SYMMETRY GROUPS, CONSERVATION LAWS AND GROUP–INVARIANT SOLUTIONS OF THE MEMBRANE SHAPE EQUATION

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Abstract. The six-parameter group of three dimensional Euclidean motions is recognized as the largest group of point transformations admitted by the membrane shape equation in Mongé representation. This equation describes the equilibrium shapes of biomembranes being the Euler-Lagrange equation associated with the Helfrich curvature energy functional under the constraints of fixed enclosed volume and membrane area. The conserved currents of six linearly independent conservation laws that correspond to the variational symmetries of the membrane shape equation and hold on its smooth solutions are obtained. All types of non-equivalent group-invariant solutions of the membrane shape equation are identified via an optimal system of one-dimensional subalgebras of the symmetry algebra. The reduced equations determining these group-invariant solutions are derived. Special attention is paid to the translationally-invariant solutions of the membrane shape equation.

1. Introduction

In aqueous solution, amphiphilic molecules (e.g., phospholipids) may form bilayers, the hydrophilic heads of these molecules being located in both outer sides of the bilayer, which are in contact with the liquid, while their hydrophobic tails remain at the interior. In many cases the bilayer form a closed membrane – that is a vesicle. Vesicles constitute a well-defined and sufficiently simple model system for studying basic physical properties of the more complex biological cells.

In 1973, Helfrich [4] has proposed that the equilibrium shapes of a lipid vesicle are determined by the extremals of the curvature (shape) energy

$$\mathcal{F}_c = \frac{k_c}{2} \int_{\mathcal{S}} (2H + \mathrm{Ih})^2 \, \mathrm{d}A + \bar{k} \int_{\mathcal{S}} K \, \mathrm{d}A$$

under the constraints of fixed enclosed volume and membrane area. Using Lagrangian multipliers, this yields the functional

$$\mathcal{F} = \frac{k_c}{2} \int_{\mathcal{S}} (2H + \mathrm{Ih})^2 \, \mathrm{d}A + \bar{k} \int_{\mathcal{S}} K \, \mathrm{d}A + \lambda \int_{\mathcal{S}} \mathrm{d}A + p \int \mathrm{d}V \qquad (1)$$

where k_c, $\bar{k}$, Ih, λ and p are real constants representing the bending and Gaussian rigidity of the membrane, the spontaneous curvature, tensile stress and osmotic pressure difference between the outer and inner media, respectively. The area element of the membrane middle-surface $\mathcal{S}$ is denoted by $\mathrm{d}A$, while H and K are the mean and Gaussian curvatures of $\mathcal{S}$, and $\mathrm{d}V$ is the volume element.

2. Membrane Shape Equation in Mongé Representation

The Euler–Lagrange equation corresponding to the functional (1) reads

$$2k_c \Delta H + k_c (2H + \mathrm{Ih})(2H^2 - \mathrm{Ih}H - 2K) - 2\lambda H + p = 0 \qquad (2)$$

where Δ is the **Laplace–Beltrami operator** on the surface $\mathcal{S}$. Equation (2), which is derived in [12] and [13], is often referred to as the membrane shape equation.

Let (x^1, x^2, x^3) be a fixed right-handed rectangular Cartesian coordinate system in the three-dimensional Euclidean space $\mathbb{R}^3$ in which our surface $\mathcal{S}$ is immersed, and let this surface be given by the equation

$$\mathcal{S} : x^3 = w(x^1, x^2), \qquad (x^1, x^2) \in \Sigma \subset \mathbb{R}^2$$

where $w : \mathbb{R}^2 \to \mathbb{R}$ is a single-valued and smooth function possessing as many derivatives as may be required in the domain Σ. Let us take x^1, x^2 to serve as Gaussian coordinates on the surface $\mathcal{S}$. Then, relative to this coordinate system, the components of the first fundamental tensor $g_{\alpha\beta}$, the second fundamental tensor $b_{\alpha\beta}$, and the alternating tensor $\varepsilon^{\alpha\beta}$ of $\mathcal{S}$ are given by the expressions

$$g_{\alpha\beta} = \delta_{\alpha\beta} + w_\alpha w_\beta, \qquad b_{\alpha\beta} = g^{-1/2} w_{\alpha\beta}, \qquad \varepsilon^{\alpha\beta} = g^{-1/2} e^{\alpha\beta} \qquad (3)$$

where

$$g = \det(g_{\alpha\beta}) = 1 + (w_1)^2 + (w_2)^2 \qquad (4)$$

$\delta_{\alpha\beta}$ is the Kronecker delta symbol and $e^{\alpha\beta}$ is the alternating symbol. The contravariant components $g^{\alpha\beta}$ of the first fundamental tensor read

$$g^{\alpha\beta} = g^{-1} \left(\delta^{\alpha\beta} + e^{\alpha\mu} e^{\beta\nu} w_\mu w_\nu \right). \qquad (5)$$

