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SOLITARY SOLUTIONS OF COUPLED KdV AND HIROTA-SATSUMA DIFFERENTIAL EQUATIONS

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Abstract. By considering the set of coupled KdV differential equations as a zero curvature representation of some fourth order linear differential equation and factorizing the linear differential equation, the hierarchy of solutions of the coupled KdV differential equations have been obtained from the eigen spectrum of constant potentials.

1. Introduction

The cKdV (coupled Korteweg-de Vries) equation is a generic example of N-component systems, energy dependent Schrödinger operators and bi-Hamiltonian structures for multi-component systems [3, 4]. Quasi-periodic and soliton solution are studied in connection with Hamitonian systems on Riemann surface in [1]. The soliton fission effect, kink to anti-kink transitions, and multi-peaked solitons extend to equations that model physical phenomena. The classical Boussinesq system and the equations governing second harmonic generation (SHG) are each connected to the cKdV system through nonsingular transformations [2]. Direct application of these transformations enables solutions of cKdV system to be interpreted in the context of these related equations. A connection between the SHG system and the cKdV system has been recently discussed [2, 14]. Therefore, in this work because of more importance of cKdV systems, we consider two kind of integrable cKdV system [5, 10] and solve them using the factorization method that it is somehow similar to the procedure of obtaining the solution of KdV equation from the free particle Schrödinger equation through the well known technique of supersymmetric quantum mechanics [6, 11, 12].

This work is organized as follows. In Section 2 we consider factorizing the fourth order linear differential equation and deform it through zero curvature representation. Section 3 is devoted to determining the set of functions that appear in the fourth order linear differential operators. In Section 4, we consider the hierarchy of the fourth order linear differential operators. Finally in Section 5, we obtain the hierarchy of the solutions of cKdV and KdV [8, 15] equations.

2. Factorization and Deformation of the Fourth Order Linear Differential Equation

Let us consider the following eigenvalue equations

$$L_1\psi_1 = \lambda\psi_1\,, \tag{1}$$

where the fourth order linear differential operator L_1 is:

$$L_1 = \partial^4 + X_1\partial^2 + Y_1\partial + Z_1\,. \tag{2}$$

The operator L_1 can be factorized as in [7]

$$L_1 = (\partial - g_4)(\partial - g_3)(\partial - g_2)(\partial - g_1) + c\,, \tag{3}$$

where c is an arbitrary constant. Hence we will have

$$L_1\psi_i = A_4A_3A_2A_1\psi_i + c\psi_i = \lambda\psi_i\,, \qquad i = 1,\dots,4 \tag{4}$$

in which A_i are obtained from periodic permutations of the functions g_i, $i = 1,\dots,4$, that is

$$g_1 \mapsto g_2 \mapsto g_3 \mapsto g_4 \mapsto g_1\,, \tag{5}$$

and

$$\begin{aligned} \psi_{i+1} &= A_i\psi_i\,,\\ \psi_1 &= A_4\psi_4\,,\\ A_j = \partial - g_j\,, &\qquad j = 1,\dots,4\,. \end{aligned} \tag{6}$$

By defining

$$F_j = (\psi_j, \psi_{jx}, \psi_{jxx}, \psi_{jxxx})^{\mathsf{T}}\,, \tag{7}$$

we can write [9]

$$F_{jx} = U_j F_j \,, \tag{8}$$

where

$$U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda - Z_j & -Y_j & -X_j & 0 \end{pmatrix} . \tag{9}$$

It must to be mentioned that we have supposed the transformations (6) are canonical ones. Now by taking derivative of the relations (6) with respect to x and using the relations (8) we will have

$$\begin{aligned} F_{j+1} &= G_j F_j \qquad j = 1, \ldots, 3\,, \\ F_1 &= G_4 F_4\,, \end{aligned} \tag{10}$$

where

$$G_k = \begin{pmatrix} -g_k & 1 & 0 & 0 \\ -g_{kx} & -g_k & 1 & 0 \\ -g_{kxx} & -2g_{kx} & -g_k & 1 \\ \lambda - Z_k - g_{kxxx} & -(Y_k + 3g_{kxx}) & -(X_k + 3g_{kx}) & 0 \end{pmatrix} \tag{11}$$

