EFFECTIVE SOLUTIONS OF AN INTEGRABLE CASE OF THE HÉNON–HEILES SYSTEM

ANGEL ZHIVKOV and IOANNA MAKAVEEVA

Faculty of Mathematics and Informatics, Sofia University
5 James Bourchier Blvd, 1164 Sofia, Bulgaria

Abstract. We solve in two-dimensional theta functions the integrable case $\ddot{r} = -ar + 2zr$, $\ddot{z} = -bz + 6z^2 + r^2$ (a and b are constant parameters) of the generalizied Hénon–Heiles system. The general solution depends on six arbitrary constants, called algebraic–geometric coordinates. Three of them are coordinates on the degree two (and dimension three) Siegel upper half-plane and define two-dimensional tori $\mathbb{T}^2$. Each trajectory of the Hénon–Heiles system lies on certain torus $\mathbb{T}^2$. Next two arbitrary constants define the initial position on $\mathbb{T}^2$. The speed of the flow depends multiplicatively on the last arbitrary constant.

Consider a galaxy which gravitational potential U_{gr} is time-independent and has an axis of symmetry. We are interested in the motion of a star in such a potential field.

Let us introduce a system of cylindrical coordinates (r, ψ, z): Oz is the axis of symmetry, z is the height of the star, $r := \sqrt{x^2 + y^2}$ is the distance between the star and the axis Oz, $\psi := \arctan \frac{y}{x}$ is the polar angle.

Two conservation laws (integrals) of the stellar motion are known:

$$I_1 = U_{gr}(r, z) + \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\psi}^2 + \dot{z}^2\right) = \text{total energy},$$

$$I_2 = mr^2\dot{\psi} = \text{angular momentum of the star around } Oz \text{ axis},$$

m is the mass of the star, $\dot{} = \dfrac{\mathrm{d}}{\mathrm{d}t}$ is the derivative with respect to the time t.
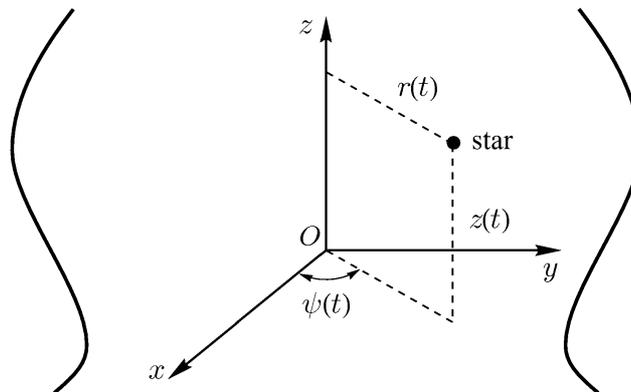
With the help of the second integral I_2 we reduce the dynamics of the star on

the meridian plane (r, z):

$$\ddot{r} = -\frac{\partial U(r, z)}{\partial r} \,, \qquad \ddot{z} = -\frac{\partial U(r, z)}{\partial z} \,,$$

$$U(r, z) := U_{gr}(r, z) + \frac{m}{2} r^2 \dot{\psi}^2 = U_{gr}(r, z) + \frac{I_2^2}{2mr^2} \,. \tag{1}$$

There also exists a variety of problems in celestial mechanics [2, 3, 7], statistical mechanics [5, 11, 4] and quantum mechanics [10], leading to the Hénon–Heiles system (1).

A third independent integral I_3 is necessary to solve the two-dimensional conservative (and potential) system. By definition, each integral I_j is either isolating or nonisolating [12]. A nonisolating integral is such that the corresponding hypersurface $I_j = \text{const}$ consists of an infinity of sheets which usually fill the phase space densely; thus from the physical point of view, nonisolating integrals does not give any information and hence have no significance. For that reason, isolating integrals are called simply integrals, and the nonisolating integrals are ignored.

In 1957 Contopoulos [2] found some particular cases of potentials $U(r, z)$ when such integral I_3 exists. In 1963 Hénon and Heiles [7] simplified the problem, canceling all terms of order ≥ 4 in the potential U. Suppose also, that U depends on r^2, or, equivalently, $U(-r, z) \equiv U(r, z)$. Every such a potential can be read as

$$U(r, z) := U_2(r, z) + U_3(r, z) = \frac{1}{2}\left(ar^2 + bz^2\right) - r^2 z - cz^3 \,,$$

where a, b and c are free parameters.