Here and in what follows: Greek indices have the range 1, 2, and the usual summation convention over a repeated indices is employed, $w_{\alpha_1 \ldots \alpha_k}$ ($k = 1, 2, \ldots$) denote the k-th order partial derivatives of the function w with respect to the variables x^1 and x^2, i.e.,

$$w_{\alpha_1 \alpha_2 \ldots \alpha_k} = \frac{\partial^k w}{\partial x^{\alpha_1} \ldots \partial x^{\alpha_k}}, \qquad k = 1, 2, \ldots .$$

The mean curvature H of the surface S and its Gaussian curvature K are given as follows

$$H = \frac{1}{2} g^{\alpha\beta} b_{\alpha\beta}, \qquad K = \frac{1}{2} \varepsilon^{\alpha\mu} \varepsilon^{\beta\nu} b_{\alpha\beta} b_{\mu\nu}$$

that is,

$$H = \frac{1}{2} g^{-3/2} \left(\delta^{\alpha\beta} + \mathrm{e}^{\alpha\mu} \mathrm{e}^{\beta\nu} w_\mu w_\nu \right) w_{\alpha\beta}, \qquad K = \frac{1}{2} g^{-2} \mathrm{e}^{\alpha\mu} \mathrm{e}^{\beta\nu} w_{\alpha\beta} w_{\mu\nu}. \tag{6}$$

In such Mongé representation of the surface S, the membrane shape equation (2) can be regarded as a fourth-order partial differential equation in two independent variables x^1, x^2 and one dependent variable – the so-called Mongé gauge w. Hereafter, when speaking about the membrane shape equation we will have in mind its Mongé representation. Now, using expressions (3)–(6) and the well-known formula

$$\Delta = g^{-1/2} \frac{\partial}{\partial x^\alpha} \left(g^{1/2} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right) = g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} + g^{-1/2} \frac{\partial}{\partial x^\alpha} \left(g^{1/2} g^{\alpha\beta} \right) \frac{\partial}{\partial x^\beta}$$

one can represent the membrane shape equation (2) in the form

$$\mathcal{E} \equiv \frac{1}{2} g^{-1/2} g^{\alpha\beta} g^{\mu\nu} w_{\alpha\beta\mu\nu} + \Phi \left(x_1, x_2, w, w_1, \ldots, w_{222} \right) = 0 \tag{7}$$

where $\Phi \left(x_1, x_2, w, w_1, \ldots, w_{222} \right)$ is a certain third order differential function, that is a function depending on the independent and dependent variables and the derivatives of the dependent variable up to third order.

3. Symmetry Groups

One of the principle results in [16], (Proposition 1) is that the ten-parameter Lie group of special conformal transformations in $\mathbb{R}^3$ is the largest group of geometric (point) transformations of the involved independent and dependent variables that a generic equation of the form (7) could admit. Using this result one can easily ascertain by inspection that the symmetry group of the membrane shape equation (2) is restricted to the group of motions in $\mathbb{R}^3$ whose basic generators $\mathbf{v}_j$ ($j = 1, \ldots, 6$) and their characteristics Q_j are given in Table 1.

Table 1. Generators and characteristics of the group of motions in $\mathbb{R}^3$

Generators	Characteristics
Translations	
$\mathbf{v}_1 = \frac{\partial}{\partial x^1}$	$Q_1 = -w_1$
$\mathbf{v}_2 = \frac{\partial}{\partial x^2}$	$Q_2 = -w_2$
$\mathbf{v}_3 = \frac{\partial}{\partial w}$	$Q_3 = 1$
Rotations	
$\mathbf{v}_4 = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$	$Q_4 = x^2 w_1 - x^1 w_2$
$\mathbf{v}_5 = -w \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial w}$	$Q_5 = x^1 + w w_1$
$\mathbf{v}_6 = -w \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial w}$	$Q_6 = x^2 + w w_2$

It should be remarked that the aforementioned symmetry group of the membrane shape equation (2) is inherent to its Mongé representation. The symmetry groups of the system of partial differential equations studied in [10], which is equivalent to the membrane shape equation (2), are completely different.

