^4$ with nonzero α-invariant.

PROPOSITION 4.4. *α: $\mathbf{EL}^m_{1,m-2} \to \pi_{m-1}(S^{m-1}) = \mathbf{Z}$ is an isomorphism ($m \geq 2$). Moreover, if $1 < p$ and $2p \leq q = m-2$ then every link of $S^p \amalg S^q$ in $\mathbf{R}^m$ or S^m is homotopically trivial.*

Now we turn to new results and some questions.

Question 4.1. Is connected sum always well-defined in $\mathbf{ELM}^m_{p,q}$? If so, is it a group operation? In particular, do inverses exist?

Whether the sum is well-defined or not, we are able to show that A is additive.

PROPOSITION 4.5. *If $L = L'\#L''$ is any connected sum of two-component links $S^p \amalg S^q \to \mathbf{R}^m, p \geq 1, q \geq 1, m \geq 1$, we have $A(L) = A(L') + A(L'')$, in the appropriate homotopy group. Consequently also $\alpha(L) = \alpha(L') + \alpha(L'')$. The same holds for connected sums of link maps.*

Proof. Recall that $A(L)$ corresponds, via the Pontrjagin map, to a framed submanifold of S^{p+q+1}. Likewise $A(L')$ and $A(L'')$. If one chooses a direction $e \in S^{m-1}$ which is parallel to the hyperplane separating L' and L'', and which is a regular value for both maps, then $A(L)$ corresponds to the union of two framed submanifolds, corresponding to L' and L'', in different hemispheres of S^{p+q+1}. This "distant union" corresponds to the usual sum in $\pi_{p+q+1}(S^m)$, proving the proposition.

Suspension of links. For a link (or link map) L there are two choices of suspension operation. If the target of L is S^m, one simply suspends the ith component into the suspension, S^{m+1}, of S^m. For a link in euclidean space,

$$L\colon S^p \amalg S^q \to \mathbf{R}^m,$$

we define the suspensions

$$\Sigma_1 L\colon S^{p+1} \amalg S^q \to \mathbf{R}^{m+1},$$
$$\Sigma_2 L\colon S^p \amalg S^{q+1} \to \mathbf{R}^{m+1}$$

in a topologically equivalent manner by regarding $\mathbf{R}^m$ as the hyperplane $x_m = 0$ of $R^{m+1} = \{(x_0, \ldots, x_m)\}$. Take $\Sigma_1 L|S^q$ equal to L. Regard S^{p+1} as the suspension $S^{p+1} \equiv \Sigma S^p$. Then on S^{p+1} we define

$$\Sigma_1 L\colon \Sigma S^p \to \mathbf{R}^m \times \mathbf{R} = \mathbf{R}^{m+1} \quad \text{by}$$
$$(x,t) \to ((1-|t|)L(x), t).$$

Suspension is not a smooth operation, although it is OK in the PL or topological category. Nevertheless, one could, by a trick, homotop a suspension of a smooth link to be a smooth link. Indeed the two suspension points are the only nonsmooth points, and the local knots measuring the singularities cancel each other in the sense of knot cobordism. Thus the suspension can be altered, in a neighbourhood of an arc joining the singularities, to be smooth, and this can clearly be done by a homotopy. Alternatively, and this is the approach we will use below, one can put a smooth structure on ΣS^p and ε-approximate $\Sigma_i L$ by a smooth link map, within its link homotopy class.

PROPOSITION 4.6. *If L is link-homotopic to L', then $\Sigma_i L$ and $\Sigma_i L'$ are also link-homotopic. Also, for any links L, L' of the same dimensions, $\Sigma_i(L\#L') \approx \Sigma_i L\#\Sigma_i L'$ in the sense that for a given realization of connected sum $L\#L'$, one can realize its suspension (up to link homotopy) as a connected sum as indicated. Finally, $\Sigma_2 \circ \Sigma_1 = r \circ \Sigma_1 \circ \Sigma_2$ where r is a reflection.*

The proof is an easy geometric argument and will be left to the reader.