Now, taking derivative of both sides of (10) with respect to x and assuming that the matrix G_k is invertible, we can write

$$\begin{aligned} U_{j+1} &= G_{jx} G_j^{-1} + G_j U_j G_j^{-1}\,, \qquad j = 1, \ldots, 3 \\ U_1 &= G_{4x} G_4^{-1} + G_4 U_4 G_4^{-1}\,. \end{aligned} \tag{12}$$

Here we have assumed that the vectors F_i depend on another parameter such as t, so that

$$F_{jt} = V_j F_j\,, \qquad j = 1, \ldots, 4\,. \tag{13}$$

By taking derivative of both sides of (10) with respect to t we conclude

$$\begin{aligned} V_{j+1} &= G_{jt} G_j^{-1} + G_j V_j G_j^{-1}\,, \qquad j = 1, \ldots, 3\,, \\ U_1 &= G_{4t} G_4^{-1} + G_4 V_4 G_4^{-1}\,. \end{aligned} \tag{14}$$

The relations (12) and (14) are just gauge transformations which preserve the zero curvature condition, i. e we have

$$U_{it} - V_{ix} + [U, V] = 0\,, \qquad j = 1, \ldots, 4\,. \tag{15}$$

Now by substituting the relations (9) and (11) in the transformations (12), we obtain

$$\begin{aligned} X_{l+1} &= X_l + 4g_{lx}\,, \\ Y_{l+1} &= Y_l + X_{lx} + 6g_{lxx} + 4g_l g_{lx}\,, \\ Z_{l+1} &= Z_l + 2X_{l+1}g_{lx} + Y_{lx} + 4g_{lxx} + g_l X_{lx} + 6g_l g_{lxx} + 4g_l^2 g_{lx}\,, \end{aligned} \tag{16}$$

for $l = 1, \ldots, 3$ and

$$\begin{aligned} X_1 &= X_4 + 4g_{4x}\,, \\ Y_1 &= Y_4 + X_{4x} + 6g_{4xx} + 4g_4 g_{4x}\,, \\ Z_1 &= Z_4 + 2X_4 g_{4x} + Y_{4x} + 4g_{4xx} + g_4 X_{4x} + 6g_4 g_{4xx} + 4g_4^2 g_{4x}\,. \end{aligned} \tag{17}$$

3. Determination of the Set of Functions g_i, X_i for Y_i and Z_i, $i = 1, 2, 3, 4$ together with their Higher Step Generalization

If we can determine the set of functions X_1, Y_1 and Z_1 by solving the eigenvalue equation $L_1\psi_1 = \lambda\psi_1$, then we can determine the set of functions X_i, Y_i and Z_i, $i = 2, 3, 4$ via the prescription of previous section, that is, by choosing $\lambda = c$ which yields

$$(\partial - g_1)\psi(c) = 0 \Longrightarrow g_1 = \frac{\partial}{\partial x}\log\psi(c)\,. \tag{18}$$

Obviously the eigenvalue equation $L_1\psi_1 = \lambda\psi_1$ has four linearly independent solutions, since it is fourth order linear differential equation. Hence we can consider the set of functions $\psi_1(c)$, $\phi_1(c)$ and $\xi_1(c)$ as three linearly independent solutions of eigenvalue equation $L_1\psi_1 = \lambda\psi_1$ for $\lambda = c$. Now, defining

$$g_1 = \frac{\partial}{\partial x}\log\psi_1(c)\,, \tag{19}$$

where the function $\psi_1(c)$ can bee chosen as the ground state of the eigenvalue equation $L_1\psi(c) = \lambda\psi(c)$. Then we have

$$\psi_2(c) = (\partial - g_1)\psi_1(c) = W\Big(\psi_1(c), \phi_1(c)\Big)\,. \tag{20}$$

Therefore, the function g_2 can be written as

$$g_2 = \frac{\partial}{\partial x}\log\frac{W\Big(\psi_1(c), \phi_1(c)\Big)}{\psi_1(c)}\,. \tag{21}$$