Using Painlevé analysis, in 1982 Bountis, Segur and Vivaldi [1] extracted all integrable cases of (1):

  i) $a = b, \ c = \dfrac{1}{3}$;

  ii) $\boxed{c = 2, \ a \text{ and } b \text{ are arbitrary};}$

  iii) $16a = b, \ c = \dfrac{16}{3}$.

In the case (i), the equations of motion decoupled in $(z+r)$, $(z-r)$ coordinates and the general solution can be expressed by elliptic functions. The question about the exact solution in the case (iii) is still open. In this article, we shall focus our attention to the second case (ii). Then the system of Hénon–Heiles takes the form

$$\ddot{r} = -ar + 2zr$$
$$\ddot{z} = -bz + 6z^2 + r^2 \tag{2}$$

and has an additional integral [1]

$$I_3 = r^4 + 4r^2z^2 + 4\dot{r}\left(\dot{r}z - r\dot{z}\right) - 4ar^2z + \left(4a - b\right)\left(\dot{r}^2 + ar^2\right).$$

Remark that if we cancel the cubic part U_3 of U, then the equations of the two-dimensional oscillator

$$\ddot{r} = -ar$$
$$\ddot{z} = -bz,$$

arises. This is the simplest system with two degrees of freedom.

The aim of this paper is to solve explicitly and effectively the system (2). It turns out that the solutions are expressed by the two-dimensional Riemann theta functions

$$\theta\left[{\alpha \atop \beta}\right](v, \Omega) := \sum_{N \in \mathbb{Z}^2} \exp\left\langle N + \frac{\alpha}{2}, \frac{1}{2}\left(N + \frac{\alpha}{2}\right)\Omega + \pi\mathrm{i}\beta + v\right\rangle,$$

where $\Omega = \left(\Omega_{ij}\right)_{i,j=1}^2$ is a Riemann matrix, i.e. a symmetric 2×2 matrix which real part is negatively defined (it is said that Ω belongs to the Siegel upper half-plane); $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ are $\mathbb{Z}^2$-integer vectors, called **characteristics**; $v = (v_1, v_2) \in \mathbb{C}^2$ is the argument of the theta function. As usual, denote by

$$\theta\left[{\alpha \atop \beta}\right] := \theta\left[{\alpha \atop \beta}\right](0, \Omega),$$

$$\dot{\theta}\left[{\alpha \atop \beta}\right] := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta\left[{\alpha \atop \beta}\right](tV, \Omega) = \partial_V \theta\left[{\alpha \atop \beta}\right](0, \Omega),$$

$$\ddot{\theta}\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\theta\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right](tV,\Omega) = \partial_V^2\theta\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right](0,\Omega)\,,$$

$\cdots\cdots\cdots\cdots$

the theta constants ($V \in \mathbb{C}^2$, ∂_V is the correspondent directional derivative).
The system (2) was integrated in 1989 by Gavrilov [6]. He studied the topology of the real level sets for all generic values of the constants of motion. Meanwhile, solving the Jacobi's inversion problem [9], Gavrilov derives the solutions

$$r = k_1\frac{\theta_0(k_2 t + t_0)}{\theta_\infty(k_2 t + t_0)}\,, \qquad z = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\theta_\infty(k_2 t + t_0) + k_3$$

where $k_{1,2,3}$ are certain constants, $\theta_{0,\infty}$ are two-dimensional theta functions with suitable characteristics (denoted by 0 and ∞) and t_0 is an arbitrary time-shift. However, these solutions are not entirely effective. For example, Jacobi's Inversion Theorem was necessary to evaluate the constants $k_{1,2,3}$. We consider the invertion of hyperelliptic integrals to be a transcendental operation. The following theorem gives the effective solutions.