Now consider a link or link map of two components, $L\colon S^p \amalg S^q \to \mathbf{R}^m$. There are suspensions $\Sigma_1 L\colon S^{p+1} \amalg S^q \to \mathbf{R}^{m+1}$ and $\Sigma_2 L\colon S^p \amalg S^{q+1} \to \mathbf{R}^{m+1}$. In both cases we have $A(\Sigma_i L) \in \pi_{p+q+1}(S^{m+1})$, whereas $A(L) \in \pi_{p+q}(S^m)$. Also consider the Freudenthal suspension map $E\colon \pi_{p+q}(S^m) \to \pi_{p+q+1}(S^{m+1})$.

PROPOSITION 4.7. *If L is a link or link map, then*

$$A(\Sigma_i L) = \pm EA(L) \quad \textit{for } i = 1, 2.$$

Proof. By a homotopy, we can arrange the suspension to be orthogonal to $\mathbf{R}^m$ in $\mathbf{R}^{m+1}$. Approximate $\Sigma_i L$ by a smooth map whose image does not intersect the hyperplane, except where the original L does. Now choose $e \in S^{m-1}$, a regular value for ϕL. Then $e \in S^{m-1} \subset S^m$ is also a regular value for $\phi(\Sigma_i L)$, and yields in both cases exactly the same framed manifold, except that the framing for the suspended link map is obtained from that of the original link map by adding a trivial 1-dimensional bundle. This corresponds to the suspension map $\pi_{p+q+1}(S^m) \to \pi_{p+q+2}(S^{m+1})$ and the proposition is proved.

Join of links. This construction, like the suspension, is most natural in the setting of links in S^m. As is well known, the join of spheres is a sphere:

$$S^p * S^{p'} \equiv S^{p+p'+1}.$$

Making a canonical choice of such a homeomorphism, the join naturally induces a multiplication

$$\pi_p(S^n) \times \pi_{p'}(S^{n'}) \xrightarrow{*} \pi_{p+p'+1}(S^{n+n'+1})$$

which satisfies:

$$\begin{aligned}
&\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma, \\
&\alpha * \beta = (-1)^{p+p'} \beta * \alpha, \\
&\alpha * (\beta \pm \beta') = \alpha * \beta \pm a * \beta', \\
&a * 0 = 0 * \alpha = 0.
\end{aligned}$$

See [W] for details.

DEFINITION 4.1. Let $L\colon S^p \amalg S^q \to S^m$ and $L'\colon S^{p'} \amalg S^{q'} \to S^{m'}$ be link maps and denote $L = L_1 \amalg L_2$, $L' = L_1' \amalg L_2'$. Define their *join* (as links) to be

$$L \circledast L' = (L_1 * L_1') \amalg (L_2 * L_2).$$

Thus $L \circledast L'$ is a link map $S^{p+p'+1} \amalg S^{q+q'+1} \to S^{m+m'+1}$. Notice that $L \circledast L'$ is a link (that is, an embedding) iff both L and L' are links. One could, of course, similarly define the join of links of arbitrarily many components, so long as the two links have the same number of components.

EXAMPLE 4.2. Let $L_0\colon S^0 \amalg S^0 \to S^1$ be the nontrivial link. Then $L_0 \circledast L_0$ is the Hopf link in S^3 (that is, equivalent to perfect circles with linking number 1).

In order to study the relation of join with the α-invariant, we need to consider, however, links in euclidean space, rather than in a sphere. Unfortunately, the join of two euclidean spaces need not even be a manifold! However, if all of the maps avoid a "point at infinity" we see that the above is homeomorphic to the following explicit construction. Consider link maps

$$L\colon S^p \amalg S^q \to \mathbf{R}^m \quad \text{and} \quad L'\colon S^{p'} \amalg S^{q'} \to \mathbf{R}^{m'}.$$