Similarly we can write

$$\psi_3(c) = (\partial - g_2)(\partial - g_1)\psi_1(c) = \frac{W\Big(\psi_1(c), \phi_1(c), \xi_1(c)\Big)}{W\Big(\psi_1(c), \phi_1(c)\Big)}, \tag{22}$$

hence the function g_3 take the following form

$$g_3 = \frac{\partial}{\partial x} \log \frac{W\Big(\psi_1(c), \phi_1(c), \xi_1(c)\Big)}{W\Big(\psi_1(c), \phi_1(c)\Big)}. \tag{23}$$

Now, considering the relation

$$g_1 + g_2 + g_3 + g_4 = 0, \tag{24}$$

and using (19), (21) and (23) we obtain

$$g_4 = -\frac{\partial}{\partial x} \log W\Big(\psi_1(c), \phi_1(c), \xi_1(c)\Big), \tag{25}$$

and consequently the ground state of the eigenvalue equation $L_1\psi(c) = \lambda\psi(c)$ corresponding to $\lambda = c$ is

$$\psi_4(c) = \frac{1}{W\Big(\psi_1(c), \phi_1(c), \xi_1(c)\Big)}. \tag{26}$$

Now, using the relations (16) and (17) we can determine the set of functions $X_i,\ i = 2, 3, 4$ in terms of the functions X_1, i. e.

$$X_2 = X_1 + 4\frac{\partial^2}{\partial x^2} \log \psi_1(c), \tag{27}$$

$$X_3 = X_1 + 4\frac{\partial^2}{\partial x^2} \log W\Big(\psi_1(c), \phi_1(c)\Big), \tag{28}$$

$$X_4 = X_1 + 4\frac{\partial^2}{\partial x^2} \log W\Big(\psi_1(c), \phi_1(c), \xi_1(c)\Big). \tag{29}$$

Even though there are not the expressions like the above ones for the set of functions Y_i and Z_i, $i = 2, 3, 4$, they can be determined in terms of the functions g_1, g_2 and g_3.

4. Hierarchy of Fourth Order Operators

In this section we introduce the following hierarchy of fourth order linear differential operators

$$L_0^1, L_0^2, L_0^3, L_0^4 = L_1^1\,; \quad L_1^2, L_1^3, L_1^4 = L_2^1\,; \ \ldots\,; \ L_n^1, L_n^2, L_n^3, L_n^4 = L_{n+1}^1 \tag{30}$$

where, the set of operators L_i^j, $j = 1, 2, 3, 4$ and $i = 1, 2, 3, \ldots$ can be factorized in the following form

$$\begin{aligned} L_n^1 &= A_n^4 A_n^3 A_n^2 A_n^1 + c_n\,, \\ L_n^2 &= A_n^1 A_n^4 A_n^3 A_n^2 + c_n\,, \\ L_n^3 &= A_n^2 A_n^1 A_n^4 A_n^3 + c_n\,, \\ L_n^4 &= A_n^3 A_n^2 A_n^1 A_n^4 + c_n\,, \end{aligned} \tag{31}$$

with

$$A_n^r = \partial - g_n^r\,, \qquad r = 1, \ldots, 4\,. \tag{32}$$

From the identity $L_n^4 = L_{n+1}^1$ we have

$$\begin{aligned} &(\partial - g_n^3)(\partial - g_n^2)(\partial - g_n^1)(\partial - g_n^4) + c_n \\ &\qquad = (\partial - g_{n+1}^4)(\partial - g_{n+1}^3)(\partial - g_{n+1}^2)(\partial - g_{n+1}^1) + c_{n+1}\,. \end{aligned} \tag{33}$$

