PACIFIC JOURNAL OF MATHEMATICS
Vol. 143, No. 2, 1990

ABOUT COMPRESSIBLE VISCOUS FLUID FLOW IN A BOUNDED REGION

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This paper deals with the question of existence for all times of the solutions of a certain class of differential equations for small initial values, and with the asymptotic behavior of these solutions. This class of equations contains different models describing the flow of viscous compressible fluids, even under the influence of a magnetic field.

1. Introduction. We consider the initial-boundary value problem on a bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions $(\partial\Omega \in C^3)$. The solutions of our equations are functions $X: \overline{\Omega} \times [0, \infty) \to \mathbb{R}^{m+1}$ $(X = X(x, t),\ m \geq n)$ representing the relevant physical variables in their dependence on space and time. For the sake of simplicity we assume that $\Omega_1 \subset \mathbb{R}^{m+1}$ is a convex domain containing all physically reasonable values of X. The set might, e.g., include only positive values for density (which is usually the $(m+1)$ st component of X), and temperature. Then our equations have the form

$$\text{(E)}\quad X_t^l + f^l(X, \nabla X) = L_X^l X + g^l(x, t) \qquad (l = 1, \dots, m),$$
$$X_t^{m+1} + \sum_{i=1}^{n} (X^{m+1} X^i)_{x_i} = 0$$

with sufficiently regular functions $f: \Omega_1 \times \mathbb{R}^{n(m+1)} \to \mathbb{R}^m$ and $g: \Omega \times [0, \infty) \to \mathbb{R}^m$, and an elliptic operator $L_Y X$ given by

$$L_Y^l X = \sum_{i=1}^{m} \sum_{j,k=1}^{n} a_i^{kjl}(Y) X_{x_k x_j}^i \qquad (l = 1, \dots, m)$$

with $a_i^{kjl} \in C^3(\Omega_1)$. The solutions we obtain are small in the sense of being close to a constant state $H_\infty \in \Omega_1$. The most important hypothesis we need for our result is expressed in Condition C, which states roughly that the linearization of our equation at H_∞ generates an analytic semigroup with exponential decay for $t \to \infty$ on a product of L_p-based Sobolev spaces for some $p > n$. This decay is the most important factor in the proof of a priori estimates for solutions of

(E) in Chapter 2, and their differences in Chapter 3, which enable us to prove the existence of a solution of E if g is sufficiently small and the initial values are sufficiently close to H_∞, and to describe its asymptotic behavior.

In contrast to the situation in one (see, e.g., **[4]**) and two space dimensions (in the latter case there are at least global weak solutions for isothermal gas flow in $\mathbb{R}^2$, as is shown in **[8]**), the only known results about existence for all times of solutions for the equations of viscous compressible fluid flow in higher dimensions impose smallness conditions on initial and boundary data, and the exterior forces. They were proved by Matsumura and Nishida (**[5]**, **[6]**, **[7]**) for domains with and without boundary. These papers also contain results about the asymptotic behavior of the solution; here a smallness condition for exterior forces and boundary values seems to be necessary, as turbulence is bound to occur at some point, preventing convergence as $t \to \infty$. For bounded domains our results go beyond those of Matsumura and Nishida. We give a rather general sufficient condition for the existence of a small solution for all times, we do not require the exterior forces to be conservative, nor the boundary values to be constant, and we do not need to confine ourselves to three dimensions. (For the incompressible case see, e.g., **[12]**.)

Let us now discuss some examples of systems of equations accessible to our method. In **[10]** Condition C is verified for compressible, viscous, and heat conducting flow. The system of equations

$$\rho_t + \operatorname{div}(\rho u) = 0,$$

$$\begin{aligned} u_t + (u \cdot \nabla) u + \frac{1}{\rho} \nabla (p(\rho, \theta)) \\ = \frac{\sigma}{\rho} (u \times B) \times B + \frac{1}{\rho} \left[\operatorname{div}(\mu \nabla u) + \nabla\left((\mu' + \mu) \operatorname{div} u\right)\right], \end{aligned}$$

$$\theta_t + (u \cdot \nabla) \theta + \frac{\theta p_\theta(\rho, \theta)}{\rho c} \operatorname{div} u = \frac{1}{\rho c} (\operatorname{div}(k \nabla \theta) + \psi)$$

with

$$\psi = \frac{\mu}{2}(u^j_{x_k} + u^k_{x_j})^2 + \mu' (\operatorname{div} u)^2 + \sigma |u \times B|^2,$$

which describes the flow of an ionized gas with density ρ, temperature θ, and velocity u (μ, μ^1 viscosity coefficients, c heat capacity, σ electric conductivity, $p = p(\rho, \theta)$ pressure) under the influence of a magnetostatic field $B(x) \in C^3(\overline{\Omega})$, which is assumed not to be perceptibly influenced by the currents in the gas, is another example (see **[2]**, **[9]**). (Note that B is not required to be small.) The proof that this system fulfills C parallels [10] closely. For the analog of Lemma 3.5 use

$$((u \times B) \times B) \cdot \overline{u} = (u \times B) \cdot (B \times \overline{u}) = -(u \times B) \cdot \left(\overline{u \times B}\right) = -|u \times B|^2 .$$

The magneto-fluid-dynamic system obtained by adding B as an unknown function, and completing the system by the equation

$$B_t = \sigma^{-1} \mu_m^{-1} \Delta B - \operatorname{curl}(u \times B)$$

(μ_m magnetic permeability) also fulfills condition C if the part of H_∞ corresponding to B is zero. Dirichlet boundary conditions for B seem, however, not very suited to this problem, as they usually interfere with the condition $\operatorname{div} B = 0$.