Consider $\mathbf{R}^{m+m'+1} = \mathbf{R}^m \times \mathbf{R}^{m'} \times \mathbf{R} = \{(x, y, t)\}$. We can realize the join of maps into $\mathbf{R}^m$ and into $\mathbf{R}^{m'}$ by considering $\mathbf{R}^m$ embedded in $\mathbf{R}^{m+m'+1}$ as $\mathbf{R}^m \times \{0\} \times \{-1\}$ and $\mathbf{R}^{m'}$ as $\{0\} \times \mathbf{R}^{m'} \times \{+1\}$, and then joining the maps via straight line segments. In other words, if $(x, y, t) \in S^p * S^{p'}$ or $S^q * S^{q'}$, then

$$(1)\quad (L \circledast L')(x, y, t) = \left(\frac{(1-t)}{2} L(x), \frac{(1+t)}{2} L'(y), t\right) \in \mathbf{R}^m \times \mathbf{R}^{m'} \times \mathbf{R}.$$

PROPOSITION 4.8. *The join $\circledast$ respects homotopy of link maps. Moreover, $L \circledast L'$ is an embedding if and only if both L and L' are embeddings. If L or L' is link-homotopically trivial, then so is $L \circledast L'$. We also have the following formulas, where $\approx$ means link-homotopic:*

$$L \circledast (L' \circledast L'') \approx (L \circledast L') \circledast L'',$$
$$L \circledast (L' \# L'') \approx (L \circledast L') \# (L \circledast L'').$$

EXAMPLE 4.3. If $H\colon S^1 \amalg S^1 \to \mathbf{R}^3$ is the Hopf link, and L is any link or link map, then $H \circledast L \approx \Sigma_2\Sigma_2\Sigma_1\Sigma_1 L$. This can be deduced from $\Sigma_2\Sigma_1 L = L \circledast L_0$, where L_0 is the link of Example 4.2.

One of our main results is the formula:

PROPOSITION 4.9. *$A(L \circledast L') = \pm A(L) * A(L')$ for any 2-component link maps L and L'. The sign depends only upon dimensions and orientation convention.*

Proof. Using the notation established above, we recall that $A(L) \in \pi_{p+q+1}(S^m)$ corresponds to the framed manifold $(\tilde{\phi}L)^{-1}(\rho) \subset S^p \times S^q$, included naturally in $S^p * S^q$ with an added trivial bundle. Here $\tilde{\phi}L\colon S^p \times S^q \to \mathbf{R}^m - 0$ is the map $(x, y) \to L(x) - L(y)$ and ρ is a ray (spanned by vector $v \in S^{m-1}$) transverse to $\tilde{\phi}L$. Similarly $A(L')$ corresponds to a pullback $(\tilde{\phi}L')^{-1}(\rho')$, in $S^{p'} \times S^{q'} \subset S^{p'} * S^{q'}$, of a ray $\rho' = (0, \infty) \cdot v'$. It follows that $A(L) * A(L')$ is represented by the product submanifold

$$(\tilde{\phi}L)^{-1}(\rho) \times (\tilde{\phi}L')^{-1}(\rho') \subset (S^p \times S^q) \times (S^{p'} \times S^{q'}),$$

with the product framing, plus three trivial line bundles, when included in $(S^p * S^q) * (S^{p'} * S^{q'}) \equiv S^{p+p'+q+q'+3}$.

Now consider $A(L \circledast L')$. It is easy to check that the ray $\overline{\rho}$ spanned by $(v, v', 0)$ in $\mathbf{R}^{m+m'+1} \equiv \mathbf{R}^m \times \mathbf{R}^{m'} \times \mathbf{R}$ is transverse to the map

$$\tilde{\phi}(L \circledast L')\colon (S^p * S^{p'}) \times (S^q * S^{q'}) \to \mathbf{R}^{m+m'+1} - \{0\}.$$

Moreover, we see that $((x, x', t_1), (y, y', t_2)) \in [\tilde{\phi}(L \circledast L')]^{-1}(\overline{\rho})$ if and only if:

(a) $t_1 = t_2 = T(x, x', y, y')$,
(b) $(x, y) \in (\tilde{\phi}L)^{-1}(\rho)$, and
(c) $(x', y') \in (\tilde{\phi}L')^{-1}(\rho')$.

