Geometry, Integrability and Quantization
September 1–10, 1999, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2000, pp 95-104

GEOMETRIC QUANTIZATION, COHOMOLOGY GROUPS AND INTERTWINING OPERATORS

PETRO I. HOLOD and T. V. SKRYPNYK

Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine

Abstract. Higher Dolbeaut cohomology groups of the flag manifolds are explicitly constructed using the technique of the intertwining operators. The integral representation for the higher order harmonic forms is obtained.

1. Introduction

Borel–Weil theory [1] gives realization of the compact group unitary representations in the space of sections of the induced linear fiber bundle over the full flag manifold. It could be viewed as direct ancestor of the geometric quantization of Kostant and Souriau [12], [4]. Generalization of the Borel–Weil theorem — famous theorem of Bott [2] served as a basis for the well known hypothesis of Landglands about the geometric realization of discrete series representations of semisimple Lie groups.

Necessity to pass to the higher cohomology groups of quantized phase space arises also in geometric quantization [5], [6]. It is connected with the notion of polarization [7] which is important technical tool that provides irreducibility of the space of quantum states. More precisely, the space of quantum states of the standard geometric quantization [7] coincides with the space of polarization-invariant sections of the line bundle over symplectic manifold. But, in some cases, the space of such sections may be equal to zero. To obtain space of quantum states in such a cases one has to pass to higher Dolbeaut cohomology groups [6]. For the reasonably wide class of situations they all but one are trivial. In the space of non trivial cohomology group the desired space of quantum space lies.

One of the cases when it is necessary to pass to the higher Dolbeaut cohomology groups is the case when the phase space coincides with the orbit O_χ of the semisimple Lie group G and the character χ does not "agree" with the chosen polarization. Let us explain this issue in more details. The choice of the polarization for the semisimple orbits is equivalent to the choice of the Borelian (parabolic) subalgebra in the complexified Lie algebra. This choice is, in turn, equivalent to the fixing of the set of positive roots. The character χ does not "agree" with the polarization when it is not dominant with respect to the described above fixed set of positive roots.

If the semisimple Lie group G is a compact (Bott theorem), all representations of G, obtained with the help of quantization in different cohomology groups, are equivalent. Nevertheless it is interesting to have these cohomology groups in explicit form. Main reason for this interest is the fact that constructing representations of non compact semisimple Lie groups in cohomologies [13], [14] one should use the cohomology groups associated with the maximally compact subgroup.

Besides, we hope that presented method of explicit construction of the representatives of the Dolbeaut cohomology classes may have some analogs in more general situation of non-invariant complex polarization (deformed complex structure).

Main tool used in this paper are the so-called intertwining operators [8]–[11]. We use them to intertwine representation of the groups G and G^C in the space of highest cohomology group with the representation in the space of lower ones. In this way we obtain integral form for representatives of cohomological classes of the different Dolbeaut cohomology groups. Cohomology group of the highest order used in the described above construction is recovered using its duality with the space of holomorphic sections of "dual" line bundle. The latter space could be find using purely geometrical considerations [15].

The structure of the present article is as follows. In the second section we remind some general facts about representations in Dolbeaut cohomology groups. In the third section we give explicit description of the space of holomorphic sections of induced linear fiber bundle. In the next section analogous description is given for the highest Dolbeaut cohomology group with the values in the dual bundle. In the last section integral representation for the representatives of the cohomological classes of other cohomology groups are constructed.

2. Generalities

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, G^c be a corresponding Lie group, G — its maximally compact subgroup. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, H be a corresponding maximal torus, λ be a some integer weight on $\mathfrak{h}$, e^{λ} — corresponding character of the maximal torus, ρ — a sum of positive roots. Let

B be a Borelian subgroup, N — its nilpotent radical, Z — contragradient to N nilpotent subgroup.

Let us consider a nondegenerated coadjoint orbit of G:

$$O = H\backslash G\,.$$

Due to the Montgomery theorem we have the following analytic diffeomorphism:

$$O = H\backslash G \simeq X = B\backslash G^c\,,$$

where $X = B\backslash G^c$ is a full flag manifold of the group G^c. For the arbitrary complex semisimple group G^c the following decomposition holds:

$$G^c = \bigcup_{w\in W} BZw\,,$$

which is connected with the standard Bruhat decomposition [16]:

$$G^c = \bigcup_{w\in W} BwZ_w\,,$$

in the sense, that that each component of the latter decomposition belongs to the corresponding component of the former one.

