Thirteenth International Conference on
Geometry, Integrability and Quantization
June 3–8, 2011, Varna, Bulgaria
Ivaïlo M. Mladenov, Andrei Ludu
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2012, pp 205–214
doi: 10.7546/giq-13-2012-205-214

ON THE INVOLUTIVE B-SCROLLS IN THE EUCLIDEAN THREE-SPACE $\mathbb{E}^3$

ŞEYDA KILIÇOĞLU

Department of Mathematics Education, Başkent University, Ankara, Turkey

Abstract. Deriving curves based on the other curves like involute-evolute curves or Bertrant curves is an old subject in geometry. In this paper, we have defined a ruled surface based on the involute curve of a given curve which is called the involutive B-scroll. We introduced the positions of the involutive B-scroll and the B-scroll relative to each other.

1. Introduction

Some of the earliest research results about plane curves were motivated by the desire to build more accurate clocks. Practical designs were based on the motion of a pendulum, requiring careful study of motion due to gravity first carried out by Galileo, Descartes, and Mersenne. The culmination of these studies was the work of Christian Huygens (1629-1695) in his 1673 treatise. He is also known for his work in optics. Some of the ideas introduced in Huygens's classic work [6], such as the involute and evolute of a curve, are part of our current geometric language. The idea of a string involute is due to Huygens, he discovered involutes while he was trying to build a more accurate clock [1].

The involute of a given curve is a well-known concept in Euclidean three-space $\mathbb{E}^3$. We can say that evolute and evolvent is a method of deriving a new curve based on a given curve. The evolvent is often called the involute of the curve. Evolvents play a part in the construction of gears [7]. Evolute is the locus of the centers of tangent circles of the given planar curve.

It is well-known that if a curve is differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves.

B-scrolls are the special ruled surfaces. *B-scroll* over null curves with null rulings in three-dimensional Lorentzian space form has been introduced by Graves [2].

In this study we will define and work on involute curves and involutive B-scroll of any curve in the *Euclidean three-space* $\mathbb{E}^3$.

Let α and β be curves in the *Euclidean three-space* $\mathbb{E}^3$. The tangent lines to a curve $\alpha(s)$ generate a surface called the tores of α. If the curve $\beta(s)$ which lies on the tores intersect the tangent lines orthogonally is called an involute of $\alpha(s)$. If a curve $\beta(s)$ is an involute of $\alpha(s)$, then by definition $\alpha(s)$ is an evolute of $\beta(s)$. Hence given $\beta(s)$, its evolutes are the curves whose tangent lines intersect $\beta(s)$ orthogonally. If $\beta(s)$ is a point on an involute β, then $\beta(s)$-$\alpha(s)$ is proportional to the tangent vector $V_1(s)$. Thus the involute $\beta(s)$ will have a representation of the form

$$\beta(s) = \alpha(s) + \lambda(s)V_1(s).$$

Theorem 1 ([5])**.** *In the Euclidean three-space $\mathbb{E}^3$, $\beta \subset \mathbb{E}^3$, if the curve $\beta(s)$ is the involute of $\alpha(s)$ with tangent vector $V_1(s)$, then we have that*

$$\beta(s) = \alpha(s) + (c - s)V_1(s), \qquad \textit{for all} \quad s \in I$$

where c is a constant.

Thus there exists an infinite number of involutes for each constant c. If the curve α is the evolute of the curve β, then the curve β is the involute of the curve α. The opposite is true locally. When the tangent line of a curve $\alpha(s) = \alpha$ is given by $\beta = \alpha + \lambda V_1, -\infty < \lambda < +\infty$, then

$$\|\frac{\mathrm{d}\beta(s)}{\mathrm{d}\lambda}\| = \|V_1\| = 1.$$

That is, λ is a natural parameter. Also since $\beta = \alpha$ for $\lambda = 0$, it follows that $|\lambda|$ is the distance between the point β on the tangent line and the point α on $\alpha(s)$ [5].