Now, using the prescription of previous section, the set of functions g_n^i, $i = 1, 2, 3$ can be determined as

$$g_n^1 = \frac{\partial}{\partial x} \log \psi_n^1(c_n)\,, \tag{34}$$

$$g_n^2 = \frac{\partial}{\partial x} \log \frac{W\Big(\psi_n^1(c_n), \phi_n^1(c_n)\Big)}{\psi_n^(c_n)}\,, \tag{35}$$

$$g_n^3 = \frac{\partial}{\partial x} \log \frac{W\Big(\psi_n^1(c_n), \phi_n^1(c_n), \xi_n^1(c)_n)\Big)}{W(\psi_n^1(c_n), \phi_n^1(c_n))}\,, \tag{36}$$

where the set of functions $\psi_n^1(c_n)$, $\phi_n^1(c_n)$ and $\xi_n^1(c_n)$ are three linearly independent solutions of the eigenvalue equation $L_n^1 \psi_n^1 = \lambda \psi_n^1$ corresponding to the eigenvalue $\lambda = c_n$. Now, if we assume that the set of functions X_0^1, Y_0^1 and Z_0^1 are arbitrary constants, then the eigen spectrum of the eigenvalue equation $L_n^1 \psi_n^1 = \lambda \psi_n^1$ can be determined right away. Hence taking its three linearly independent eigen functions $\psi_0^1(c_0)$, $\phi_0^1(c_0)$ and $\xi_0^1(c_0)$ together, and using the relations (34), (35) and (36), we get

$$g_0^1 = \frac{\partial}{\partial x} \log \psi_0^1(c_0)\,, \tag{37}$$

$$g_0^2 = \frac{\partial}{\partial x} \log \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0)\Big)}{\psi_0^(c_0)}\,, \tag{38}$$

$$g_0^3 = \frac{\partial}{\partial x} \log \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0\Big)}{W(\psi_0^1(c_0), \phi_0^1(c_0))}\,. \tag{39}$$

Similarly for $\lambda = c_1$ from the set functions $\psi_0^1(c_1)$, $\phi_0^1(c_1)$ and $\xi_0^1(c_1)$, we obtain

$$\psi_0^2(c_1) = \frac{W\Big(\psi_0^1(c_0), \psi_0^1(c_1)\Big)}{\psi_0^1(c_0)}\,, \tag{40}$$

$$\psi_0^3(c_1) = \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \psi_0^1(c_1)\Big)}{W\Big(\psi_0^1(c_0), \phi_0^1(c_0)\Big)}\,, \tag{41}$$

$$\psi_0^4(c_1) = \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_1), \psi_0^1(c_1)\Big)}{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_1)\Big)}\,. \tag{42}$$

Obviously the relations (40), (41) and (42) hold true for the set of functions $\phi_0^1(c_1)$ and $\xi_0^1(c_1)$, too, where all we need only is to replace the function $\psi_0^1(c_1)$ with $\phi_0^1(c_1)$ and $\xi_0^1(c_1)$, respectively. Now, for $n = 1$ using (42) and taking the fact that $H_0^4 = H_1^1$, we get

$$g_1^1 = \frac{\partial}{\partial x} \log \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \psi_0^1(c_1)\Big)}{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0)\Big)}\,. \tag{43}$$

Indeed we choose the function $\psi_1^1(c_1) = \psi_0^4(c_1)$ as the ground state of eigenvalue equation $L_1^1\psi_1^1 = \lambda\psi_1^1$. Also

$$\psi_1^2 = (\partial - g_1^1)\phi_1^1 = \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \psi_0^1(c_1), \phi_0^1(c_1)\Big)}{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \psi_0^1(c_1)\Big)}\,, \tag{44}$$

which leads to

$$g_1^2 = \frac{\partial}{\partial x} \log \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \psi_0^1(c_1), \phi_0^1(c_1)\Big)}{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \psi_0^1(c_1)\Big)}\,. \tag{45}$$

Finally, the function $\psi_1^3(c_1)$ can be written as

$$\psi_1^3 = (\partial - g_1^2)(\partial - g_1^1)\xi_1^1$$
$$= \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \psi_0^1(c_1), \phi_0^1(c_1), \xi_0^1(c_1)\Big)}{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \psi_0^1(c_1), \phi_0^1(c_1)\Big)} \,. \tag{46}$$

Therefore, for the function g_1^3 we get the following expression

$$g_1^3 = \frac{\partial}{\partial(x)} \log \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \psi_0^1(c_1), \phi_0^1(c_1), \xi_0^1(c_1)\Big)}{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \psi_0^1(c_1), \phi_0^1(c_1)\Big)} \,. \tag{47}$$