This decomposition yields the system of local complex charts on the flag manifold X:

$$X = B\backslash G^c = \bigcup_{w\in W} Z\,.$$

Here parameters of the group Z plays the role of the local complex coordinates on X. Let us consider the line bundle:

$$C \to E_\lambda \to X$$

constructed from the set of pairs (v, g) via the identification $(v, g) \sim (\mathrm{e}^\lambda(b)v, b^{-1}g)$, where e^λ is the character of subgroup H, extended trivially to the nilpotent radical N. Let us introduce antiholomorphic differential forms with the values in the fiber bundle E_λ. In the described above system of local charts they have the following form: $\varphi = \{\varphi_\alpha\}$

$$\varphi_\alpha = \sum_{i_1,\cdots,i_k} \varphi_{\alpha,i_1,\cdots,i_k}\, \mathrm{d}\bar{z}_\alpha^{i_1} \wedge \ldots \wedge \mathrm{d}\bar{z}_\alpha^{i_k}$$

where z_α^i are the parameters of the group Z.

Action of the group in the space of E_λ-valued antiholomorphic differential forms is the following:

$$T_g\varphi = \mathrm{e}^\lambda(b(z_\alpha w_\alpha g)) \sum_{i_1,\ldots,i_k} \varphi_{i_1,\ldots,i_k}(z_\alpha)^g\, \mathrm{d}(\bar{z}_\alpha^g)^{i_1} \wedge \ldots \wedge \mathrm{d}(\bar{z}_\alpha^g)^{i_k} \qquad (1)$$

where $z_\alpha w_\alpha g = b(z_\alpha w_\alpha g) z_\alpha^g w_\alpha$.

Let $C^k(X, E_\chi)$ be a set of E_χ-valued differential forms of the order k on the complex manifold X. Let us consider the complex

$$\xrightarrow{\bar\partial} C^{k-1} \xrightarrow{\bar\partial} C^k \xrightarrow{\bar\partial} C^{k+1} \xrightarrow{\bar\partial}$$

Cohomology group of this complex

$$H_{\bar\partial} = \sum_{k\geq 0} H^k(X, E_\chi)$$

is usually called Dolbeaut cohomology group. Let $\flat$ be an operator adjoint to $\bar\partial$ with respect to G-invariant scalar product. It yields the following sequence [12]:

$$\xleftarrow{\flat} C^{k-1} \xleftarrow{\flat} C^k \xleftarrow{\flat} C^{k+1} \xleftarrow{\flat} .$$

It is evident that the space $C^k(X, E_\chi)$ is invariant under the action of the operator $\square = (\flat\bar\partial + \bar\partial\flat)$. Let $\mathcal{H}^k(E_\chi)$ be a space of eigenvectors of $\square = (\flat\bar\partial + \bar\partial\flat)$ in $C^k(X, E_\chi)$ that corresponds to zero eigenvalue, i. e. let $\mathcal{H}^k(E_\chi)$ be the space of harmonic forms. It is known [12] that $\mathcal{H}^k(E_\chi) \cong H^k(X, E_\chi)$ as a linear spaces, i. e. each cohomology class contains one and only one harmonic form. This isomorphism extends also to isomorphism of G-modules. Indeed, due to the G-invariance of the operator $\square$, the space of harmonic forms constitutes a subrepresentation of G in the space of all forms, contrary to the group $H^k(X, E_\chi)$, which by the very definition is a subquotient representation of the groups G and G^c. Irreducibility of these representations is provided by the Bott theorem [2], which state that each cohomology group $H^k(X, E_\chi)$ is irreducible G and G^c module, if $k =$ length w, where w is such an element of the Weyl group W, that $w\lambda$ is a dominant weight, or zero otherwise. If the weight λ is dominant then we come to the Borel–Weil theorem which states that each irreducible unitary representation of the group G could be realized in the space of globally holomorphic sections of the fiber bundle E_λ. Indeed, in the case of dominant weight λ we obtain that $w = e$ and length $w = 0$. On the other hand, it is easy to see that $H^0(X, E_\lambda) = \Gamma(E_\lambda)$, where $\Gamma(E_\lambda)$ is a space of globally holomorphic sections of E_λ.