All involutes of a given curve are parallel to each other. This property also makes it easy to see that evolute of a curve is the envelope of its normals. If we calculate the distance between the respective congruent points of two involutes $\beta_1(s) = \alpha(s) + (c_1 - s)V_1(s)$ and $\beta_2(s) = \alpha(s) + (c_2 - s)V_1(s)$ we have remains constant for all s and equal to $|c_1 - c_2|$, for all $s \in I$.

Theorem 2. *In the Euclidean three-space $\mathbb{E}^3$, α, $\beta \subset \mathbb{E}^3$, if the curve $\beta(s)$ is the involute of $\alpha(s)$, then for all $s \in I$*

$$d(\alpha(s), \beta(s)) = |c - s|, \qquad c = \text{const}$$

is the distance between the arclengthed curves $\alpha(s)$ and $\beta(s)$ and c is constant.

Example 1 (see Fig. 1)**.** *Along the circle* $\alpha(t) = (a\cos t, a\sin t), a > 0$*, we have the involute curve*

$$\beta(t) = (a\cos t - (c - at)\sin t, a\sin t + (c - at)\cos t).$$

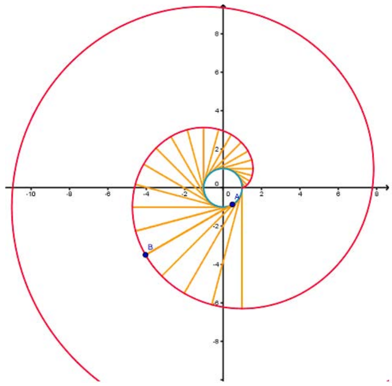

Figure 1. A circle and its involute.

Example 2 (see Fig. 2)**.** *Let us consider the circular helix* $\alpha(t) = (a\cos t, a\sin t, bt)$, $a > 0$, *then the involute curve of* $\alpha(t)$ *is*

$$\beta(t) = (a[(\cos t + t\sin t) - \gamma\sin t], a[(\sin t - t\cos t) + \gamma\cos t], \gamma b)$$
$$\gamma = \frac{c}{\sqrt{a^2+b^2}}, \qquad t = \frac{s}{\sqrt{a^2+b^2}}.$$

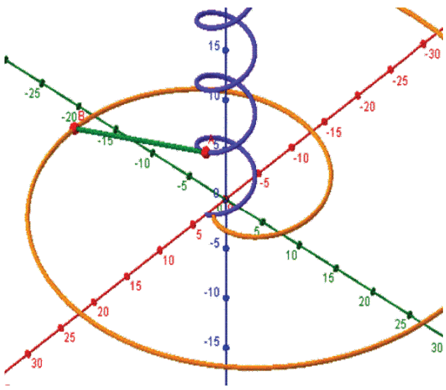

Figure 2. A helix and its involute.

As shown in the Figure the involute of a helix is a planar curve, whose plane is $z = \gamma b$.

2. Frenet Apparatus and Frenet Formulas of the Involute Curve

The set, whose elements are frame vectors and curvatures of a curve, is called Frenet apparatus of the curves. The following result shows that we can write the Frenet apparatus of the involute curve based on its evolute curve. And also we can introduce the Frenet formulas of the involute curve based on the Frenet apparatus of its evolute curve.

Theorem 3. *In the Euclidean three-space $\mathbb{E}^3$ α, $\beta \subset \mathbb{E}^3$, $\alpha(s)$ and $\beta(s^*)$ are the arclengthed curves with the arc-parameters s and s^*, respectively. Let V_1, V_2, V_3 and V_1^*, V_2^*, V_3^* be the Frenet vectors belonging to the the curve $\alpha(s)$ and its involute $\beta(s^*)$, respectively, and we have the equation* [3]

$$\langle V_1, V_1^* \rangle = 0.$$

Theorem 4 ([3])**.** *Let $\alpha(s)$ and $\beta(s^*)$ are the arclengthed curves in the Euclidean three-space $\mathbb{E}^3$, with the arc-parameters s and s^*, respectively. Let the first and second curvatures of the curves $\alpha(s)$ and $\beta(s^*)$ be k_1, k_2 and k_1^*, k_2^*, respectively. In these settings we have the following equations*

$$\begin{aligned} V_1^* &= V_2, \qquad \lambda k_1 > 0, \qquad \lambda = c - s \\ V_2^* &= \frac{-k_1 V_1 + k_2 V_3}{\lambda k_1 k_1^*} \\ V_3^* &= \frac{k_2 V_1 + k_1 V_3}{\lambda k_1 k_1^*} \end{aligned}$$

where c is a constant.