Theorem 1. *The general solution of the Hénon–Heiles system (2) is*

$$r(t) = p\frac{\theta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right](v,\Omega)}{\theta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](v,\Omega)}\,, \qquad z(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\theta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](v,\Omega) + q\,,$$

where

- *Ω is an arbitrary 2×2 Riemann matrix*
- *$v := v_0 + tV$, $v_0 := (v_{01}, v_{02}) \in \mathbb{C}^2$ is an arbitrary vector and*

$$V := k\left(\frac{\partial}{\partial z_2}\theta\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right], -\frac{\partial}{\partial z_1}\theta\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right]\right)$$

- *$k \in \mathbb{C}$ is an arbitrary constant*
- *the constants p and q must satisfy the quadratic equations*

$$p^2 = 4\frac{\ddot{\theta}\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]\dot{\theta}\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]^2} - 4\frac{\ddot{\theta}\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]\dot{\theta}\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]^2}{\dot{\theta}\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]^2}\,,$$

$$6q^2 - 4q\frac{\ddot{\theta}\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]}{\dot{\theta}\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]} - \frac{\dddot{\theta}\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]} - 3\frac{\ddot{\theta}\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]^2}{\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]^2} + 4\frac{\ddot{\theta}\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]\ddot{\theta}\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]}{\dot{\theta}\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]} = 0$$

- *the constants a and b are defined by*

$$a = 2q - \frac{\ddot{\theta}\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right]} - \frac{\ddot{\theta}\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right]}\,, \qquad b = 12q - 4\frac{\ddot{\theta}\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]}{\dot{\theta}\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]}\,.$$

Proof: Following Mumford [9], for each $(m, n) \in \mathbb{Z}^2$ we introduce the linear spaces of order two theta functions

$$
\mathcal{R}^2 \left[\begin{smallmatrix} m \\ n \end{smallmatrix} \right] := \begin{cases} \text{entire functions } f \colon \mathbb{C}^2 \to \mathbb{C}, \text{ satisfying} \\ f(v + 2\pi i N) = f(v) \exp \pi i \langle m, N \rangle, \\ f(v + \Omega M) = f(v) \exp \langle -M, 2\Omega M + 2v + \pi i n \rangle, \\ \text{for all } M, N \in \mathbb{Z}^2 \text{ and } v \in \mathbb{C}^2. \end{cases}
$$

It is no difficult to prove that [9]

$$
\dim \mathcal{R}^2 \left[\begin{smallmatrix} ab \\ cd \end{smallmatrix} \right] = 4
$$

for all $a, b, c, d \in \mathbb{Z}$, and to show that the functions

$$
f_1 := p\ddot{\theta} \left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix} \right](v)\theta \left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix} \right](v) + p\ddot{\theta} \left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix} \right](v)\theta \left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix} \right](v) - 2p\dot{\theta} \left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix} \right](v)\dot{\theta} \left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix} \right](v),
$$

$$
f_2 := p\theta \left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix} \right](v)\theta \left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix} \right](v),
$$

$$
f_3 := \theta \left[\begin{smallmatrix} 10 \\ 01 \end{smallmatrix} \right](v)\theta \left[\begin{smallmatrix} 01 \\ 00 \end{smallmatrix} \right](v),
$$

$$
f_4 := \theta \left[\begin{smallmatrix} 00 \\ 00 \end{smallmatrix} \right](v)\theta \left[\begin{smallmatrix} 11 \\ 01 \end{smallmatrix} \right](v),
$$

$$
f_5 := \theta \left[\begin{smallmatrix} 10 \\ 00 \end{smallmatrix} \right](v)\theta \left[\begin{smallmatrix} 01 \\ 01 \end{smallmatrix} \right](v)
$$

belongs to $\mathcal{R}^2 \left[\begin{smallmatrix} 11 \\ 01 \end{smallmatrix} \right]$. Hence these five functions are linearly dependent — there exist some constants $c_1, \ldots, c_5$ (not all of them vanish), such that

$$
c_1 f_1(v) + c_2 f_2(v) + c_3 f_3(v) + c_4 f_4(v) + c_5 f_5(v) = 0
$$

for each $v \in \mathbb{C}$.

Next, for every $(m, n) \in \mathbb{Z}^2$ let

$$
\left[\begin{smallmatrix} m \\ n \end{smallmatrix} \right] := \pi i n + \frac{1}{2} m\Omega \in \mathbb{C},
$$

Ω is the Riemann matrix. Using the properties of the thetas, we set successively

$$
v = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \qquad v = \begin{bmatrix} 00 \\ 10 \end{bmatrix}, \qquad v = \begin{bmatrix} 01 \\ 01 \end{bmatrix}, \qquad v = \begin{bmatrix} 00 \\ 00 \end{bmatrix},
$$

and find respectively that

$$
c_5 = 0, \qquad c_4 = 0, \qquad c_3 = 0, \qquad c_1 f_1(0) + c_2 f_2(0) = 0.
$$

The last identity proves that

$$
2q - a = \frac{\ddot{\theta} \left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix} \right]}{\theta \left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix} \right]} + \frac{\ddot{\theta} \left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix} \right]}{\theta \left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix} \right]},
$$

see the Theorem, and has been equivalent to the first equation $\ddot{r} = -ar + 2zr$ of the Hénon–Heiles system (2).