Repeating the above procedure and using

$$\frac{W(\phi_1, \ldots, \phi_n, f), W(\phi_1, \ldots, \phi_n, g))}{W(\phi_1, \ldots, \phi_n)} = W(\phi_1, \ldots, \phi_n, f, g)\,, \tag{48}$$

we can evaluate all g_n, and we have

$$g_n^1 = \frac{\partial}{\partial x} \log(\Omega_n^1)\,, \tag{49}$$

$$\Omega_n^1 = \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \ldots}{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \ldots}$$
$$\frac{\ldots, \psi_0^1(c_{n-1}), \phi_0^1(c_{n-1}), \xi_0^1(c_{n-1}), \psi_0^1(c_n)\Big)}{\ldots, \psi_0^1(c_{n-1}), \phi_0^1(c_{n-1}), \xi_0^1(c_{n-1})\Big)} \,, \tag{50}$$

$$g_n^2 = \frac{\partial}{\partial x} \log(\Omega_n^2)\,, \tag{51}$$

$$\Omega_n^2 = \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \ldots}{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \ldots}$$
$$\frac{\ldots, \psi_0^1(c_{n-1}), \phi_0^1(c_{n-1}), \xi_0^1(c_{n-1}), \psi_0^1(c_n), \phi_0^1(c_n)\Big)}{\ldots, \psi_0^1(c_{n-1}), \phi_0^1(c_{n-1}), \xi_0^1(c_{n-1}), \psi_0^1(c_n)\Big)} \,, \tag{52}$$

$$g_n^3 = \frac{\partial}{\partial x} \log(\Omega_n^3)\,, \tag{53}$$

$$\Omega_n^3 = \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \dots}{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \dots} \frac{\dots, \psi_0^1(c_{n-1}), \phi_0^1(c_{n-1}), \xi_0^1(c_{n-1}), \psi_0^1(c_n), \phi_0^1(c_n), \xi_0^1(c_n)\Big)}{\dots, \psi_0^1(c_{n-1}), \phi_0^1(c_{n-1}), \xi_0^1(c_{n-1}), \psi_0^1(c_n), \phi_0^1(c_n)\Big)} . \tag{54}$$

Now, using the identity $L_{n+1}^1 = L_n^4$, we obtain

$$\begin{aligned} X_{n+1}^1 &= X_n + 4\frac{\partial}{\partial x}(g_n^1 + g_n^2 + g_n^3) \\ &= X_n + 4\frac{\partial^2}{\partial x^2} \log \frac{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \dots}{W\Big(\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \dots} \\ &\quad \frac{\dots, \psi_0^1(c_{n-1}), \phi_0^1(c_{n-1}), \xi_0^1(c_{n-1}), \psi_0^1(c_n), \phi_0^1(c_n), \xi_0^1(c_n)\Big)}{\dots, \psi_0^1(c_{n-1}), \phi_0^1(c_{n-1}), \xi_0^1(c_{n-1})\Big)} . \end{aligned} \tag{55}$$

Repeating the relation (55) n times, we obtain

$$X_n^2 = X_0^1 + 4\frac{\partial^2}{\partial x^2} \log \tau_n^2 , \tag{56}$$

$$\begin{aligned} \tau_n^2 = W\Big(&\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \dots \\ &\dots, \psi_0^1(c_{n-1}), \phi_0^1(c_{n-1}), \xi_0^1(c_{n-1}), \psi_0^1(c_n)\Big) , \end{aligned} \tag{57}$$

$$X_n^3 = X_0^1 + 4\frac{\partial^2}{\partial x^2} \log \tau_n^3 , \tag{58}$$

$$\begin{aligned} \tau_n^3 = W\Big(&\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \dots \\ &\dots, \psi_0^1(c_{n-1}), \phi_0^1(c_{n-1}), \xi_0^1(c_{n-1}), \psi_0^1(c_n), \phi_0^1(c_n)\Big) , \end{aligned} \tag{59}$$

$$X_{n+1}^1 = X_n^4 = X_0^1 + 4\frac{\partial^2}{\partial x^2} \log \tau_n^4 , \tag{60}$$

$$\begin{aligned} \tau_n^4 = W\Big(&\psi_0^1(c_0), \phi_0^1(c_0), \xi_0^1(c_0), \dots \\ &\dots, \psi_0^1(c_{n-1}), \phi_0^1(c_{n-1}), \xi_0^1(c_{n-1}), \psi_0^1(c_n), \phi_0^1(c_n), \xi_0^1(c_n)\Big) . \end{aligned} \tag{61}$$