Theorem 5. *In the Euclidean three-space $\mathbb{E}^3$, α, $\beta \subset \mathbb{E}^3$, $\alpha(s)$ and $\beta(s^*)$ are the arclengthed curves with the arc-parameters s and s^*, respectively. Let the first and second curvatures of the curve $\alpha(s)$ and $\beta(s^*)$ be k_1, k_2 and k_1^*, k_2^*, respectively. The first curvature of the involute β is*

$$k_1^* = \sqrt{\frac{k_1^2 + k_2^2}{\lambda^2 k_1^2}}, \qquad \lambda = c - s, \qquad k_1 \neq 0.$$

If we use this result in the equation for V_2^* and V_3^*, we have

$$\begin{aligned} V_1^* &= V_2, \qquad \lambda k_1 > 0 \\ V_2^* &= \frac{-k_1 V_1 + k_2 V_3}{\sqrt{k_1^2 + k_2^2}} \\ V_3^* &= \frac{k_2 V_1 + k_1 V_3}{\sqrt{k_1^2 + k_2^2}}. \end{aligned}$$

Corollary 6. *If the second curvature k_2 of the curve $\alpha(s)$ is equal to zero, that is $\alpha(s)$ is a planar curve, then*

$$k_1^* = \frac{1}{\lambda}, \qquad \lambda = c - s > 0.$$

Corollary 7. *If the second curvature k_2 of the curve $\alpha(s)$ is constant but not equal to zero, then $\dot{k}_2 = 0$. Hence we have*

$$(k_1^*)^2 = \frac{k_1^2 + k_2^2}{\lambda^2 k_1^2}, \qquad k_1 \neq 0.$$

Theorem 8. *Let $\alpha(s)$ and $\beta(s^*)$ are the arclengthed curves in the Euclidean three-space $\mathbb{E}^3$, with the arc-parameters s and s^*, respectively. Let $\beta(s^*)$ be the involute of the curve $\alpha(s)$. Then, the differentials of the Frenet vector fields can be expressed in the form $V_1^*(s), V_2^*(s), V_3^*(s)$ as*

$$\begin{bmatrix} \dot{V}_1^* \\ \dot{V}_2^* \\ \dot{V}_3^* \end{bmatrix} = \begin{bmatrix} \frac{-1}{\lambda} & 0 & \frac{k_2}{\lambda k_1} \\ \frac{k_2 k_2^*}{\lambda k_1 k_1^*} & -k_1^* & \frac{k_2^*}{\lambda k_1^*} \\ \frac{1}{\lambda} & 0 & -\frac{k_2}{\lambda k_1} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Theorem 9. *Let $\alpha(s)$ and $\beta(s^*)$ are the arclengthed curves in the Euclidean three-space $\mathbb{E}^3$, with the arc-parameters s and s^*, respectively. If $\beta(s^*)$ is the involute of the curve $\alpha(s)$ for $\lambda = c - s$, then*

$$k_2^* = \frac{k_1 k_2' - k_1' k_2}{\lambda k_1 (k_1^2 + k_2^2)}, \qquad c = \text{const}\,.$$

Theorem 10. *If the second curvature k_2 of the curve α is equal to zero, $k_2 = 0$, then $k_2^* = 0$, i.e., if the curve α is a planar curve, then the involute of α is a planar curve too.*