In the next sections we shall find the groups $H^k(X, E_\chi)$ and $\mathcal{H}^k(E_\chi)$ explicitly in the above described local trivialization.

3. Space of the Holomorphic Sections — Explicit Description

Let us explicitly describe the group $H^0(X, E_\lambda)$. Hereafter the weight λ will be dominant. For an arbitrary integer weight χ it is possible to introduce a function $\mathrm{e}^\chi : G^c \to \mathbb{C}$, $\mathrm{e}^\chi(g) = \mathrm{e}^\chi(h(g))$, $g = n(g)h(g)z(g)$, where $n(g) \in N$, $h(g) \in H$, $z(g) \in Z$. The following proposition holds true.

Proposition 1. *The space of the globally holomorphic sections of the fiber bundle E_λ in the local trivialization on the subgroup Z consists of the polynomials of the form $\mathrm{e}^\lambda(z_\alpha g)$, where $g \in G$, $z_\alpha \in Z$ (α here denotes the chosen local chart).*

Proof: It is easy to see that for dominant weight λ functions $\mathrm{e}^\lambda(zg)$ are polynomials on the subgroup Z. Besides, they form subrepresentation in the space of all functions on the subgroup Z. Let us prove this fact. For the purpose of convenience we shall consider the action of the group on the functions of the type $\mathrm{e}^\lambda(z_\alpha g^{w_\alpha})$, where $g^{w_\alpha} = w_\alpha g w_\alpha^{-1}$ and w_α is the element of the Weyl group, corresponding to the α local chart. Using (1) we obtain

$$\begin{aligned} T_{\acute{g}} \cdot \mathrm{e}^\lambda(z_\alpha g^{w_\alpha}) &= \mathrm{e}^\lambda(z_\alpha \acute{g}^{w_\alpha})\, \mathrm{e}^\lambda(z_\alpha^{\acute{g}^{w_\alpha}}\, g^{w_\alpha}) \\ &= \mathrm{e}^\lambda(z_\alpha \acute{g}^{w_\alpha})\, \mathrm{e}^\lambda(b^{-1}(z_\alpha \acute{g}^{w_\alpha}) z_\alpha (\acute{g} g)^{w_\alpha}) = \mathrm{e}^\lambda(z_\alpha (\acute{g} g)^{w_\alpha}) \end{aligned}$$

where $z_\alpha^{\acute{g}^{w_\alpha}}\, w_\alpha = b^{-1}(z_\alpha \acute{g}^{w_\alpha}) z_\alpha \acute{g}^{w_\alpha} w_\alpha$.

Let us prove that $\mathrm{e}^\lambda(z_\alpha g) \in \Gamma(E_\lambda)$, i. e. that $\mathrm{e}^\lambda(zg)$ is a local component of a globally analytic section. Let $f_\alpha = \mathrm{e}^\lambda(z_\alpha g)$. It is easy to show [15], that transition to the other chart could be made with the help of the elements of Weyl group. Hence, on the intersection of two charts we shall have

$$\begin{aligned} f_\beta = g_{\beta\lambda} f_\alpha &= \mathrm{e}^\lambda(b(z_\beta w_\beta w_\alpha^{-1}))\, \mathrm{e}^\lambda(z_\alpha g) \\ &= \mathrm{e}^\lambda(b(z_\beta w_\beta w_\alpha^{-1}) z_\beta^{w_\beta w_\alpha^{-1}}\, g) = \mathrm{e}^\lambda(z_\beta w_\beta w_\alpha^{-1} g) = \mathrm{e}^\lambda(z_\beta \acute{g}) \end{aligned}$$

To complete the proof it is sufficient to notice, that according to the theorem of Borel and Weil $\Gamma(E_\lambda)$ is an irreducible G^c module.

In the case of the classical compact Lie groups Proposition 1 could be made more explicit.