5. Solitary Solutions of Coupled KdV Differential Equations

In this section using the results of previous section, we will obtain solutions of KdV and Hirota–Satsuma differential equations. We choose the operators such that, they satisfy $g_n^1 + g_n^2 = 0$ and $g_n^3 + g_n^4 = 0$. We can choose

$$\begin{aligned} g_n^1 &= -v_n^1 \,, \qquad & g_n^2 &= v_n^1 \,, \\ g_n^3 &= -v_n^2 \,, \qquad & g_n^4 &= v_n^2 \,, \end{aligned} \tag{62}$$

and then we can write

$$\begin{aligned} L_n^1 &= (\partial + v_n^2)(\partial - v_n^2)(\partial - v_n^1)(\partial + v_n^1) + c_n \,, \\ L_n^2 &= (\partial + v_n^2)(\partial - v_n^2)(\partial - v_n^1)(\partial + v_n^1) + c_n \,, \\ L_n^3 &= (\partial - v_n^1)(\partial + v_n^1)(\partial + v_n^2)(\partial - v_n^2) + c_n \,, \\ L_n^4 &= (\partial - v_n^2)(\partial - v_n^1)(\partial + v_n^1)(\partial + v_n^2) + c_n \,. \end{aligned} \tag{63}$$

By defining the functions [5]

$$\begin{aligned} \varphi_n &= \frac{1}{2}(v_{nx}^1 - v_{nx}^2 - (v_n^1)^2 - (v_n^2)^2) \,, \\ u_n &= \frac{1}{2}(v_{nx}^1 + v_{nx}^2 - (v_n^1)^2 + (v_n^2)^2) \,, \\ f_n &= v_n^1 + v_n^2 \,, \qquad h_n = v_{nx}^1 v_{nx}^2 - v_{nx}^1 \,. \end{aligned} \tag{64}$$

they reduce to

$$\begin{aligned} L_n^1 &= \partial^4 + 2u_n\partial^2 + 2(u_{nx} + \varphi_{nx})\partial + u_n^2 \\ &\quad - \varphi_n^2 + u_{nxx} + \varphi_{nxx} + c_n \,, \\ L_n^2 &= \partial^4 + (2h_n - f_{nx} - f_n^2)\partial^2 + (2h_{nx} - f_{nxx} - 2f_n f_{nx})\partial \\ &\quad + h_n^2 + f_n h_{nx} + h_n f_{nx} + c_n \,, \\ L_n^3 &= \partial^4 + 2u_n\partial^2 + 2(u_{nx} - \varphi_{nx})\partial + u_n^2 - \varphi_n^2 \\ &\quad + u_{nxx} - \varphi_{nxx} + c_n \,, \\ L_n^4 &= \partial^4 + (2h_n + 3f_{nx} - f_n^2)\partial^2 + (2h_{nx} + 3f_{nxx} - 2f_n f_{nx})\partial + h_{nxx} \\ &\quad + f_{nxxx} + h_n^2 - f_n h_{nx} + h_n f_{nx} + c_n \,. \end{aligned} \tag{65}$$

In (65) the transformation $\varphi_n \to -\varphi_n$ maps the first and third operators into each other, while the transformations $f_n \to -f_n$ and $h_n \to h_n + f_n$ map the second operator to the fourth one. Since, we are interested in obtaining the solitary solutions of coupled KdV differential equations, we choose the operator M_n^1 in the following form [5]

$$M_n^1 = 2\partial^3 + 3u_n\partial + \frac{3}{2}(u_n + 2\phi_n) \,. \tag{66}$$

The Lax's equations associated with the pairs of operators M_n^1 and L_n^1 lead to Hirota–Satsuma equation