Now we make our statements precise. (For the definitions of some of the objects mentioned here see 1.3.) We denote the two groups of variables of f by H and P $(f = f(H, P))$, with $H = (H^1, \dots, H^{m+1}) \in \mathbb{R}^{m+1}$, $P = (P_j^i)_{i=1}^{m+1}{}_{j=1}^{n} \in \mathbb{R}^{n(m+1)}$. Then we suppose $f \in C^3$, $f(H_\infty, 0) = 0$, $f_{H^{m+1}}(H_\infty, 0) = 0$, and finally $\pi_n H_\infty = 0$, which is reasonable if the first n components of X represent the velocity of the flow. We also assume the ellipticity condition

$$\sum_{i=1}^{m} a_i^{kjl}(H)\, \eta_i \eta_l \xi_k \xi_j > 0 \qquad (H \in \Omega_1 ,\ \eta \in \mathbb{R}^m ,\ \xi \in \mathbb{R}^n ,\ \eta \neq 0 \neq \xi).$$

As a final step we make a number of definitions needed to state Condition C. For technical reasons the Sobolev spaces in the following section are complex, although our solutions are real. We first define the linear operator

$$A_1 \colon (H_p^2)^m \times H_p^1 \to (L_p)^m \times H_p^1$$

by

$$(A_1 X)^l = f_{H_i}^l(H_\infty, 0)\, X^i + \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} f_{P_\nu^\mu}^l(H_\infty, 0)\, X_{y_\nu}^\mu - L_{H_\infty} X \qquad (l = 1, \dots, m),$$

and $(A_1 X)^{m+1} = H_\infty^{m+1} \sum_{i=1}^n X_{x_i}^i$. Now we need two spaces defined in analogy to [10]:

$$B_p = \left\{ X \colon \Omega \to \mathbb{C}^{m+1} | X \in (L_p)^m \times H_p^1 ,\ \int_\Omega X^{m+1}\, dx = 0 \right\},$$

$$D_A^p = B_p \cap \left\{ X \in (H_p^2)^m \times H_p^1 \ | \pi_m X | \partial\Omega = 0 \right\}.$$

These sets are Banach spaces with respect to any norm of $(L_p)^m \times H_p^1$ and $(H_p^2)^m \times H_p^1$, respectively. We choose one of these in each case and denote them by $\|\cdot\|_{B_p}$ and $\|\cdot\|_{D_A^p}$. Then let $A = A_1|D_A^p$; note that $A(D_A^p) \subset B_p$. This allows us to formulate

Condition C. The operator $A\colon D_A^p \to B_p$ is closed, and there are numbers $\eta > 0$, and $K_2 < \infty$ such that the resolvent $(A + zI)^{-1}$ of A exists at least for all elements z of

$$\mathscr{R} = \left\{ z \in \mathbb{C} \mid \operatorname{Re} z \geq -2\eta, \text{ or } \operatorname{Re} z \geq -|\operatorname{Im} z| \text{ and } |z| \geq \eta^{-1} \right\},$$

and

$$\left\| (A + zI)^{-1} X \right\|_{B_p} \leq K_2 (1 + |z|)^{-1} \|X\|_{B_p}$$

for all $X \in B_p$, $z \in \mathscr{R}$. In addition,

$$\|X\|_{D_p} \leq K_2 \|(A + zI) X\|_{B_p},$$

$$\|X\|_{(H_p^1)^m \times H_p^2} \leq K_2 (1 + |z|)^{-1/2} \|(A + zI) X\|_{(H_p^1)^m \times H_p^2}$$

and

$$\|X\|_{(H_p^3)^m \times H_p^2} \leq K_2 (1 + |z|)^{1/2} \|(A + zI) X\|_{(H_p^1)^m \times H_p^2}$$

for all $z \in \mathscr{R}$, $X \in D_A^p$, and $X \in (H_p^3)^m \times H_p^2$ in addition for the last two inequalities. Now we can state our two theorems.

1.1. THEOREM. *Assume the system of equations* (E) *together with* $H_\infty \in \Omega_1$ *fulfill Condition* C, *and the other hypotheses stated as yet. Then there are numbers* $\varepsilon_1 > 0$, *and* $K_1 < +\infty$ *such that if* $X_0 \in H_p^2$, $(\beta, g) \in \mathscr{B}$ (*see* 1.3) *with* $\beta(0) = X_0|\partial\Omega$, $\pi_n \beta = 0$, *and*

$$\|X_0 - H_\infty\|_{H_p^2} + \|(\beta - \pi_m H_\infty, g)\|_{\mathscr{B}} \leq \varepsilon_1,$$

then there is exactly one function $X = X(x, t)$ *belonging to*

$$C^0([0, \infty), H_p^2) \cap C^1([0, \infty), L_p) \cap C^1((0, \infty), H_p^1)$$

with $\pi_m X \in C^0((0, \infty), H_p^3)$, *and solving* (E) *in the classical sense for* $t > 0$, *such that* $\pi_m X|\partial\Omega = \beta$, *and* $X(x, 0) = X_0(x)$ *for* $x \in \overline{\Omega}$. *For* $t \geq 1$ *this function fulfills the inequality*

$$\|X(x, t)\|_{H_p^2} + \|\pi_m X(x, t)\|_{H_p^3} \leq K_1 \left(\|X_0 - H_\infty\|_{H_p^2} + \|(\beta - \pi_m H_\infty, g)\|_{\mathscr{B}} \right).$$

1.2. THEOREM. *There is an* $\varepsilon_2 > 0$ *such that if*

$$\|X_0 - H_\infty\|_{H_p^2} + \|(\beta - \pi_m H_\infty, g)\|_{\mathscr{B}} \le \varepsilon_2$$

in addition to the assumptions of 1.1, *and* $\beta(t) \to \hat{\beta}$ *in* $H_p^2(\partial\Omega)$, $\beta_t \to 0$ *in* $L_p(\partial\Omega)$, *and* $g(t) \to \hat{g}$ *in* L_p $(t \to \infty)$, *then the solution* $X(x, t)$ *mentioned in* 1.1 *converges weakly in* H_p^2 *to a function* $\widehat{X}$, *and* $\pi_m X(x, t)$ *even converges weakly in* H_p^3 *as* $t \to \infty$. *Also* $\widehat{X}$ *is a time-independent solution of* (E) *with boundary values* $\hat{\beta}$ *and right side* $\hat{g}$. *It is the only such solution fulfilling the inequality*

$$\|\widehat{X} - H_\infty\|_{H_p^2} + \|\pi_m(\widehat{X} - H_\infty)\|_{H_p^3} \le K_1\varepsilon_2.$$