Corollary 11. *If the second curvature k_2 of the curve $\alpha(s)$ is constant but not equal to zero, then $\dot{k}_2 = 0$. Hence we have that*

$$k_2^* = -\frac{k_1' k_2}{\lambda k_1 (k_1^2 + k_2^2)}.$$

Corollary 12. *If the curve $\alpha(s)$ is a helix, then the involute $\beta(s)$ of the curve $\alpha(s)$ is a planar curve. If the curve $\alpha(s)$ is a helix*

$$k_2^* = \frac{k_1 k_2' - k_1' k_2}{\lambda k_1 (k_1^2 + k_2^2)} = \frac{\left(\frac{k_2}{k_1}\right)'}{\frac{\lambda (k_1^2 + k_2^2)}{k_1}} = 0.$$

3. Involutive B-scroll in the Euclidean Three-Space $\mathbb{E}^3$

Definition 1. *Let $\alpha(s)$ be an arclengthed curve in the Euclidean three-space $\mathbb{E}^3$. The equation*

$$\varphi(s,u) = \alpha(s) + uV_3(s)$$

*is the parametrization of the ruled surface which is called **B-scroll** (binormal scroll)* [4]. *The directrix of this B-scroll is the curve $\alpha(s)$. The generating space of this B-scroll is spanned by binormal subvector V_3 and $\mathrm{Sp}\{V_1, V_2\}$ is the osculator plane of the curve $\alpha(s)$.*

Definition 2. *In the Euclidean three-space $\mathbb{E}^3$, let $\alpha(s)$ and $\beta(s^*)$ be the arclengthed curves. If the curve $\beta(s)$ is the involute of the curve $\alpha(s)$. The equation*

$$\varphi^*(s,v) = \beta(s) + vV_3^*(s)$$

*is the parametrization of the ruled surface which is called **involutive B-scroll** (binor-mal scroll) of the curve α. The directrix of this involutive B-scroll is the involute curve $\beta(s) = \alpha(s) + (c-s)V_1(s)$ of the curve $\alpha(s)$. The generating space of B-scroll is spaned by binormal subvector V_3^* and $Sp\{V_1^*, V_2^*\}$ is the osculator plane of the curve β.*

Theorem 13. *Let the sets V_1, V_2, V_3, k_1, k_2 and $V_1^*, V_2^*, V_3^*, k_1^*,\ k_2^*$ in the Euclidean three-space $\mathbb{E}^3$ be the Frenet apparatus of the curve α and its involute curve β, respectively. The parametrization of the involutive B-scroll of the curve $\alpha(s)$ is*

$$\varphi^*(s,v) = \alpha(s) + \left(\lambda + \frac{vk_2(s)}{\sqrt{k_1^2+k_2^2}}\right)V_1(s) + \frac{vk_1(s)}{\sqrt{k_1^2+k_2^2}}V_3(s)$$
$$\lambda = c - s, \qquad c = \text{const}, \qquad \lambda k_1 > 0.$$

Proof: It is trivial. ■

Theorem 14. *Let the sets V_1, V_2, V_3, k_1, k_2 and $V_1^*, V_2^*, V_3^*, k_1^*,\ k_2^*$ in the Euclidean three-space $\mathbb{E}^3$ be the Frenet apparatus of the non-planar curve α and the involute curve β, respectively. The intersection of the involute B-scroll of $\alpha(s)$ and B-scroll of the curve $\alpha(s)$ is a curve with parametrization*

$$\varphi(s) = \alpha(s) - \lambda\frac{k_1(s)}{k_2(s)}V_3(s), \qquad \lambda = c - s, \qquad c = \text{const}\,.$$