$$\begin{aligned} u_{nt} &= \frac{1}{2}u_{nxxx} + 3uu_{nx} - 6\phi_n\phi_{nx}\,, \\ \phi_{nt} &= -\phi_{nxxx} + 3u_n\phi_{nx}\,, \end{aligned} \tag{67}$$

which are invariant with respect to the transformation $\phi \longrightarrow -\phi$. Now, defining the operator M_n^2 as [5]

$$M_n^2 = 2\partial^3 + \frac{3}{2}(2h_n - f_{nx} - f_n^2)\partial + \frac{3}{4}(2h_{nx} - f_{nxx} - 2f_nf_{nx})\,, \tag{68}$$

then, the Lax's equation associated with the pair of operators M_n^2 and L_n^2 (the second operator in (65)) lead to coupled KdV equation

$$\begin{aligned} f_{nt} = -\frac{1}{2}\Big(&2f_{nxxx} + 3f_nf_{nxx} + f_{nx}^2 - 3f_n^2f_{nx} + h_nf_{nx} \\ &+ 6f_nh_{nx}\Big)\,, \end{aligned} \tag{69}$$

$$\begin{aligned} h_{nt} = -\frac{1}{4}\Big(&2h_{nxxx} + 12h_nh_{nx} + 6f_nh_{nxx} + 12h_nf_{nxx} \\ &+ 18f_{nx}h_{nx} - 6h_nf_{nx} + 3f_{nxxxx} + 3f_nf_{nxxx} \\ &+ 18f_{nx}f_{nxx} - 6f_n^2f_{nxx} - 6f_nf_{nx}^2\Big)\,. \end{aligned} \tag{70}$$

The equations (67) are invariant under the transformation $\phi_n \longrightarrow \phi_n$, while the equations (69) and (70) are invariant under the transformations $f_n \longrightarrow -f_n$ and $h_n \longrightarrow h_n + f_n$. Also, through the above transformations in (66) and (68), the Lax's partners of the operators L_n^3 and L_n^4 take the following form

$$M_n^3 = 2\partial^3 + 3u_n\partial + \frac{3}{2}(u_n - 2\phi_n)\,, \tag{71}$$

$$M_n^4 = 2\partial^3 + \frac{3}{2}(2h_n + 3f_{nx} - f_n^2)\partial + \frac{3}{4}(2h_{nx} + 3f_{nxx} - f_nf_{nx})\,. \tag{72}$$

We should mention that if we consider $\xi_n^1(c_n)$, $\varphi_n^1(c_n)$ and $\psi_n^1(c_n)$ as three linearly independent solutions of the eigenvalue equation $L_n^1\psi_n^1 = \lambda\psi_n^1$, then according to (62), we have

$$W\Big(\psi_n^1(c_n), \phi_n^1(c_n)\Big) = \text{const} \neq 0\,, \tag{73}$$

that is, in every step we should choose the linearly independent solutions $\varphi_n^1(c_n)$ and $\psi_n^1(c_n)$ with a constant Wronskian. According to (65), the equality $L_n^4 =$

L^1_{n+1} implies that

$$\begin{aligned} \varphi_{n+1} &= \text{const}\,, \\ u_{n+1} &= \frac{1}{2}(2h_n + 3f_{nx} - f_n^2)\,. \end{aligned} \tag{74}$$

Hence the solutions of coupled KdV equation give the solution of KdV equation via exploiting the relation (74). Now we consider an example. If the potentials u_0 and φ_0 are arbitrary constants, then the solutions of the eigenvalue equation $L^1_0\psi_0 = \lambda\psi_0$ for $\lambda = c_k$ are

$$\exp[\pm\alpha_k x + m_t] \quad \text{and} \quad \exp[\pm\beta_k x + n_t] \tag{75}$$

where α_k and β_k are

$$\begin{aligned} \alpha_k &= \left[-u_0 + \left(\varphi_0^2 + c_0 - c_k\right)^{1/2}\right]^{1/2}, \\ \beta_k &= \left[-u_0 - \left(\varphi_0^2 + c_0 - c_k\right)^{1/2}\right]^{1/2}. \end{aligned} \tag{76}$$