1.3. *Notation.* In the symbols used for function spaces the set on which the functions are defined is often omitted, this means usually that the set is Ω, except in statements like $f \in C^3$, where f, and its domain, have been given before. In general we take over the notation of **[11]**, with the exception that X represents here what would have been $X + H_\infty$ there. Let

$$\mathscr{B} = \Big\{(\beta, g) \,|\, \beta\colon \partial\Omega \to \mathbb{R}^m,\ g\colon \overline{\Omega} \to \mathbb{R}^m,\ \beta \in C^1\big([0, \infty), H_p^3(\partial\Omega)\big) \cap C^2\big([0, \infty), H_p^1(\partial\Omega)\big),\ \text{and}\ g \in C^0\big([0, \infty), H_p^2\big) \cap C^1\big([0, \infty), H_p^1\big)\Big\},$$

for $(\beta, g) \in \mathscr{B}$ we define

$$\|(\beta, g)\|_{\mathscr{B}} = \|\beta\|_{C^1([0,\infty), H_p^3(\partial\Omega))} + \|\beta\|_{C^2([0,\infty), H_p^1(\partial\Omega))} + \|g\|_{C^0([0,\infty), H_p^2)} + \|g\|_{C^1([0,\infty), H_p^1)}.$$

For $\mu \ge \nu$ let $\pi_r\colon \mathbb{R}^\mu \to \mathbb{R}^r$ be defined by $\pi(x_1, \dots, x_\mu) = (x_1, \dots, x_\nu)$. For $h\colon \Omega \to \mathbb{R}^{m+1}$, $h \in L_1$ let

$$M(h) = \left(0, \dots, 0, |\Omega|^{-1} \int_\Omega h^{m+1}\, dx\right) \quad \text{and}$$
$$\widetilde{H}_\infty(h) = \left(\pi_m H_\infty, M^{m+1}(h)\right).$$

All constants in this paper are independent of T unless otherwise stated.

2. Existence of a global solution. We begin this chapter by stating an easy consequence of the local existence theorem proved in

[11]. Then we show that every solution of (E)—expressed in Lagrange coordinates—solves the equation $X_t + A_1 X = G(X)$ with the $G(X)$ defined in 2.2. We use this information to prove an a priori estimate which—together with the local existence result 2.1—then gives the existence of a global solution.

2.1. LEMMA. *To every* $T > 0$ *there is a* $\delta_1 > 0$ *such that if* $(\beta, g) \in \mathscr{B}$, $X_0 \in H_p^2$, $X_0|\partial\Omega = \beta(0)$, $\pi_n \beta = 0$, *and*

$$\|(\beta - \pi_m H_\infty, g)\|_{\mathscr{B}} + \|X_0 - H_\infty\|_{H_p^2} \leq \delta_1,$$

then there is exactly one function $X \in C^0([0, T], H_p^2)$ *with* $\pi_m X \in C^0((0, T], H_p^3)$ *which is a classical solution of* (E) *for* $t > 0$ *with* $X(x, 0) = X_0(x)$ *for* $x \in \overline{\Omega}$, $X(x, t) = \beta(x, t)$ *for* $x \in \partial\Omega$, *and* X *fulfills the inequality*

$$\begin{aligned} &\|\pi_m (X - H_\infty)\|_{H_p^3} \sqrt{t} + \|X - H_\infty\|_{H_p^2} \\ &\quad \leq K_3(T) \left(\|(\beta - \pi_m H_\infty, g)\|_1 + \|X_0 - H_\infty\|_{H_p^2} \right). \end{aligned}$$

Proof. This can be obtained by repeated application of Theorem 1.1 and Theorem 5.2 of **[11]**.

By Lemma 2.6 and Theorem 2.7 of **[11]**, it is clear that $X(x, t)$ can be transformed into Lagrange coordinates in all of $[0, T]$ if

$$\|(\beta - \pi_m H_\infty, g)\|_{\mathscr{B}} + \|X_0 - H_\infty\|_{H_p^2}$$

is small enough, and that the transformed solution $X(y, t)$ has the property

$$\|X(y, t) - H_\infty\|_{\mathscr{S}(T)} \leq \widetilde{K}_3(T) \left(\|(\beta - \pi_m H_\infty, g)\|_{\mathscr{B}} + \|X_0 - H_\infty\|_{H_p^2} \right)$$

with a suitable constant $\widetilde{K}_3(T)$. (For the definition of $\mathscr{S}(T)$ see **[11]**, Def. 3.1.)

The following definitions will be applied to such solutions.

2.2. *Definition and Lemma.* To every $T > 0$ there is a $\delta_2 > 0$ such that if $X - H_\infty \in \mathscr{S}(T)$, $\|X - H_\infty\|_{\mathscr{S}(T)} \leq \delta_2$, and $\pi_n X|\partial\Omega = 0$, then

$$\underline{T}_t^X(y) = y + \int_0^t \pi_n X(y, \tau)\, d\tau \qquad (y \in \overline{\Omega})$$

is an admissible family of transformations on $\overline{\Omega}$ (see Chap. 2 of [**11**]). With $x = \underline{T}_t^X(y)$ we then define $G(X)$ as follows: For $l = 1, \ldots, m$ let

$$\begin{aligned} G^l(X) &= a_i^{kjl}(X(y, t))X_{y_\nu}^i((\underline{T}_t^X)^{-1})_{x_k x_j}^\nu(y, t) + f_{H_i}^l(H_\infty, 0)X^i \\ &\quad + \sum_{\nu=1}^n \sum_{\mu=1}^m f_{P_\nu^\mu}^l(H_\infty, 0)X_{y_\nu}^\mu - \overline{f}(X, X_{y_\nu}^i((\underline{T}_t^X)^{-1})_{x_\mu}^\nu) \\ &\quad + [a_i^{kjl}(Y)((\underline{T}_t^X)^{-1})_{x_k}^\nu((\underline{T}_t^X)^{-1})_{x_j}^\mu - a_i^{\nu\mu l}(H_\infty)]X_{y_\mu y_\nu}^i \\ &\quad + g(\underline{T}_t^X(y), t), \qquad (\overline{f}^l(H, P) = f^l(H, P) - H^j P_j^l), \end{aligned}$$

and, completing the definition,

$$G^{m+1}(X) = (H_\infty^{m+1} - X^{m+1}(y, t))X_{y_i}^i + X^{m+1}[\delta_i^\nu - ((\underline{T}_t^X)^{-1})_{x_i}^\nu]X_{y_\nu}^i.$$

Now we estimate $G(X)$ in terms of X.