Here

$$T(x, x', y, y') = \frac{|L(x) - L(y)| - |L'(x') - L'(y')|}{|L(x) - L(y)| + |L'(x') - L'(y')|}.$$

This means that the submanifold of $(S^p * S^{p'}) \times (S^q * S^{q'})$ corresponding to $A(L \circledast L')$ is naturally diffeomorphic to the product of the manifolds $(\tilde{\phi}L)^{-1}(\rho)$ and $(\tilde{\phi}L')^{-1}(\rho)$. We leave the reader to check that the framings correspond as well, up to orientation of the additional trivial bundles, and we conclude $A(L) * A(L') = \pm A(L \circledast L')$.

Precomposition. We wish to relate the A-invariant of a link map $f\colon S^p \amalg S^q \to \mathbf{R}^m$ with that of a composite $F\colon S^{p'} \amalg S^{q'} \to \mathbf{R}^m$, where $F = f(g \amalg h)$ is the composite of f with maps $g\colon S^{p'} \to S^p$ and $h\colon S^{q'} \to S^q$. Note that $A(f) \in \pi_{p+q+1}(S^m)$, while $A(F) \in \pi_{p'+q'+1}(S^m)$. Let $[g * h] \in \pi_{p'+q'+1}(S^{p+q+1})$ denote the class of the join of g and h.

PROPOSITION 4.10. $A(F) = A(f) \cdot [g * h]$, *where* $\cdot$ *is the usual homotopy-theoretic composition operation.*

Proof. This is a direct consequence of the definition of the A-invariants and the commutativity of the following diagram:

$$\begin{tikzcd}
S^{p'} \times S^{q'} \arrow[rr] \arrow[dr, "\phi F"] \arrow[dd, "g\times h"'] & & S^{p'+q'+1} \arrow[dr, "G\phi F"] \arrow[dd, "g*h"] & \\
& S^{m-1} \arrow[rr, "\Sigma"] & & S^m \\
S^p \times S^q \arrow[rr] \arrow[ur, "\phi f"] & & S^{p+q+1} \arrow[ur, "G\phi f"'] &
\end{tikzcd}$$

We conclude this section with some examples. We saw (§3) that there is a link map $f\colon S^2 \amalg S^2 \to \mathbf{R}^4$ which has $A(f)$ equal to the nonzero element, say η, of $\pi_5(S^4) = \mathbf{Z}/2\mathbf{Z}$. Let $h\colon S^3 \to S^2$ be the Hopf map (so that η is represented by the stable suspension of h). Now precompose f by the map h on the first S^2 and by the identity on the second S^2. This gives a link map $F\colon S^3 \amalg S^2 \to \mathbf{R}^4$. We compute $A(F) \in \pi_6(S^4) = \mathbf{Z}/2\mathbf{Z}$ and using Proposition 4.10: $A(F) = A(f) \cdot [h * i] = \eta^2$ which is the generator of the stable 2-stem, and we conclude the following.

EXAMPLE 4.4. There is a link map $F\colon S^3 \amalg S^2 \to \mathbf{R}^4$ realizing the nonzero element of $\pi_6(S^4) = \mathbf{Z}/2\mathbf{Z}$.

Precomposing the same f by the Hopf map on both components gives a link map $S^3 \amalg S^3 \to \mathbf{R}^4$ whose A-invariant lies in the *un*stable group $\pi_7(S^4)$. However, if we stabilize, it goes to the element η^3 in the stable 3-stem, $\mathbf{Z}/24\mathbf{Z}$, and it is known to be nonzero there (see **[Toda]**, p. 190).

EXAMPLE 4.5. There is a link map $S^3 \amalg S^3 \to \mathbf{R}^4$ (described above) whose A-invariant is the element $(0, 6)$ in $\pi_7(S^4) = \mathbf{Z} \oplus \mathbf{Z}/12\mathbf{Z}$.

Question. Which other nonzero elements of $\pi_7(S^4)$ may be realized as invariants of link maps of two 3-spheres in $\mathbf{R}^4$? The following argument, pointed out by K. Lam, shows that the $\mathbf{Z}$-component, which is the Hopf invariant, of such an element must vanish. First, since one of the components, say the first, of the image of a link map $S^3 \amalg S^3 \to \mathbf{R}^4$ must be accessible from ∞, a simple geometric argument shows that the ϕ-construction gives a map $S^3 \times S^3 \to S^3$ whose restriction $* \times S^3 \to S^3$ has degree zero. It follows that the Hopf construction gives a map $S^7 \to S^4$ with Hopf invariant zero (see **[St]**, p. 13).