Let G be one of the classical Lie groups of the type $SU(n+1)$, $SO(2n)$, $Sp(n)$, $SO(2n+1)$ and the last three groups are realized as subgroups in $SU(d)$ ($d = 2n$ and $d = 2n+1$ correspondingly), so that their maximal nilpotent subgroups are the subgroups of the group of upper triangular matrix. Then the following proposition [15] holds true:

Proposition 2. *The space of the holomorphic sections of the line bundle E_λ in a local trivialization at the subgroup Z coincides with the space of polynomials of the degree not higher then λ_k in minor determinants of order k of the matrix z.*

Remark: Hereafter we shall consider only the special local chart that correspond to the trivial element of the Weyl group $w = e$. That is why we shall omit index λ everywhere.

Example. Let $G = SU(3)$, then $z = \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{pmatrix}$. The space of the sections of E_λ in a local trivialization on the subgroup Z consists of the linear combination of monomials: $P_{k_1 k_2 l_1 l_2}(z) = z_{12}^{k_1} z_{13}^{k_2} z_{23}^{l_1} (z_{23} z_{12} - z_{13})^{l_2}$

$$k_1 + k_2 \le \lambda_1 \,, \qquad l_1 + l_2 \le \lambda_2 \,.$$

4. Highest Dolbeaut Cohomology Group — Explicit Description

Let us consider the highest cohomology group $H^n(X, E_{-\lambda-\rho})$. Let $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ denote holomorphic and antiholomorphic part of the measure on the subgroup Z, respectively. It is evident that $\varphi \in C^n(X, E_\chi)$ has the following form $\varphi = f(z, \bar{z})\,\mathrm{d}\bar{z}$. It is also evident that each such a form is $\bar{\partial}$-closed. To describe the group $H^n(X, E_{-\lambda-\rho})$ it is sufficient to describe the set of exact forms in the local trivialization. The following statement is true.

Proposition 3. *The differential form $\varphi \in H^n(X, E_{-\lambda-\rho})$ is exact if and only if the function $f(z, \bar{z})$ has trivial moments, i. e.*

$$\int\limits_Z f(z, \bar{z}) p(z)\,\mathrm{d}z \wedge \mathrm{d}\bar{z} = 0 \,, \qquad \text{for all } p \in H^0(X, E_\lambda)|_Z \,.$$

Proof: It is evident that correspondence $p \to p\,\mathrm{d}z$ yields a local isomorphism of the groups $H^0(X, E_\lambda)$ and $H^0(X, E_{\lambda+\rho} \otimes \Omega(n))$, where $\Omega(n)$ is a fiber bundle of the holomorphic forms of higherst order. Groups $H^0(X, E_{\lambda+\rho} \otimes \Omega(n))$ and $H^n(X, E_{-\lambda-\rho})$ are dual G^c-modules. The duality is established with the help of the following coupling: $\langle \varphi, p \rangle = \int\limits_X \varphi \wedge p$. Taking into account that Z is dense in X one can rewrite the coupling in the following way: $\langle \varphi, p \rangle = \int\limits_Z f\,\mathrm{d}\bar{z} \wedge p\,\mathrm{d}z$. Due to the fact that each differential form $p(z)\,\mathrm{d}z$, where $p(z) \in \Gamma(E_\lambda)$ is closed and not exact, and the equality $\int\limits_Z f\,\mathrm{d}\bar{z} \wedge p\,\mathrm{d}z = 0$ it follows, that φ is a trivial linear functional on the space $H^0(X, E_{\lambda+\rho} \otimes \Omega(n))$, and, hence $SL(2, \mathbb{C})$ is a trivial element of the group $H^n(X, E_{-\lambda-\rho})$.

Example. If $G = SL(2, \mathbb{C})$, then $\lambda = n > 0$. Globally $\bar{\partial}$-exact differential forms in local trivialization on the subgroup Z have the form: $f(z, \bar{z})\,\mathrm{d}\bar{z}$, where function $f(z, \bar{z})$ has zero moments: $\int\limits_C f(z, \bar{z}) z^k\,\mathrm{d}z \wedge \mathrm{d}\bar{z} = 0$.