2.3. Lemma. *For $T > 0$ there is a $\delta_3 > 0$ and a constant $K_4(T) < \infty$ such that if $X \in \mathscr{S}(T)$, $\pi_n X|\partial\Omega = 0$, and $\|X - H_\infty\|_{\mathscr{S}(T)} \le \delta_2$ then $G(X) \in C^0((0, T], (H_p^1)^m \times H_p^2)$, and*

$$\begin{aligned} &\|G\|_{C^0([t, T], (H_p^1)^m \times H_p^2)} + \|G\|_{C^{1/4}([t, T], (L_p)^m \times H_p^1)} \\ &\qquad \le t^{-1/2} K_4(T) \left(\|X - H_\infty\|_{\mathscr{S}(T)}^2 + \|(0, g)\|_{\mathscr{B}}\right). \end{aligned}$$

Proof. This is an easy consequence of Lemma 3.5 of [**11**].

2.4. Lemma. *The operator $-A$ generates an analytic semigroup on B_p, and there is a number $K_5 < +\infty$ such that for $X \in B_p$ we have*

$$\|e^{-tA}X\|_{B_p} \le K_5 e^{-\eta t}\|X\|_{B_p}, \tag{1}$$

$$\|e^{-tA}X\|_{D_A^p} \le K_5(1 + t^{-1})e^{-\eta t}\|X\|_{B_p}, \tag{2}$$

and if X also belongs to $(H_p^1)^m \times H_p^2$ the additional inequalities

$$\|e^{-tA}X\|_{(H_p^1)^m \times H_p^2} \le K_5(1 + t^{-1/2})e^{-\eta t}\|X\|_{(H_p^1)^m \times H_p^2}, \tag{3}$$

$$\|e^{-tA}X\|_{(H_p^3)^m \times H_p^2} \le K_5(1 + t^{-3/2})e^{-\eta t}\|X\|_{(H_p^1)^m \times H_p^2}, \tag{4}$$

$$\|e^{-tA}X\|_{H_p^2} \le K_5 e^{-\eta t}(\|X\|_{B_p}(1 + t^{-1}) + \|X\|_{(H_p^1)^m \times H_p^2}(1 + t^{-1/2})), \tag{5}$$

$$\|e^{-tA}X\|_{H_p^2} \le K_5 e^{-\eta t}(1 + t^{-1})\|X\|_{H_p^2} \ (X \in H_p^2) \tag{6}$$

are valid.

Proof. The inequalities (1)–(4) are easily obtained from Condition C using the integral representation from p. 103 in **[3]** for $A - \eta I$. These then imply (5) as

$$\|\pi_m e^{-tA} X\|_{H_p^2} \le C_1(t^{-1}+1)e^{-\eta t}\|X\|_{B_p} \quad \text{and}$$
$$\|(e^{-tA}X)^{m+1}\|_{H_p^2} \le C_2(t^{-1/2}+1)e^{-\eta t}\|X\|_{(H_p^1)^m \times H_p^2};$$

(6) is just a weakening of (5).

2.5. LEMMA. *If $X(y, t) \in \mathscr{S}(T)$ is a solution of* (E) *in Lagrange coordinates on $[0, T]$, if $\phi \in C^1([0,\infty), H_p^3) \cap C^2([0,\infty), H_p^1)$, $\phi^\nu \equiv 0$ for $\nu \in \{1, \dots, n, m+1\}$, and $\pi_m\phi|\partial\Omega = \pi_m X|\partial\Omega$, then for $t \in [0, T]$ we have $X_t + A_1 X = G(X)$ and*

$$\begin{aligned} X(y,t) &= M(X(y,t)) + \phi(y,t) \\ &\quad + e^{-tA}(X(y,0) - \phi(y,0) - M(X(y,0))) \\ &\quad + \int_0^t e^{-(t-s)A}(G(X) - M(G(X)) - \phi_t - A_1\phi)\, ds. \end{aligned}$$

Proof. Using Theorem 2.7 of **[11]**, we see that $X(y,t)$ fulfills the equation $X_t + A_1 X = G(X)$; but X does not usually even belong to B_p. However, $X - \phi - M(X-\phi) \in D_A^p$ for $t > 0$ and

$$\begin{aligned} &(X - \phi - M(X-\phi))_t + A(X - \phi - M(X-\phi)) \\ &\quad = G(X) - M(X_t - \phi_t) - A_1 M(X-\phi) - A_1\phi - \phi_t. \end{aligned}$$

Now $M\phi = 0$ as $\phi^{m+1} \equiv 0$, $A_1 M X = 0$ because $f_{H^{m+1}}(H, 0) = 0$, and

$$|\Omega|\,(MA_1X)^{m+1} = H_\infty^{m+1}\int_\Omega \operatorname{div}\pi_n X\, dy = H_\infty \int_{\partial\Omega}(\pi_n X, n)\, do = 0,$$

so $MX_t = -MA_1X + MG = MG$, and

$$(X - \phi - MX)_t + A(X - \phi - MX) = G(X) - MG(X) - A_1\phi - \phi_t.$$

As the right side of the above equation belongs to $C^{1/4}([\hat\delta, T], B_p)$ for all $\hat\delta > 0$ we get the desired representation from Theorem 3.2 (p. 109) of **[3]** for the interval $[\hat\delta, T]$, and can then let $\hat\delta$ go to zero.

2.6. LEMMA. *There is a constant $K_6 < +\infty$ such that if $h(t)\colon (0, T] \to B_p$ fulfills the inequality*

$$\|h\|_{C^{1/4}([t,T],B_p)} + \|h\|_{C^0([t,T],(H_p^1)^m\times H_p^2)} \le \frac{\alpha}{\sqrt{t}} \qquad (t>0),$$

then for $t \geq 2$ *we have*

$$\hat{h}(t) = \int_0^t e^{-(t-s)A} h(s)\, ds \in H_p^2$$

and $\|\hat{h}(t)\|_{H_p^2} \leq K_6 \alpha$.