We can also employ suspensions of these examples to create further nontrivial link maps. For example, suspending the first component of the link map $f: S^2 \amalg S^2 \to \mathbf{R}^4$ with $\alpha(f) = \eta$ in the stable 1-stem gives a link map $\Sigma_1 f: S^3 \amalg S^2 \to \mathbf{R}^5$ with $\alpha(\Sigma_1 f) = \Sigma\eta = \eta$ in the 1-stem. This can be repeated indefinitely, suspending either component.

EXAMPLE 4.6. Given any integers $p > 1$ and $q > 1$, there is a nontrivial link map of $S^p \amalg S^q$ into $\mathbf{R}^{p+q}$.

EXAMPLE 4.7. If $p > 1$ and $q > 2$, there is a nontrivial link map of $S^p \amalg S^q$ into $\mathbf{R}^{p+q-1}$.

EXAMPLE 4.8. If $p > 2$ and $q > 2$, there is a nontrivial link map of $S^p \amalg S^q$ into $\mathbf{R}^{p+q-2}$.

If we try to go on to $\mathbf{R}^{p+q-3}$, we are stuck, due to the vanishing of the stable 4-stem.

Another way of constructing examples is through the join. Taking again the nontrivial map $f: S^2 \amalg S^2 \to \mathbf{R}^4$, we can join it with itself to obtain the link map $f \circledast f: S^5 \amalg S^5 \to \mathbf{R}^9$ with $\overline{\alpha}(f \circledast f) = \eta^2$ the generator of the stable 2-stem. We ask: is this homotopic with the link map of Example 4.7 with $p = q = 5$?

5. Filtration of homotopy groups. Consider the set of all elements of $\pi_r(S^m)$ which are realized as the A-invariants of link maps $S^p \amalg S^q \to \mathbf{R}^m$:

$$\mathbf{A}^m_{p,q} \equiv A(\mathbf{ELM}^m_{p,q}) \subset \pi_r(S^m), \qquad p + q + 1 = r.$$

These sets are well-behaved with respect to the operations $+, -, *$ (join), E (suspension) on homotopy groups of spheres.

PROPOSITION 5.1. *For $p+q+1 = r$, the subsets $\mathbf{A}^m_{p,q} \subset \pi_r(S^m)$ satisfy the following*:

(i) $\mathbf{A}^m_{p,q}$ *is nonempty iff $p \geq 0, q \geq 0$, and $m \geq 1$*;

(ii) *If* $\beta \in \mathbf{A}^m_{p,q}$, *then* $-\beta \in \mathbf{A}^m_{p,q}$;
(iii) $\mathbf{A}^m_{p,q}$ *is a subgroup of* $\pi_r(S^m)$ *if* $p, q, m \geq 1$;
(iv) *Compatibility with suspension*:

$$\mathbf{A}^{m+1}_{p+1,q} \supset \mathbf{E}A^m_{p,q} \subset \mathbf{A}^{m+1}_{p,q+1}$$

(v) *Symmetry*: $\mathbf{A}^m_{p,q} = \mathbf{A}^m_{q,p}$
(vi) *Multiplicativity*: $(r' = p' + q' + 1)$

$$\mathbf{A}^m_{p,q} * \mathbf{A}^{m'}_{p',q'} \subset \mathbf{A}^{m+m'+1}_{p+p'+1,q+q'+1} \subset \pi_{r+r'+1}(S^{m+m'+1}).$$

Proof. (i) is obvious and (ii) can be seen by reflecting $\mathbf{R}^m$. Then (iv) follows, since $\mathbf{A}^m_{q,p} = \pm\mathbf{A}^m_{p,q}$. The rest of the proposition follows easily from the results of the previous section.