4. Conservation Laws

All vector fields $\mathbf{v}_j$ ($j = 1, \ldots, 6$) are variational symmetries of the membrane shape equation (2) and hence, in virtue of Noether's theorem, six linearly independent conservation laws

$$D_\alpha P_j^\alpha = Q_j E(L), \qquad \alpha = 1, 2, \qquad j = 1, \ldots, 6$$

exist that hold on the smooth solutions of this equation. The respective conserved currents P_j^α are

$$P_j^\alpha = N_j^\alpha L.$$

Here

$$N_j^\alpha = \xi_j^\alpha + Q_j \frac{\partial}{\partial w_\alpha} - \frac{1}{2} Q_j D_\mu \frac{\partial}{\partial w_{\alpha\mu}} - \frac{1}{2} Q_j D_\mu \frac{\partial}{\partial w_{\mu\alpha}}$$
$$+ \frac{1}{2}(D_\mu Q_j) \frac{\partial}{\partial w_{\alpha\mu}} + \frac{1}{2}(D_\mu Q_j) \frac{\partial}{\partial w_{\mu\alpha}}$$

are the so-called Noether operators (see, e.g., [5]), corresponding to the vector fields $\mathbf{v}_j$ with characteristics Q_j,

$$E = \frac{\partial}{\partial w} - D_\mu \frac{\partial}{\partial w_\mu} + D_\mu D_\nu \frac{\partial}{\partial w_{\mu\nu}} - \cdots$$

is the Euler operator,

$$D_\alpha = \frac{\partial}{\partial x^\alpha} + w_\alpha \frac{\partial}{\partial w} + w_{\alpha\mu} \frac{\partial}{\partial w_\mu} + \cdots$$

are the total derivative operators and L denotes the Lagrangian density of the functional (1) taken in Mongé representation.

5. Group–Invariant Solutions

Once a group G is found to be a symmetry group of a given differential equation, it is possible to look for the so-called group-invariant (G-invariant) solutions of the equation, i.e., the solutions, which are invariant under the transformations of the symmetry group G (see [11], [15]). The main advantage that one can gain when looking for this kind of particular solutions of the given differential equation consists in the fact that each group-invariant solution is determined by a reduced equation obtained by a symmetry reduction of the original one and involving less independent variables than the latter.

The vector fields

$$\langle \mathbf{v}_1 \rangle, \qquad \langle a\mathbf{v}_3 + \mathbf{v}_4 \rangle, \qquad a = \text{constant}$$

constitute an optimal system of one-dimensional subalgebras of the symmetry algebra. So, the essentially different group-invariant solutions correspond to the groups generated by the vector fields $\mathbf{v}_1$ and $a\mathbf{v}_3 + \mathbf{v}_4$.

The $G(\mathbf{v}_1)$-invariant solutions are sought in the form

$$w = W(x^1)$$

and the corresponding reduced equation is

$$\frac{k_c}{(v^2 + 1)^{5/2}} v_{11} - \frac{5 k_c v}{2(v^2 + 1)^{7/2}} v_1^2 - \frac{1}{2} \frac{(k_c \mathrm{lh}^2 + 2\lambda)v}{(v^2 + 1)^{1/2}} + px^1 = C_1 \qquad (8)$$

where C_1 is a real number, $v = \mathrm{d}W/\mathrm{d}x^1$, $v_1 = \mathrm{d}v/\mathrm{d}x^1$ and $v_{11} = \mathrm{d}v_1/\mathrm{d}x^1$.