Proof. We can write $\hat{h}(t)$ as the sum of the two functions

$$h_1(t) = \int_0^t e^{-(t-s)A}\, ds\, h(t)$$

and

$$h_2(t) = \int_0^t e^{-(t-s)A} (h(s) - h(t))\, ds.$$

As

$$\frac{d}{d\tau}\left(A^{-1} e^{-\tau A} h(t)\right) = -e^{-\tau A} h(t)$$

we have $h_1(t) = A^{-1}h(t) - A^{-1}e^{-tA}h(t)$; therefore

$$\|h_1(t)\|_{H_p^2} \leq C_1 \|h(t)\|_{(H_p^1)^m \times H_p^2} \leq C_2 \alpha \qquad (t \geq 2).$$

Using 2.4 we also obtain

$$\begin{aligned}
&\|h_2(t)\|_{H_p^2} \\
&\quad \leq C_3 \int_0^t e^{-\eta(t-s)} \Big[\|h(s) - h(t)\|_{B_p} \left(|s-t|^{-1} + 1\right) \\
&\qquad\qquad + \|h(s) - h(t)\|_{(H_p^1)^m \times H_p^2} \left(|s-t|^{-1/2} + 1\right)\Big]\, ds \\
&\quad \leq C_4 \int_0^{t/2} e^{-\eta(t-s)} \left[\|h(s)\|_{(H_p^1)^m \times H_p^2} + \|h(t)\|_{(H_p^1)^m \times H_p^2}\right] ds \\
&\qquad + C_5 \alpha \int_{t/2}^t e^{-\eta(t-s)} \left(|s-t|^{-3/4} + 1\right) ds \\
&\quad \leq C_6 \alpha \int_0^{t/2} e^{-\eta(t-s)} (1 + s^{-1/2})\, ds \\
&\qquad + C_7 \alpha \int_{t/2}^t e^{-\eta(t-s)} \left(1 + |s-t|^{-3/4}\right) ds \\
&\quad \leq C_8 \alpha \quad \text{for } t \geq 2.
\end{aligned}$$

This immediately implies our assertion.

2.7. Lemma. *For every* $T > 0$ *there are constants* K_7, $\widetilde{K}_7(T)$, *and a* $\delta_5 \in (0, \delta_1)$ *such that if* X_0, β, g *fulfill the conditions of* 2.1 *and*

$$\max\left(\|X_0 - H_\infty\|_{H_p^2}, \|(\beta - \pi_m H_\infty, g)\|\right) = \alpha \leq \delta_5,$$

then for the solution X *of* (E) *corresponding to* X_0, β, g *proven to exist in* 2.1 *we have the inequality*

$$\begin{aligned}
&\left\|X(x,t)-\widetilde{H}_\infty(X_0)\right\|_{H_p^2}\\
&\quad\le K_7\left(1+\widetilde{K}_7(T)\,\alpha\right)\\
&\qquad\times\left[e^{-\eta t}\left\|X(x,0)-\widetilde{H}_\infty(X_0)\right\|_{H_p^2}\right.\\
&\qquad\left.+\widetilde{K}_7(T)\left(\|(\beta-\pi_m H_\infty,\,g)\|_{\mathscr{B}}+\|X_0-H_\infty\|_{H_p^2}^2\right)\right]\\
&\hspace{20em}(t\in[2,T]).
\end{aligned}$$

Proof. First we choose $\delta_5>0$ so small that the solution obtained in 2.1 can be transformed into Lagrange coordinates over the whole interval $[0,T]$ by means of an admissible family of transformations $\underline{T}_t(y)$. Let $X(y,t)$ be this solution written in Lagrange coordinates. Note that there is a constant C_1 depending solely on the geometry of Ω such that a $\phi\in C^1([0,\infty),\,H_p^3)\cap C^2([0,\infty),\,H_p^1)$ can be chosen with $\pi_m\phi|\partial\Omega=\beta$, $\phi^{m+1}\equiv 0$, and

$$\|\pi_m(\phi-H_\infty)\|_{H_p^3}+\|\phi_t\|_{H_p^3}+\|\phi_{tt}\|_{H_p^1}\le C_1\,\|(\beta-\pi_m H_\infty,\,0)\|_{\mathscr{B}}\,.$$

Using 2.3, 2.4, 2.5, and 2.6, we get

$$\begin{aligned}
&\left\|X(y,t)-\widetilde{H}_\infty(X(y,t))\right\|_{H_p^2}\\
&\quad\le C_2e^{-\eta t}\left(\|X(x,0)-\widetilde{H}_\infty(X_0)\|_{H_p^2}+\|\pi_m(\phi(x,0)-H_\infty)\|_{H_p^2}\right)\\
&\qquad+\|\pi_m(\phi(y,t)-H_\infty)\|_{H_p^2}\\
&\qquad+\left\|\int_0^t e^{-(t-s)A}\,(G(X)-MG(X)-\phi_t-A_1\phi)\,ds\right\|_{H_p^2}\\
&\quad\le C_2e^{-\eta t}\|X(x,0)-\widetilde{H}_\infty(X_0)\|_{H_p^2}+C_3(T)\,\|(\beta-\pi_m H_\infty,\,g)\|_{\mathscr{B}}\\
&\qquad+C_4(T)\,\|X-H_\infty\|_{\mathscr{S}(T)}^2\,.
\end{aligned}$$

As $X(x,t)=X(\underline{T}_t^{-1}(x),t)$ we also get

$$\begin{aligned}
&\left\|X(x,t)-\widetilde{H}_\infty(X(y,t))\right\|_{H_p^2}\\
&\quad\le(1+C_5(T)\,\alpha)\left\|X(y,t)-\widetilde{H}_\infty(X(y,t))\right\|_{H_p^2}
\end{aligned}$$

and, using the time-independence of $\widetilde{H}_\infty(X(x, t))$,

$$\begin{aligned}
|\Omega| \cdot \left|\widetilde{H}_\infty(X(y, t)) - \widetilde{H}_\infty(X_0)\right|
&= \left|\int_\Omega X^{m+1}(y, t) - H_\infty^{m+1}\, dy - \int_\Omega X^{m+1}(x, t) - H_\infty^{m+1}\, dx\right| \\
&= \left|\int_\Omega \left(X^{m+1}(y, t) - H_\infty^{m+1}\right)\left(1 - J_{\underline{T}t}(y)\right) dy\right| \\
&\le C_6(T)\left\|X^{m+1}(y, t) - H_\infty\right\|_{L_\infty} \cdot \int_0^T \|\nabla \pi_m X\|_{L_\infty}\, d\tau \\
&\le C_7(T)\,\|X(y, t) - H_\infty\|^2_{\mathscr{S}(T)}.
\end{aligned}$$

By 2.1 we have

$$\|X - H_\infty\|_{\mathscr{S}(T)} \le C_8(T)\left(\|X(x, 0) - H_\infty\|_{H_p^2} + \|(\beta - \pi_m H_\infty, g)\|_{\mathscr{B}}\right),$$

which implies our claim.