EXAMPLE 5.1. As a trivial example, $\mathbf{A}^1_{0,0}$ is the subset $\{-1, 0, 1\}$ of the group $\pi_1(S^1) = Z$, thus not a subgroup. On the other hand $\mathbf{A}^2_{1,0} = \mathbf{A}^2_{0,1} = \pi_2(S^2) = Z$.

Now let us pass to stable homotopy by applying stable suspension E^∞. In the stable n-stem

$$\Pi^S_n \equiv E^\infty \pi_{m+n}(S^m)$$

we define the subsets

$$\Pi^{p,q}_n \equiv \alpha(\mathbf{ELM}^{p+q+1-n}_{p,q}) \subset \Pi^S_n.$$

From the relation $E^\infty(\mathbf{A}^m_{p,q}) = \Pi^{p,q}_n, n = p + q + 1 - m$, we see that these sets form a symmetric bifiltration of the stable homotopy ring $\Pi^S_* = \bigoplus_{n=0}^\infty \Pi^S_n$.

PROPOSITION 5.2. *The subsets* $\Pi^{p,q}_n$ *of* Π^S_n *satisfy*:
(i) $\Pi^{p,q}_n$ *is nonempty iff* $p, q \geq 0$ *and* $p + q \geq n$;
(ii) *if* $\beta \in \Pi^{p,q}_n$, *then* $-\beta \in \Pi^{p,q}_n$;
(iii) $\Pi^{p,q}_n$ *is a subgroup of* Π^S_n, *at least if* $p, q \geq 1$ *and* $p + q \geq n$;
(iv) *Double filtration property*:

$$\Pi^{p+1,q}_n \supset \Pi^{p,q}_n \subset \Pi^{p,q+1}_n;$$

(v) *Symmetry*: $\Pi^{p,q}_n = \Pi^{q,p}_n$
(vi) $\Pi^{p,q}_n * \Pi^{p',q'}_{n'} \subset \Pi^{p+p'+1,q+q'+1}_{n+n'}$.

Calculations and further properties of this bifiltration will appear in **[Ko2]**, where link maps into S^m are also studied. One can similarly define a finer bifiltration using embedded links rather than link maps.

References

[E] B. Eckmann, *Systeme von Richtungsfeldern auf Sphacren und stetige Lösengen linearer Gleichungen*, Comment. Math. Helv., **15** (1942), 1–26.

[F-R] R. A. Fenn and D. Rolfsen, *Spheres may link homotopically in 4-space*, J. London Math. Soc., (2) **34** (1986), 177–184.

[K] M. Kervaire, *An interpretation of G. Whitehead's generalization of H. Hopf's invariant*, Annals of Math., **69** (1959), 345–365.

[Ki] P. Kirk, *A link homotopy invariant for* $S^k \cup S^{2k-2} \to S^{2k}$, Brandeis preprint.

[Ko] U. Koschorke, *Higher-order homotopy invariants for higher-dimensional link maps*, Lecture Notes in Math v. 1172, Springer-Verlag (1985), 116–129.

[Ko2] ——, *On link maps and the geometry of their invariants*, Manuscripta Math., **61** (1988), 383–415.

[M-R] W. S. Massey and D. Rolfsen, *Homotopy classification of higher-dimensional links*, Indiana U. Math. J., **34** (1985), 375–391.

[P] L. Pontrjagin, *Classification homotopique des applications de la sphère à* $(n+2)$*-dimensions sur celle à n-dimension*, Comptes rendus de l'Académie des Sciences de l'U.R.S.S., **70** (1950), 957–959.

[S] P. Scott, *Homotopy links*, Abh. Math. Sem. Hamburg, **32** (1968), 186–190.

[St] N. E. Steenrod, *Cohomology operations* (notes by D. B. A. Epstein), Ann. of Math. Studies, **50** (1962).

[T] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv., **28** (1954), 17–86.

[W] G. W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, 1978.

Received July 15, 1987, and in revised form May 18, 1988. Research by the first author was supported in part by Siegen University granting a sabbatical term. Research by the second author was supported in part by the Natural Sciences and Engineering Research Council of Canada.

University of Siegen
Siegen, West Germany

and

University of British Columbia
Vancouver, B.C. V6T 1Y4, Canada