Proof: Under the conditions

$$\lambda + \frac{vk_2}{\sqrt{k_1^2+k_2^2}} = 0 \qquad \text{and} \qquad \frac{vk_1}{\sqrt{k_1^2+k_2^2}} = 0$$

we get

$$v = -\lambda \frac{k_1}{k_2}, \qquad k_2 \neq 0.$$

■

Theorem 15. *In the Euclidean three-space $\mathbb{E}^3$, let the Frenet vectors of the curve α be V_1, V_2, V_3 and*

$$\varphi(s,u) = \alpha(s) + uV_3(s)$$

is the parametrization of the ruled surfaces which is called B-scroll (binormal scroll). Then the normal vector field [4] *of ruled surface B-scroll is*

$$N = \frac{-uk_2V_1 - V_2}{\sqrt{1+u^2k_2^2}}.$$

Theorem 16. *In the Euclidean three-space $\mathbb{E}^3$, the normal vector field of involute B-scroll of the curve $\alpha(s)$ is*

$$\begin{aligned} N^* = {} & \frac{\lambda k_1^2\sqrt{k_1^2+k_2^2}}{\sqrt{(\lambda k_1(k_1^2+k_2^2))^2 + v^2(k_1k_2' - k_1'k_2)^2}} V_1 \\ & + \frac{-v(k_1k_2' - k_1'k_2)}{\sqrt{(\lambda k_1(k_1^2+k_2^2))^2 + v^2(k_1k_2' - k_1'k_2)^2}} V_2 \\ & - \frac{\lambda k_1k_2\sqrt{k_1^2+k_2^2}}{\sqrt{(\lambda k_1(k_1^2+k_2^2))^2 + v^2(k_1k_2' - k_1'k_2)^2}} V_3. \end{aligned}$$

Proof: We already have the equation of the involute B-scroll of the curve $\alpha(s)$, and also it is well known that the normal vector field N^* of any B-scroll surface [4] is

$$N^* = \frac{-vk_2^*V_1^* - V_2^*}{\sqrt{1+v^2k_2^{*2}}}$$

so that normal vector field N^* of the involute B-scroll is

$$\begin{aligned} N^* = {} & \frac{-v(k_1k_2' - k_1'k_2)}{\sqrt{(\lambda k_1(k_1^2+k_2^2))^2 + v^2(k_1k_2' - k_1'k_2)^2}} V_2 \\ & - \frac{-k_1V_1 + k_2V_3}{\left(\frac{\sqrt{k_1^2+k_2^2}}{\lambda k_1(k_1^2+k_2^2)}\right)\sqrt{(\lambda k_1(k_1^2+k_2^2))^2 + v^2(k_1k_2' - k_1'k_2)^2}}. \end{aligned}$$

This completes the proof. ■

Theorem 17. *In the Euclidean three-space $\mathbb{E}^3$, let us consider the involutive B-scroll of the curve $\alpha(s)$ given by $\varphi^*(s,v)=\beta(s)+vV_3^*(s)$. If the normal vector field N^* of involute B-scroll of the curve $\alpha(s)$ and the normal vector field N of B-scroll of the curve $\alpha(s)$ are perpendicular to each other, then*

$$v = u\frac{\lambda k_1^2\sqrt{k_1^2+k_2^2}}{k_1k_2^{'}-k_1^{'}k_2}.$$

Proof: Let us rename the coefficients of the normal vector fields N^* of involute B-scroll of the curve $\alpha(s)$ as δ, ϵ and η, we get $N^*=\delta V_1+\epsilon V_2+\eta V_3$. Using the orthogonality condition; If $N^*\perp N$, then $\langle N^*,N\rangle=0$ and

$$\delta\frac{-uk_2}{\sqrt{1+u^2{k_2}^2}}=\epsilon\frac{1}{\sqrt{1+u^2{k_2}^2}},\qquad \sqrt{1+u^2{k_2}^2}\neq 0$$

$$uk_2\delta=-\epsilon,\qquad \sqrt{(\lambda k_1(k_1^2+k_2^2))^2+v^2(k_1k_2^{'}-k_1^{'}k_2)^2}\neq 0$$

$$\frac{u}{v}=\frac{k_1k_2^{'}-k_1^{'}k_2}{\lambda k_1^2k_2\sqrt{k_1^2+k_2^2}}.$$

■

Corollary 18. *If $\alpha(s)$ is helix, then $\left(\frac{k_2}{k_1}\right)^{'}=0$ the normal vector field N^* of involute B-scroll of the curve $\alpha(s)$ and the normal vector field N of B-scroll of the curve $\alpha(s)$ can not be perpendicular to each other and*

$$\frac{u}{v}=\frac{1}{\lambda k_2\sqrt{k_1^2+k_2^2}}\left(\frac{k_2}{k_1}\right)^{'}=0,\qquad v\neq 0,\qquad u=0.$$

That is there are not any B-scroll surfaces.