Now we can prove Theorem 1.1.

First we choose $T \ge 2$ large enough so that $K_7 e^{-\eta T} \le \frac{1}{4}$. Once this is done, we need no longer indicate the dependence of our constants on T. With $\alpha_1 = \min(\delta_5, (4\widetilde{K}_7)^{-1}, (K_7\widetilde{K}_7 8)^{-1})$ we obtain from Lemma 2.7 that if

$$\max\left(\|X_0 - H_\infty\|_{H_p^2}, \|(\beta - \pi_m H_\infty, g)\|_1\right) \le \alpha_1,$$

then there is a solution X of (E) on $[0, T]$ fulfilling the inequality

$$\begin{aligned}
\|X(x, T) - \widetilde{H}_\infty(X_0)\|_{H_p^2}
&\le \frac{1}{3}\left\|X(x, 0) - \widetilde{H}_\infty(X_0)\right\|_{H_p^2} \\
&\quad + C_1\left(\|(\beta - \pi_m H_\infty, g)\|_{\mathscr{B}} + |H_\infty - \widetilde{H}_\infty(X_0)|\right) \\
&\quad + K_7\widetilde{K}_7\frac{5}{4}\alpha_1\left\|X(x, 0) - \widetilde{H}_\infty(X_0)\right\|_{H_p^2} \\
&\le \frac{1}{2}\|X(x, 0) - \widetilde{H}_\infty(X_0)\|_{H_p^2} \\
&\quad + C_1\left(\|(\beta - \pi_m H_\infty, g)\|_{\mathscr{B}} + |H_\infty - \widetilde{H}_\infty(X_0)|\right).
\end{aligned}$$

This immediately gives us

$$\begin{aligned}
\|X(x, T) - H_\infty\|_{H_p^2} &\le \frac{1}{2}\|X(x, 0) - H_\infty\|_{H_p^2} \\
&\quad + C_2\left(\|(\beta - \pi_m H_\infty, g)\|_1 + |H_\infty - \widetilde{H}_\infty(X_0)|\right).
\end{aligned}$$

If we choose ε_1 small enough to assure

$$C_2\left(\|(\beta - \pi_m H_\infty, g)\|_1 + |H_\infty - \widetilde{H}_\infty(X_0)|\right) \le \alpha_1/2,$$

then $\|X(x, 0) - H_\infty\|_{H_p^2} \le \alpha_1$ implies $\|X(x, T) - H_\infty\|_{H_p^2} \le \alpha_1$. So we can continue our solution to $[0, 2T]$. As $\widetilde{H}_\infty(X(x, t))$ is constant and $\|(\beta - \pi_m H_\infty, g)\|_{\mathscr{B}}$ cannot be larger for these new initial value problems than it was for the original one, this can be repeated indefinitely.

3. Asymptotic behavior. In order to prove Theorem 1.2 we consider solutions $\widetilde{X}_1$ and $\widetilde{X}_2$ of (E) with right sides $\tilde{g}_1$, $\tilde{g}_2$, and assume these, and the initial and boundary values fulfill the conditions of Theorem 1.1. We define $X_i(x, t) = \widetilde{X}_i(x, t+1)$, $\beta_i = X_i|\partial\Omega$, and $g_i(x, t) = \tilde{g}_i(x, t+1)$ $(x \in \overline{\Omega}, t \ge 0, i = 1, 2)$. We shall derive some estimates for $Z(x, t) = X_1(x, t) - X_2(x, t)$ assuming $M(X_1(x, t)) = M(X_2(x, t))$; i.e., $M(Z(x, t)) = 0$. With

$$\omega = \sum_{i=1}^{2}\left(\|\widetilde{X}_i(x, 0) - H_\infty\|_{H_p^2} + \left\|\left(\pi_m(\widetilde{X}_i|\partial\Omega - H_\infty), \tilde{g}_i\right)\right\|_{\mathscr{B}}\right)$$

then we have

$$\sum_{i=1}^{2}\left(\|X_i(x, t)\|_{H_p^2} + \|\pi_m X_i(x, t)\|_{H_p^3}\right) \le K_1\omega$$

by Theorem 1.1. For given $T < +\infty$ the transformation of any function into Lagrange coordinates with respect to X_1 on the interval $[0, T]$ is therefore possible by means of an admissible family of transformations $x = \underline{T}_t(y)$, if ω is small enough. In what follows we always assume this to be the case. Then we have

3.1. Lemma. *The transformation* $Z(y, t)$ *of* $Z(x, t)$ *into Lagrange coordinates with respect to* X_1 *fulfills the equation*

$$Z_t(y, t) + \widetilde{A}_1(t) Z(y, t) = \gamma(y, t) + B(t) Z(y, t)$$

$(\gamma = (g_1^1 - g_2^1, \dots, g_1^m - g_2^m, 0))$ *on* $[0, T]$ *with linear operators*

$$\widetilde{A}_1(t) : (H_p^2)^m \times H_p^1 \to (L_p)^m \times H_p^1 \quad and$$
$$B(t) : (H_p^2)^m \times H_p^1 \to (L_p)^m \times H_p^1$$

having the property that

$$\text{(1)} \qquad B(t)h \in C^0([0, T], (L_p)^m \times H_p^1)$$

$$\text{(2)} \qquad \|(\widetilde{A}_1(t) - \widetilde{A}_1(s))h\|_{B_p} \le K_8(T)\omega|t-s|^{1/2}\|h\|_{(H_p^2)^m \times H_p^1},$$

$$\text{(3)} \qquad \|(\widetilde{A}_1(t) - A_1)h\|_{B_p} \le K_8(T)\omega\|h\|_{(H_p^2)^m \times H_p^1},$$

$$\text{(4)} \qquad \|B(t)h\|_{(L_p)^m \times H_p^1} \le K_8(T)\omega\|h\|_{H_p^1}$$

for $t, s \in [0, 1]$, $h \in (H_p^2)^m \times H_p^1$ *with* $\pi_n h|\partial\Omega = 0$.