The $G(a\mathbf{v}_3 + \mathbf{v}_4)$-invariant solutions are sought in the form

$$w = \tilde{w}(r) + a\arctan\left(\frac{x^2}{x^1}\right), \qquad r = \sqrt{(x^1)^2 + (x^2)^2}$$

and the corresponding reduced equations can be written in the form

$$E_r(L_2) + pr = C_2$$

where

$$E_r = \frac{\partial}{\partial v} - \left(\frac{\partial}{\partial r} + v_1 \frac{\partial}{\partial v} + v_{11} \frac{\partial}{\partial v_1} \right) \frac{\partial}{\partial v_1}$$

$$L_2 = \frac{k_c F^2}{G^5} + \frac{2k_c \mathrm{lh} F}{G^2} + (k_c \mathrm{lh}^2 + 2\lambda) G$$

$$G^2 = r^2(v^2 + 1) + a^2, \qquad F = r(r^2 + a^2)v_1 + (G^2 + a^2)v$$

and C_2 is an arbitrary real number, $v = \mathrm{d}\tilde{w}/\mathrm{d}r$, $v_1 = \mathrm{d}v/\mathrm{d}r$, $v_{11} = \mathrm{d}v_1/\mathrm{d}r$.

6. Translationally-Invariant Solutions

The functions $v(x^1)$ satisfying equation (8) for a given set of the parameters k_c, lh, λ and p describe translationally-invariant solutions of the respective membrane shape equation (2) that are cylindrical surfaces in $\mathbb{R}^3$ whose generatrices are parallel to the x^2-axis and whose directrices are curves Γ given by the equation $\Gamma : x^3 = \tilde{w}(x^1)$ where

$$\frac{\mathrm{d}\tilde{w}}{\mathrm{d}x^1} = v(x^1).$$

The problem that will be studied in the present Section is the determination of the curves Γ. For that purpose, it is convenient to rewrite equation (8) in the equivalent form

$$2\frac{\mathrm{d}^2 \mathrm{k}}{\mathrm{d}s^2} + \mathrm{k}^3 - \mu \mathrm{k} - \sigma = 0 \qquad (9)$$

where

$$\mu = \mathrm{lh}^2 + \frac{2\lambda}{k_c}, \qquad \sigma = -\frac{2p}{k_c}$$

with $\mathrm{k} = \mathrm{k}(s)$ and s being respectively the curvature and the arc length of the sought directrices Γ. The mean curvature $H(s)$ of the corresponding solution surfaces of the membrane shape equation is related to k by the equation $\mathrm{k}(s) = 2H(s)$. Once a solution of equation (9) is known in an explicit form, it is possible to recover the corresponding curve Γ. Because, given a curve Γ with curvature $\mathrm{k}(s)$, the embedding $s \longmapsto (X(s), Y(s))$ of that curve in the plane $\mathbb{R}^2$ (up to a rigid motion) can be found by solving the system

$$\frac{\mathrm{d}X(s)}{\mathrm{d}s}\frac{\mathrm{d}^2 Y(s)}{\mathrm{d}s^2} - \frac{\mathrm{d}^2 X(s)}{\mathrm{d}s^2}\frac{\mathrm{d}Y(s)}{\mathrm{d}s} = \mathrm{k}(s)$$

$$\left(\frac{\mathrm{d}X(s)}{\mathrm{d}s} \right)^2 + \left(\frac{\mathrm{d}Y(s)}{\mathrm{d}s} \right)^2 = 1. \qquad (10)$$

Thus, the main problem is to find the solutions to equation (9).

This equation is studied by Arreaga *et al* [1] with the aim to determine the equilibria of an elastic loop in the plane subject to the constraints of fixed length and fixed enclosed area. In the three dimensional case considered here, each such loop

will determine a directrix Γ of a cylindrical surface that is a solution of the membrane shape equation. In [1] the determination of the curvature $\Bbbk$ at equilibrium is reduced to the study of the motion of a particle in a quartic potential. Indeed, equation (9) is the Euler–Lagrange equation associated with the functional

$$L(\Bbbk) = \int (T - U)\mathrm{d}s, \qquad T = \frac{1}{2}\left(\frac{\mathrm{d}\Bbbk}{\mathrm{d}s}\right)^2, \qquad U = \frac{1}{8}\Bbbk^4 - \frac{1}{4}\mu\Bbbk^2 - \frac{1}{2}\sigma\Bbbk$$

in which T and U can be thought of as the one-dimensional kinetic and potential energies, respectively, of some fictitious particle. In this setting $\Bbbk$ gets an interpretation of its displacement and s plays the role of the time. Using this analogy, the authors succeeded in obtaining a purely geometric construction for determination of the embedding without a reference to explicit expressions for the solutions of equation (9). Nevertheless, our opinion is that the knowledge of the solutions of this equation in an explicit form is an important and powerful tool in determination of the surfaces that are translationally-invariant solutions of the membrane shape equation. Therefore, several classes of such solutions in elementary and elliptic functions are presented below.