Theorem 19. *In the Euclidean three-space $\mathbb{E}^3$, if the normal vector field N^* of involute B-scroll of the curve $\alpha(s)$ and the normal vector field N of B-scroll of the curve $\alpha(s)$ cannot be perpendicular to each other.*

Proof: Using the condition of parallelism

$$\text{If}\quad N//N^*,\qquad \frac{\delta}{\frac{-uk_2}{\sqrt{1+u^2k_2^2}}}=\frac{\epsilon}{\frac{-1}{\sqrt{1+u^2k_2^2}}}\quad\text{and}\qquad \eta=0$$

we get

$$uv=\frac{-\lambda k_1^2\sqrt{k_1^2+k_2^2}}{k_2(k_1k_2^{'}-k_1^{'}k_2)}\qquad\text{and}\qquad \lambda k_1k_2\sqrt{k_1^2+k_2^2}=0$$

$$uv=0.$$

That is there are not any B-scroll surface. ■

Example 3. *In the Euclidean three-space* $\mathbb{E}^3$*, along the helix* $\alpha(t) = (a\cos t, a\sin t, bt)$*,* $a > 0$*, we have the involute B-scroll of the helix* $\alpha(t)$ *is*

$$\begin{aligned}\varphi^*(t,v) &= \beta(t) + vV_3^*(t)\\ &= (a[(\cos t + t\sin t) - \gamma\sin t], a[(\sin t - t\cos t) + \gamma\cos t], \gamma b + v)\end{aligned}$$

$$\gamma = c(a^2+b^2)^{\frac{-1}{2}} \quad \textit{and} \quad t = s(a^2+b^2)^{\frac{-1}{2}}.$$

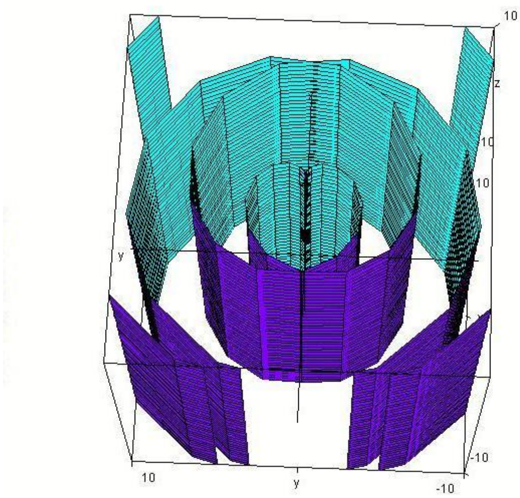

Figure 3. Involutive B-scroll of the circle.

Example 4. *Then we obtain the intersection of the involute B-scroll of the helix* $\alpha(s)$ *and B-scroll of a non-planar curve* $\alpha(s)$ *as a curve with parametrization*

$$\varphi(s) = \left(a\cos t + \frac{\lambda a\sin t}{\sqrt{a^2+b^2}}, a\sin t - \frac{\lambda a\cos t}{\sqrt{a^2+b^2}}, bt + \frac{\lambda a^2}{b\sqrt{a^2+b^2}}\right)$$

$$t = s(a^2+b^2)^{\frac{-1}{2}}.$$

Example 5 (see Fig. 3)**.** *Along the circle* $\alpha(t) = (a\cos t, a\sin t, 0)$*,* $a > 0$ *we have the involute B-scroll of the circle* $\alpha(t)$ *with the parametrization*

$$\begin{aligned}\varphi^*(t,v) &= \beta(t) + vV_3^*(t)\\ &= (a\cos t - (c-at)\sin t, a\sin t + (c-at)\cos t, v).\end{aligned}$$

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