Proof. In the Euler coordinates we have for $l = 1, \dots, m$,

$$Z_t^l + f^l(X_1, \nabla X_1) - f^l(X_2, \nabla X_2) = L_{X_1}^l X_1 - L_{X_2}^l X_2 + \gamma^l.$$

With $X_\tau = (2-\tau)X_1 + (\tau - 1)X_2$ we then obtain

$$Z_t^l + \int_1^2 f_{H^i}^l(X_\tau, \nabla X_\tau)\, d\tau Z^i + \sum_{\nu=1}^n \sum_{\mu=1}^m \int_1^2 f_{P_\nu^\mu}^l(X_\tau, \nabla X_\tau)\, d\tau Z_{x_\nu}^\mu$$
$$= L_{X_1}^l Z + \int_1^2 (a_k^{\lambda\mu l})_{H^\nu}(X_\tau)\, d\tau Z^\nu (X_2^k)_{x_\lambda x_\mu} + \gamma^l.$$

Transforming this into Lagrange coordinates we see that

$$\widetilde{A}_1^l(t) Z = f_{H_i}^l(H_\infty, 0) Z^i + \sum_{\nu=1}^n \sum_{\mu=1}^m f_{P_\nu^\mu}^l(H_\infty, 0) Z_{y_\nu}^\mu$$
$$- a_\lambda^{k\mu l}(X_1)(\underline{T}_t^{-1})_{x_k}^\nu (\underline{T}_t^{-1})_{x_\mu}^\pi Z_{y_\nu y_\pi}^\lambda,$$

and the $B^l(t)$ determined thereby for $l = 1, \dots, m$ fulfill our conditions. For the $(m+1)$ st component we have in Euler coordinates

$$X_{it}^{m+1} + (X_i^{m+1} X_i^j)_{x_j} = 0 \qquad (i = 1, 2)$$

which implies

$$Z_t^{m+1} + X_1^j Z_{x_j}^{m+1} + Z^{m+1} X_{1x_j}^j + X_{2x_j}^{m+1} Z^j + X_2^{m+1} Z_{x_j}^j = 0,$$

so in Lagrange coordinates with respect to X_1 we get

$$0 = Z_t^{m+1}(y, t) + Z^{m+1}(\underline{T}_t^{-1})_{x_\nu}^\mu X_{1y_\mu}^\nu$$
$$+ X_{2y_\nu}^{m+1}(\underline{T}_t^{-1})_{x_j}^\nu Z^j + X_2^{m+1} Z_{y_\nu}^j (\underline{T}_t^{-1})_{x_j}^\nu.$$

With

$$\widetilde{A}^{m+1}(t)Z = X_2^{m+1} Z_{y_\nu}^j (\underline{T}_t^{-1})_{x_j}^\nu - M\left((0, \dots, 0, X_2^{m+1} Z_{y_\nu}^j (\underline{T}_t^{-1})_{x_j}^\nu)\right)$$

and the $B^{m+1}(t)$ thus determined our claims are easily verified as $H_\infty^{m+1} \int_\Omega Z_{y_\nu}^\nu \, dy = 0$ by Gauss's theorem, and therefore

$$\left\| M(0, \ldots, 0, X_2^{m+1} Z_{y_\nu}^j (\underline{T}_t^{-1})_{x_j}^\nu) \right\|_{L_p} \le C_2(T) \omega \|Z\|_{H_p^1}.$$

3.2. LEMMA. *There is a constant $K_9 < +\infty$ such that to every $T > 0$ there exists a $\delta_6 > 0$ having the following property: If $\omega \le \delta_6$, then the fundamental solution $\Gamma(t, s)$ on B_p (see* **[3]**, *part 2, chap. 4–6) of the equation*

$$h_t + \widetilde{A}(t) h = 0 \qquad \left(\widetilde{A}(t) = \widetilde{A}_1(t) \,|D_A^p \right)$$

exists for $t, s \in [0, T]$, $t \ge s$, and we have the inequality

$$\|\Gamma(t, s) h\|_{H_p^1} \le K_9 e^{-\eta(t-s)/2} \|h\|_{B_p} (1 + |t-s|^{-1/2})$$
$$(h \in B_p ; t, s \in [0, T], t \ge s).$$

Proof. From Theorem 3.1 (p. 109), Lemma 7.1 (p. 127), and Theorem 10.1 (p. 27) of **[3]** it is clear that the fundamental solution exists, and the estimate is valid for $t - s \le 2$. Also we can obviously assume

$$\|e^{-\tau \widetilde{A}(t)} h\|_{B_p} \le C_1 e^{-2\eta\tau/3} \|h\|_{B_p} \qquad (\tau \in [0, \infty))$$

with a suitable constant $C_1 < +\infty$. As $\Gamma(t, s) = \Gamma(t, t-1)\Gamma(t-1, s)$ for $t - s \ge 1$ we only need to prove

$$\|\Gamma(t, s) h\|_{B_p} \le \widetilde{C}_1 e^{-\eta\tau/2} \|h\|_{B_p},$$

which can be done using equation 4.8 (p. 111) of **[3]** and the method employed to obtain 13.15 on p. 154 of **[3]**.