Since these solutions should satisfy the time evolution equation $\psi^1_{0t}(c_k) = M^1_0\psi^1_0(c_k)$ too, they can be written in the form

$$a_k \exp[\pm\alpha_k(x + (2\alpha_k^2 + 3u_0)t] \quad \text{and} \quad b_k \exp[\pm\alpha_k(x + (2\beta_k^2 + 3u_0)t] \tag{77}$$

As an example for $k = 0$ and using (73) we choose $\psi^1_0(c_0)$, $\varphi^1_0(c_0)$ and $\xi^1_0(c_0)$ in the following form

$$\begin{aligned} \psi^1_0(c_0) &= \cosh(\alpha_0(x + (2\alpha_0^2 + 3u_0)t)\,, \\ \phi^1_0(c_0) &= \sinh(\alpha_0(x + (2\alpha_0^2 + 3u_0)t)\,, \\ \xi^1_0(c_0) &= \cosh(\beta_0(x + (2\beta_0^2 + 3u_0)t)\,. \end{aligned} \tag{78}$$

By choosing $U_0 = -3$ and $\varphi_0 = 1$, in the first step of factorization, using the relations (64) and (74) we obtain the following results for the solutions of coupled KdV equations and KdV itself

$$f_0 = -\tanh[\sqrt{2}(-x + 5t)] - 2\tanh(-2x + 2t)\,, \tag{79}$$

$$h_0 = -2\sqrt{2}\tanh(-2x + 2t)\tanh[\sqrt{2}(-x + 5t)] + \frac{4}{\cosh^2(-2x + 2t)}\,, \tag{80}$$

$$u_1 = -3 + \frac{4}{\cosh^2[\sqrt{2}(-x + 5t)]}\,. \tag{81}$$

References

[1] Alber M., Luther G. and Marsden J., *Energy Dependent Schrödinger Operators and Complex Hamiltonian Systems on Riemann Surface*, Nonlinearity **10** (1997) 223–242.

[2] Alber M., Luther G. and Miller C., *On Soliton-Type Solutions of Equations Associated with N-Component Systems*, J. Math. Phys. **41** (2000) 284–316.

[3] Antonowicz M. and Fordy A., *A Family of Completly Integrable Multi-Systems*, Phys Lett A **122** (1987) 95–99.

[4] Antonowicz M. and Fordy A., *Coupled KdV Equations with Multi-Hamiltonian Structures*, Physica D **28** (1987) 345–357.

[5] Baker S., Enolskii V. and Fordy A., *Integrable Quartic Potentials and Coupled KdV Equations*, hep-th/9504087.

[6] Cooper F., Khare A. and Sukhatme U., *Supersymmetry and Quantum Mechanics*, Phys. Rep. **251c** (1995) 267.

[7] Dickey L., *Soliton Equations and Hamiltonian Systems*, World Scientific, Singapore 1991.

[8] Drazin P. and Johnson R., *Solitons: An Introduction*, Cambridge Univ. Press, Cambridge 1989.

[9] Faddeev L. and Takhtadjan L., *Hamiltonian Methods in The Theory of Solitons*, Springer, Berlin 1980.

[10] Geng X. and Wu Y., *New Finite-Dimensional Integrable Systems and Explicit Solutions of Hirota-Satsuma Coupled Korteweg-de Vries Equation*, J. Math. Phys. **38**(6) (1997) 3069.

[11] Gront A. and Rosner J., *Supersymmetric Quantum Mechanics and the Korteweg-de Vries Hierarchy*, J. Math. Phys. **35** (1994) 2142.

[12] Jafarizadeh M., Esfandyari A. and Panahi-Talemi H., *Isospectral Deformation of Some Shape Invariant Potentials*, J. Math. Phys. **41** (2000) 675.

[13] Kersten P. and Krasil'shehile J., *Complete Integrability of the Coupled-mKdV System*, nlin/0010041.

[14] Khusnutdinova K. and Steudel H., *Second Harmonic Generation: Hamiltonian Structures and Particular Solutions*, J. Math. Phys. **39** (1998) 3754–3764.

[15] Lamb G., *Elements of Soliton Theory*, John Wiley and Sons, New York 1980.