Highest Group of Harmonic Forms — Explicit Description

Let us find explicitly group of harmonic forms $\mathcal{H}^n(E_{-\lambda-\rho})$. Let "$^\sigma$" is the anti-involution on G^c that coincides with the exponented Cartan anti-involution. The following proposition holds true.

Proposition 4. *The function* $\mathrm{e}^{\chi}(zz^{\sigma})$ *is a multiplicative invariant of the group* G *with the multiplier* $|\,\mathrm{e}^{-\chi}|^2$*:*

$$\mathrm{e}^{\chi}(z^g(z^g)^{\sigma}) = |\,\mathrm{e}^{-\chi}(zg)|^2\,\mathrm{e}^{\chi}(zz^{\sigma})\,.$$

Proof: Taking into account that $z^g = b^{-1}(zg)zg$, $(z^g)^{\sigma} = g^{\sigma}z^{\sigma}(b^{-1}(zg))^{\sigma}$, $gg^{\sigma} = 1$ hence $z^g(z^g)^{\sigma} = b^{-1}(zg)zz^{\sigma}(b^{-1}(zg))^{\sigma}$.
Taking into account the properties of the function e^{χ} [11] we obtain

$$\mathrm{e}^{\chi}(z^g(z^g)^{\sigma}) = \mathrm{e}^{\chi}(b^{-1}(zg)zz^{\sigma}(b^{-1}(zg))^{\sigma}) = |\,\mathrm{e}^{-\chi}(zg)|^2\,\mathrm{e}^{\chi}(zz^{\sigma})\,.$$

Now we are ready to describe the group of harmonic forms $\mathcal{H}^n(E_{-\lambda-\rho})$.

Proposition 5. *The group* $\mathcal{H}^n(E_{-\lambda-\rho})$ *in a local trivialization based on the subgroup* Z *consists of the differential forms of the type*

$$\Phi_g = \frac{\mathrm{e}^{\lambda}(\overline{zg})\,\mathrm{d}\bar{z}}{\mathrm{e}^{\lambda+\rho}(zz^{\sigma})}\,.$$

Proof: It is evident that all differential forms of the type Φ_g are $\bar{\partial}$-closed. Let us prove that they are not exact. Indeed, according to the Proposition 4 the differential form is not exact if there exists $p_g \in H^0(X, E_{\lambda+\rho}\otimes\Omega(n))$, such that $I_g = \int p_g \wedge \Phi_g \neq 0$. Let $p_g = \mathrm{e}^{\lambda}(zg)\,\mathrm{d}z$. Then we shall have

$$I_g = \int\limits_Z \frac{|\,\mathrm{e}^{\lambda}(zg)|^2 d\bar{z}\wedge dz}{\mathrm{e}^{\lambda+\rho}(zz^{\sigma})}\,,$$

But, according to [15] $I_g = ||p_g||^2 \neq 0$.
To prove the proposition one has only to show that the differential forms of the type Φ_g form representation in the space of all forms of the order n. Indeed by the direct calculation we obtain $T_{\acute{g}}\Phi_g = \mathrm{e}^{-\lambda-\rho}(z\acute{g})\Phi_g(z^{\acute{g}})$. Taking into account that $\mathrm{d}\bar{z}^{\acute{g}} = \mathrm{e}^{-\rho}(\overline{z\acute{g}})\,\mathrm{d}\bar{z}$, $\mathrm{e}^{\lambda}(z^{\acute{g}}g) = \mathrm{e}^{-\lambda}(z\acute{g})\,\mathrm{e}^{\lambda}(z g\acute{g})$, and using previous Proposition we obtain

$$T_{\acute{g}}\Phi_g = \mathrm{e}^{-\lambda-\rho}(z\acute{g})|\,\mathrm{e}^{\lambda+\rho}(z\acute{g})|^2\,\mathrm{e}^{-\lambda-\rho}(\overline{z\acute{g}})\frac{\mathrm{e}^{\lambda}(\overline{z\acute{g}g})\,\mathrm{d}\bar{z}}{\mathrm{e}^{\lambda+\rho}(zz^{\sigma})} = \Phi_{\acute{g}g}\,.$$

That completes the proof of the Proposition.