3.3. LEMMA. *There are numbers K_{10}, δ_7, $T \in (0, \infty)$ such that $\omega \le \delta_7$ implies the inequality*

$$\begin{aligned} &\|Z(x, T)\|_{H_p^1} \\ &\quad \le \frac{1}{2} \|Z(x, 0)\|_{H_p^1} \\ &\qquad + K_{10} \max_{\tau \in [0, T]} \Big[\|g_1(\tau) - g_2(\tau)\|_{L_p} + \|\beta_1(\tau) - \beta_2(\tau)\|_{H_p^2(\partial\Omega)} \\ &\qquad\qquad + \|\beta_{1t}(\tau) - \beta_{2t}(\tau)\|_{L_p(\partial\Omega)} \Big]. \end{aligned}$$

Proof. We start out by deriving an estimate for fixed arbitrary T. There is a constant C_1 only depending on the geometry of Ω such that we can always find a $\phi \in C^1([0, \infty), H_p^3) \cap C^2([0, \infty), H_p^1)$ with the properties

$$\|\phi\|_{H_p^2} + \|\phi_t\|_{L_p} \leq C_1 \left(\|\beta_1 - \beta_2\|_{H_p^2} + \|\beta_{1t} - \beta_{2t}\|_{L_p} \right),$$

and $\pi_m \phi | \partial\Omega = \beta_1 - \beta_2$, $\phi^{m+1} \equiv 0$. In Lagrange coordinates with respect to X_1 now Z is easily seen to fulfill the equation

$$\begin{aligned}
&(Z - \phi - MZ)_t + \widetilde{A}(t)(Z - \phi - MZ) \\
&\quad = \gamma + B(t) Z - \widetilde{A}_1(t) \phi - \phi_t \\
&\qquad + \left(A_1 - \widetilde{A}_1(t) \right) MZ(y, t) - MB(t) Z,
\end{aligned}$$

as $AMZ = MA_1 Z = M\gamma = 0$. This gives us the representation

$$\begin{aligned}
Z(y, t) = \phi(y, t) + M(Z(y, t)) + \Gamma(t, 0)(Z(x, 0) - \phi(x, 0)) \\
+ \int_0^t \Gamma(t, s) \Big[\gamma + B(s) Z - MB(s) Z - \widetilde{A}_1(s) \phi - \phi_t \\
+ \left(A_1 - \widetilde{A}_1(s) \right) MZ(y, s) \Big] \, ds,
\end{aligned}$$

from which, using 3.2, we can derive the inequality

$$\begin{aligned}
&\sqrt{t} e^{\eta t/2} \|Z(y, t)\|_{H_p^1} \\
&\quad \leq \sqrt{t} e^{\eta t/2} \|MZ(y, t)\|_{H_p^1} + 2K_9 \|Z(x, 0)\|_{H_p^1} \\
&\qquad + C_2(T) \sup_{\tau \in [0, T]} \left(\|\gamma(\tau)\|_{L_p} + \|\phi_t(\tau)\|_{L_p} + \|\phi(\tau)\|_{H_p^2} \right) \\
&\qquad + C_3(T) \omega \sup_{\tau \in [0, T]} \left(\sqrt{t} e^{\eta\tau/2} \|Z(y, t)\|_{H_p^1} \right).
\end{aligned}$$

As in the proof of Theorem 2.7 we see that

$$|M(Z(y, t))| \leq C_4(T) \omega \|Z\|_{H_p^1},$$

so taking the supremum on both sides of the inequality for $t \in [0, T]$ we get

$$\begin{aligned}
&\sup_{t \in [0, T]} \left(\sqrt{t} e^{\eta t/2} \|Z(y, t)\|_{H_p^1} \right) \\
&\quad \leq C_5 \|Z(x, 0)\|_{H_p^1} \\
&\qquad + C_6(T) \sup_{\tau \in [0, T]} \left(\|\gamma(t)\|_{L_p} + \|\phi_t(t)\|_{L_p} + \|\phi(t)\|_{H_p^2} \right)
\end{aligned}$$

with a C_5 independent of T, provided ω is small enough. To obtain our statement we now choose T so large that $e^{-\eta T/2} C_5 \frac{1}{\sqrt{T}} \leq \frac{1}{4}$, and then δ_7 small enough to make the above estimate true for $\omega \leq \delta_7$ and in addition

$$\|Z(x, T)\|_{H_p^1} \leq 2\|Z(y, T)\|_{H_p^1}, \quad \|\gamma(y, t)\|_{L_p} \leq 2\|\gamma(x, t)\|_{L_p}$$

for $t \in [0, T]$.

Now we can prove 1.2. First consider the solution $\widetilde{X}(x, t)$ of the problem with $\beta(t) = \hat{\beta}$, $g(t) = \hat{g}$, and a suitable initial value, and let $\widetilde{X}_1 = \widetilde{X}$, $\widetilde{X}_2(t) = \widetilde{X}(t + \Delta t)$ with a $\Delta t > 0$. We choose $\varepsilon_2 > 0$ small enough to be able to apply 3.3 starting at any point $t \in [0, \infty)$ instead of 0. This gives us

$$\left\|\widetilde{X}(T + t + \Delta t) - \widetilde{X}(T + t)\right\|_{H_p^1} \leq \frac{1}{2}\left\|\widetilde{X}(t + \Delta t) - \widetilde{X}(t)\right\|_{H_p^1} \qquad (t \in [0, \infty)),$$

as our equation is autonomous in this case. Dividing by Δt and letting $\Delta t \to 0$ we get

$$\|\widetilde{X}_t(T + t)\|_{H_p^1} \leq \frac{1}{2}\|\widetilde{X}_t(t)\|_{H_p^1},$$

so $\widetilde{X}_t(t) \to 0$ in H_p^1 and $\int_1^\infty \|\widetilde{X}_t(t)\|_{H_p^1} < +\infty$.

So $\widetilde{X}$ converges in H_p^1, and its limit $\widehat{X}$ must belong to H_p^2 and $\pi_m \widehat{X} \in H_p^3$. As $\widehat{X}$ is a stationary solution of (E), it is also unique by 3.3.

For arbitrary $\beta(t)$, $g(t)$ we apply 3.3 to $\widetilde{X}_1 = X$, $\widetilde{X}_2 = \widehat{X}$, and obtain

$$\begin{aligned} &\|X(x, T + t) - \widehat{X}(x)\|_{H_p^1} \\ &\quad \leq \frac{1}{2}\|X(x, t) - \widehat{X}(x)\|_{H_p^1} \\ &\qquad + K_{10} \sup_{\tau \in [t, t+t]} \left(\|g(\tau) - \hat{g}\|_{L_p} + \|\beta(\tau) - \hat{\beta}\|_{H_p^2} + \|\beta_t\|_{L_p}\right). \end{aligned}$$

Taking the lim sup on both sides this implies $X(x, t) \to \widehat{X}(x)$ in H_p^1, from which the remainder follows easily.

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Received December 15, 1987.

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