Evidently, equation (9) admits the one-parameter group of translations of the independent variable s as a variational symmetry group and hence, by virtue of Noether's theorem, there is a conservation law and respectively a first integral

$$\frac{\mathrm{d}E}{\mathrm{d}s} = 0, \qquad E = T + U \tag{11}$$

that holds on its smooth solutions. In characteristic form, the above conservation law reads

$$\frac{\mathrm{d}E}{\mathrm{d}s} = \frac{1}{2}\frac{\mathrm{d}\Bbbk}{\mathrm{d}s}\left(2\frac{\mathrm{d}^2\Bbbk}{\mathrm{d}s^2} + \Bbbk^3 - \mu\Bbbk - \sigma\right).$$

Therefore, each solution to the equation (9), which is not identically a constant, corresponds to a certain value of the real constant E, further referred to as the total energy, and satisfies the equation

$$\left(\frac{\mathrm{d}\Bbbk}{\mathrm{d}s}\right)^2 - P(\Bbbk) = 0, \qquad P(\Bbbk) = 2E - \frac{1}{4}\Bbbk^4 + \frac{1}{2}\mu\Bbbk^2 + \sigma\Bbbk$$

i.e.,

$$\frac{\mathrm{d}\Bbbk}{\mathrm{d}s} = \pm\sqrt{P(\Bbbk)} \tag{12}$$

with an appropriate sign (plus or minus), and vice versa. Let us remark here that the polynomial $P(\Bbbk)$ takes non-negative values for each solution of equation (9). Integrating equation (12) one obtains

$$s = \pm \int \frac{\mathrm{d}\Bbbk}{\sqrt{2E - \frac{1}{4}\Bbbk^4 + \frac{1}{2}\mu\Bbbk^2 + \sigma\Bbbk}} - \overset{\circ}{s} \tag{13}$$

where $\overset{\circ}{s}$ is a real constant, which, without loss of generality, can be taken equal to zero due to the translational invariance of the equation (9).

Let us remark that under the scaling transformation $\tau : (s, \Bbbk) \longmapsto (\pm s/\tau, \tau \Bbbk)$, where τ is an arbitrary real number, each equation of form (9) corresponding to certain constants μ and σ transforms into an equation of the same form but with new coefficients $\mu \longmapsto \mu/\tau^2$, $\sigma \longmapsto \sigma/\tau^3$, that is τ is an equivalence transformation for equation (9). At the same time, τ is an equivalence transformation for equation (12) too, provided that $E \longmapsto E/\tau^4$.

7. Explicit Solutions

Due to the absence of a term with $\Bbbk^3$ in the polynomial $P(\Bbbk)$ it can be represented through three of its roots, denoted below as α, β and γ, in the form

$$P(\Bbbk) = -(\Bbbk - \alpha)(\Bbbk - \beta)(\Bbbk - \gamma)(\Bbbk + \alpha + \beta + \gamma)/4 \tag{14}$$

where by Vieta's formulas we have additionally

$$\begin{aligned}
\mu &= (\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma)/2 \\
\sigma &= -(\alpha + \beta)(\alpha + \gamma)(\beta + \gamma)/4 \\
E &= \alpha\beta\gamma(\alpha + \beta + \gamma)/8.
\end{aligned} \tag{15}$$

Because the case $\sigma = 0$ was thoroughly studied elsewhere (see [6] and [8]) our main concern here will be the case when $\sigma \neq 0$. It should be stressed also that this assumption implies

$$\alpha + \beta \neq 0, \qquad \alpha + \gamma \neq 0, \qquad \beta + \gamma \neq 0. \tag{16}$$

Depending on various types of the roots we will have to distinguish the cases of multiple and simple roots.