Analogously, consider the functions

$$f_1 := \dddot{\theta} \left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](v)\theta \left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](v) - 4\ddot{\theta} \left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](v)\dot{\theta} \left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](v) + 3\ddot{\theta} \left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](v)^2,$$

$$f_2 := \ddot{\theta} \left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](v)\theta \left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](v) - \dot{\theta} \left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](v)^2,$$

$$f_3 := \theta \left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](v)^2,$$

$$f_4 := p^2\theta \left[\begin{smallmatrix}11\\11\end{smallmatrix}\right](v)^2,$$

$$f_5 := \theta \left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](v)^2.$$

They belong to the space $\mathcal{R}^2 \left[\begin{smallmatrix}00\\00\end{smallmatrix}\right]$ and are linearly dependent: $\sum c_j f_j(v) \equiv 0$. In the case under consideration we set consequently

$$v = \begin{bmatrix}10\\00\end{bmatrix}, \qquad v = \begin{bmatrix}01\\01\end{bmatrix}, \qquad v = \begin{bmatrix}10\\10\end{bmatrix}, \qquad v = \begin{bmatrix}01\\11\end{bmatrix}$$

to conclude that

$$c_5 = 0, \qquad 12q - b = 4\frac{\ddot{\theta} \left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]}{\dot{\theta} \left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]},$$

$$q(6q - b) = -\frac{\dddot{\theta} \left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]}{\theta \left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]} - 3\frac{\ddot{\theta} \left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]^2}{\theta \left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]^2} + 4\frac{\ddot{\theta} \left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]\ddot{\theta} \left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]}{\dot{\theta} \left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]\theta \left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]},$$

$$p^2 = 4\frac{\ddot{\theta} \left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]\dot{\theta} \left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]}{\theta \left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]^2} - 4\frac{\ddot{\theta} \left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]\dot{\theta} \left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]^2}{\dot{\theta} \left[\begin{smallmatrix}01\\11\end{smallmatrix}\right]\theta \left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]^2},$$

and the second equation $\ddot{z} = -bz + 6z^2 + r^2$ from (2) is fulfilled.

This finishes the proof of the Theorem. Actually we have verified two identities between theta functions. $\square$

Every solution $\left(r(t), z(t)\right)$ of the Hénon–Heiles system lies on a two-dimensional torus

$$\mathbb{T}^2 := \mathbb{C}^2 \Big/ \left\{2\pi \mathrm{i}N + \Omega M \,;\, (N, M) \in \mathbb{Z}^2\right\},$$

defined by the constants $\Omega_{11}, \Omega_{12}, \Omega_{22}$. For any fixed Riemann matrix Ω, the points on the corresponding torus $\mathbb{T}^2$ are parametrized by the vector v and the flow $v = v_0 + tV$ is linear on $\mathbb{T}^2$. The speed depends multiplicatively on the

constant k and $V = V(k, \Omega)$ is the winding vector on the torus $\mathbb{T}^2$. The six constants

$$\Omega_{11}, \Omega_{12}, \Omega_{22}, k, v_{01}, v_{02}$$

are called algebraic-geometric coordinates of the problem. Remark that the system (2) depends on 4 initial conditions $\left(r(0), z(0), \dot{r}(0), \dot{z}(0)\right)$ and two parameters a and b. Thus the sum $4 + 2$ coinsides with the number of the constants $\Omega_{11}, \Omega_{12}, \Omega_{22}, k, v_{01}, v_{02}$.

Remark finally, that using the modular transformations [9] on Siegel's upper half-plane, it is possible [8] to replace the characteristics $\left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix}\right]$ (see the Theorem) with any different two characteristics $\left[\begin{smallmatrix} ab \\ cd \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} pq \\ rs \end{smallmatrix}\right]$.

Acknowledgments

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