7.1. Repeated Roots

Let us suppose first, that the polynomial $P(\Bbbk)$ has a double root. Without loss of generality we can take this to be α and that $\gamma \equiv \alpha$. Then the expressions (14) and (15) take the form

$$P(\Bbbk) = -(\Bbbk - \alpha)^2(\Bbbk - \beta)(\Bbbk + 2\alpha + \beta)/4$$

and

$$\begin{aligned}
\mu &= \alpha^2 + (\alpha + \beta)^2/2 \\
\sigma &= -\alpha(\alpha + \beta)^2/2 \\
E &= \alpha^2\beta(2\alpha + \beta)/8.
\end{aligned}$$

Now, conditions (16) reduce to $\alpha \neq 0$ and $\alpha + \beta \neq 0$. In addition, we are forced to assume also that $\alpha \neq \beta$ and $-2\alpha - \beta \neq \alpha$ which exclude exactly the case of a triple root that will be studied separately. Besides, it can be easily shown that α and β should be real numbers in order that μ, σ, E and $\Bbbk$ be real too.

Indeed, let us assume for the moment that α is a complex number. Then, the other two roots β and $-(2\alpha + \beta)$ should be complex conjugates of α, and this means that $2(\alpha + \beta) = 0$. The latter however contradicts the first condition $\alpha + \beta \neq 0$ in (16) and therefore α mist be a real number.

Let us suppose next that β is a complex number. Then $-(2\alpha + \beta)$ should be a complex conjugate of β, which means $\mathrm{Re}(\beta) = -\alpha$ and simultaneously implies that the polynomial $P(\Bbbk)$ could be written in the form

$$P(\Bbbk) = -\frac{1}{4}(\Bbbk - \alpha)^2 \left((\Bbbk + \alpha)^2 + \mathrm{Im}(\beta)^2 \right).$$

Evidently, the polynomial $P(\Bbbk)$ above does not take any positive values and therefore this possibility should be excluded as well.

Consider now the only possible case in which both α and β are real numbers. Instead of dealing with β, it is convenient to introduce a new parameter $\kappa \neq 0$ such that $\beta = -\alpha + \sqrt{2}\kappa$. Then, $\alpha = -\sigma/\kappa^2$ and the polynomial $P(\Bbbk)$ can be expressed entirely in terms of the parameters σ and κ

$$P(\Bbbk) = \frac{1}{4\kappa^8}(\Bbbk\kappa^2 + \sigma)^2 \left(2\kappa^6 - (\Bbbk\kappa^2 - \sigma)^2 \right).$$

The corresponding integral in (13) can be computed by making use of the substitution

$$\Bbbk = \frac{\sigma}{\kappa^2} + \sqrt{2}\kappa + \frac{1}{t} \tag{17}$$

and this gives

$$s = \pm\frac{4\kappa^2}{\sqrt{4\sigma^2 - 2\kappa^6}} \arctan \sqrt{\frac{(\sqrt{2}\kappa^3 + 2\sigma)(2\sqrt{2}\kappa t + 1)}{\sqrt{2}\kappa^3 - 2\sigma}}.$$

By solving the last equation with respect to t one obtains immediately

$$t = -\frac{1}{2\sqrt{2}\kappa} - \frac{1}{2\kappa}\frac{\sqrt{2}\sigma - \kappa^3}{2\sigma + \sqrt{2}\kappa^3} \tan^2 \left(\frac{s\sqrt{2\sigma^2 - \kappa^6}}{2\sqrt{2}\kappa^2} \right).$$

In the same time the substitution of the above expression for t into (17) leads to the following expression

$$\mathbb{k}(s) = \frac{\sqrt{2}\kappa^3 + \sigma}{\kappa^2} - \frac{2\sqrt{2}\kappa(\sqrt{2}\sigma + \kappa^3)}{\sqrt{2}\sigma + \kappa^3 + (\sqrt{2}\sigma - \kappa^3)\tan^2\left(\dfrac{s\sqrt{2\sigma^2 - \kappa^6}}{2\sqrt{2}\kappa^2}\right)} \cdot \tag{18}$$

This curvature satisfies equation (9) with

$$\mu = \frac{\sigma^2 + \kappa^6}{\kappa^4}$$

and corresponds to a "motion" with total energy

$$E = \frac{\sigma^2(2\kappa^6 - \sigma^2)}{8\kappa^8} \cdot$$

Obviously, the just obtained curvature $\mathbb{k}$ will be a non-constant solution if the condition $2\sigma^2 - \kappa^6 \neq 0$ is satisfied.

Next, suppose that the polynomial $P(\mathbb{k})$ has a triple root. Again, without loss of generality we will take this to be α but this time we will have also $\beta = \gamma = \alpha$. Then

$$P(\mathbb{k}) = -(\mathbb{k} - \alpha)^3(\mathbb{k} + 3\alpha)/4$$

and

$$\mu = 3\alpha^2, \quad \sigma = -2\alpha^3, \quad E = 3\alpha^4/8. \tag{19}$$

Apparently, we have to notice that α should be nonzero as otherwise σ will vanish as well. In the above setting, the integral in (13) can be computed and one obtains

$$s = \frac{\sqrt{(\alpha - \mathbb{k})(3\alpha + \mathbb{k})}}{\alpha(\alpha - \mathbb{k})}$$

which can be readily solved for $\mathbb{k}$ giving the expression

$$\mathbb{k} = -\alpha\frac{3 - \alpha^2 s^2}{1 + \alpha^2 s^2} \cdot \tag{20}$$

This second class of solutions in elementary functions corresponds to the energy specified in (19).

Let us notice also that solutions corresponding to quadruple root of $P(\mathbb{k})$ will be not discussed here because such a root should be identically zero which implies $\sigma \equiv 0$ and therefore presents a case that is outside the class in which we are interested.

Actually, there exists a procedure based on the so-called invariants of the equation $P(\mathbb{k}) = 0$ by which one can check if the polynomial $P(\mathbb{k})$ has a repeated root. Unfortunately, the respective expressions are not quite compact but the interested reader could find them in the book by Chandrasekharan [2].
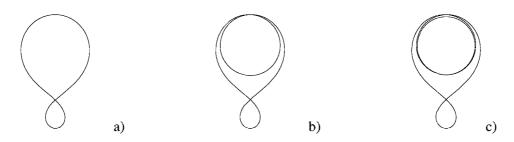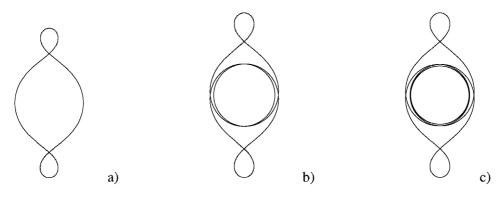
Figure 1. Closed self-intersecting one-fold symmetry curves obtained by the solution (18) and corresponding to: a) $\mu = 0.208$, $\sigma = 0.036$; b) $\mu = 0.195$, $\sigma = 0.033$; c) $\mu = 0.192$, $\sigma = 0.032$.

Figure 2. Closed self-intersecting two-fold symmetry curves obtained by the solution (18) and corresponding to: a) $\mu = 0.238$, $\sigma = 0.042$; b) $\mu = 0.199$, $\sigma = 0.034$; c) $\mu = 0.193$, $\sigma = 0.033$.

Figure 3. Closed self-intersecting curves with three-fold symmetry obtained by the solution (18) and corresponding to: a) $\mu = 0.268$, $\sigma = 0.047$; b) $\mu = 0.223$, $\sigma = 0.039$ and c) $\mu = 0.202$, $\sigma = 0.035$.
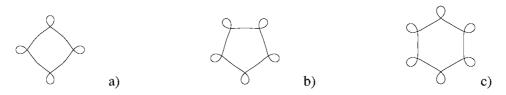

Figure 4. Other closed self-intersecting curves obtained by the solution (18) and corresponding to: a) $\mu = 0.299$, $\sigma = 0.052$; b) $\mu = 0.330$, $\sigma = 0.057$; c) $\mu = 0.361$, $\sigma = 0.061$.

7.2. Simple Roots

Being a quartic function with real coefficients the polynomial $P(\Bbbk)$ can be always represented as a product of a pair of quadratic factors $Q_1(\Bbbk) = a_1\Bbbk^2 + b_1\Bbbk + c_1$ and $Q_2(\Bbbk) = a_2\Bbbk^2 + b_2\Bbbk + c_2$ in which all coefficients $a_1, b_1, \ldots, c_2$ are real. Then, irrespectively of the various possibilities about the presence of complex and real roots of the equation $P(\Bbbk) = Q_1(\Bbbk)Q_2(\Bbbk) = 0$, the integral in (13) can be reduced by means of the substitution

$$\Bbbk = \frac{p + q\eta}{1 + \eta}, \qquad p, q \in \mathbb{R} \tag{21}$$

to the form

$$\int \frac{(q - p)\mathrm{d}\eta}{\sqrt{(a\eta^2 + b)(\tilde{a}\eta^2 + \tilde{b})}} = \pm s. \tag{22}$$

This integral belongs to the class of **elliptic integrals** of the first kind which can be uniformized by the so-called Jacobi **elliptic functions**. More details about elliptic integrals and functions can be found in [2, 3, 7, 9] and references therein.

Assuming that $\mu > 0$ and introducing

$$\nu^2 = 2\mu, \qquad \sigma = \nu\sqrt{2E}$$

where ν is a real number and E ($E > 0$) is the total energy, we have found the following solution to equation (9)

$$\Bbbk(s) = -\frac{4\sqrt{2E}\,\mathrm{sn}(us, k)}{\nu(c + \mathrm{sn}(us, k))} \tag{23}$$

in which the **elliptic module** k and the parameters c and u are given by the formulae
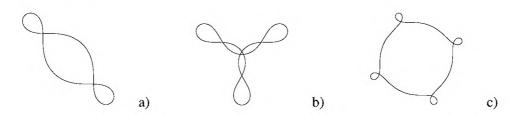
$$k = \sqrt{\frac{\nu^2 - 8\sqrt{2E}}{\nu^2 + 8\sqrt{2E}}}, \qquad c = \sqrt{1 + \frac{8\sqrt{2E}}{\nu^2}}, \qquad u = \frac{c\nu}{4}.$$

Alternatively, making use of the genuine parameters μ and σ of our problem the solution (23) can be written in the form
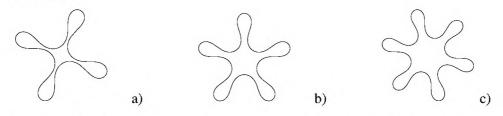
$$\Bbbk(s) = -\frac{2\sigma\,\mathrm{sn}(us, k)}{\mu(c + \mathrm{sn}(us, k))} \tag{24}$$

where

$$k = \sqrt{\frac{\mu\sqrt{\mu} - 2\sqrt{2}\sigma}{\mu\sqrt{\mu} + 2\sqrt{2}\sigma}}, \qquad c = \sqrt{1 + \frac{2\sqrt{2}\sigma}{\mu\sqrt{\mu}}}, \qquad u = \pm\frac{1}{4}c\sqrt{2\mu}$$

Figure 5. Closed self-intersecting curves generated by the solution (23) and corresponding to the set of parameters: a) $\mu = 1.449$, $\sigma = 1$; b) $\mu = 0.732$, $\sigma = 1$; c) $\mu = 0.5$, $\sigma = 0.046$.

Figure 6. Closed non self-intersecting curves obtained by the solution (23) and corresponding to: a) $\mu = 0.477$, $\sigma = 1$; b) $\mu = 0.347$, $\sigma = 1$ and c) $\mu = 0.269$, $\sigma = 1$.
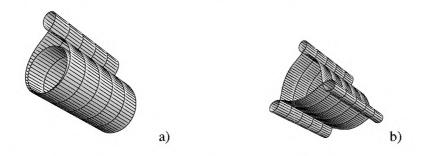

Figure 7. Surfaces generated by the directrices shown in: a) Fig. 1b); b) Fig. 3a).
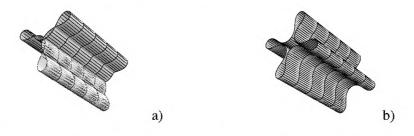

Figure 8. Surfaces generated by the directrices shown in: a) Fig. 5b); b) Fig. 6a).

and one should remember that $\mu > 0$. The above solution corresponds to the total energy

$$E = \frac{\sigma^2}{4\mu} \cdot$$

The directrices associated with the solutions (18) and (23) are reconstructed for several values of the parameters μ and σ by solving numerically the system (10) starting with appropriate initial conditions. The so obtained curves are displayed in Figures 1–6. The surfaces corresponding to directrices depicted in Fig. 1b and Fig. 3a are shown in Fig. 7 and that ones corresponding to Fig. 5b and Fig. 6a are shown in Fig. 8.

Acknowledgements

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