Example. Let $G \equiv SU(3)$, then automorphism σ coincides with the hermitian transposition. The group $H^3(X, E_{(-\lambda_2-2,-\lambda_1-2)})$ contains the following forms:

$$\Phi = \frac{p(\bar{z}_{12},\bar{z}_{23},\bar{z}_{13})\prod_{i<j}\mathrm{d}\bar{z}_{ij}}{(1+|z_{12}|^2+|z_{13}|^2)^{\lambda_1+2}(1+|z_{23}|^2+|z_{23}z_{12}-z_{13}|^2)^{\lambda_2+2}}\,,$$

where the section $p(z_{12},z_{23},z_{13}) \in \Gamma(E_{\lambda})$ is defined as in the first example.

5. Dolbeaut Cohomology Groups — Integral Representation

In this section we shall use higher cohomology group $H^n(X, E_{-\lambda-\rho})$ that was constructed in the previous chapters to find the explicit form of the groups $H^k(X, E_\chi)$, $k < n$. To do this we shall essentially use the so called intertwining operators [9, 11]. The following theorem holds true.

Theorem. *Let λ be a dominant integer weight, w be the element of the Weyl group, such that $w^{-1}Nw \cap Z = Z_P$, where Z_P is a maximally nilpotent subgroup of the Levi factor G_P of some parabolic subgroup P. Let $\rho = \sum\limits_{\alpha\in\triangle^+} \lambda$, $\rho_w = \sum\limits_{\alpha\in w^{-1}\triangle^- w\cap\triangle^+} \alpha$, $\Phi = \phi\,\mathrm{d}\bar{z} \in H^n(X, E_{-\lambda-\rho})$. Then the map*

$$
\begin{aligned}
A_w &\colon H^n(X, E_{-\lambda-\rho}) \to H^{n-l(w)}(X, E_{-w(\lambda+\rho)-\rho_w});\\
A_w\Phi &= \int\limits_{Z_P} \mathrm{e}^{\lambda+\rho-\rho_w}(yw)\phi(yz)\,\mathrm{d}y \wedge \mathrm{d}\overline{yz}
\end{aligned}
$$

gives the local isomorphism between the representations of the group G^c in the corresponding cohomology groups.

Proof: Intertwining properties of the operators A_w follow from the intertwining properties of the standard intertwining operators [10, 9]. Let us prove that the constructed forms are $\bar{\partial}$-closed. With this purpose we shall rewrite them in the following form:

$$
A_w \cdot \Phi = \int\limits_{Z_P} \mathrm{e}^{\lambda+\rho-\rho_w}(yw)\phi((yz_P)z(p))\,\mathrm{d}y \wedge \mathrm{d}\overline{yz}_P \wedge \mathrm{d}\bar{z}(p)\,,
$$

where $z = z_p z(p)$ is the decomposition of the element z, that corresponds to group decomposition: $Z = Z_P Z(P)$. Making the replacement of variables $x = yz_P$ and taking into account Z_P-invariance of the measure $\mathrm{d}yz_P = \mathrm{d}x$ we obtain:

$$
A_w \cdot \Phi = \Big(\int\limits_{Z_P} \mathrm{e}^{\lambda+\rho-\rho_w}(x(z_p)^{-1}w)\phi(xz(p)\Big)\,\mathrm{d}x \wedge \mathrm{d}\bar{x}) \wedge \mathrm{d}\bar{z}(p)\,.
$$

From the last expression it became obvious that dependence of the coefficient function

$$
\phi(z,\bar{z}) = \int\limits_{Z_P} \mathrm{e}^{\lambda+\rho-\rho_w}(x(z_p^{-1})w)\phi(xz(p))\,\mathrm{d}x \wedge \mathrm{d}\bar{x}
$$

on the parameters of the group Z_P is purely holomorphic. That's why we obtain:

$$
\partial_{\bar{z}}\Phi = (\partial_{\bar{z}}\phi) \wedge \bar{z}(p) = (\partial_{\bar{z}_P}\phi) \wedge \mathrm{d}\bar{z}(p) = 0\,.
$$

Now to prove the theorem it is necessary only to show that constructed forms are not exact. It follows from the fact that their restriction to the flag manifold $X_P = G^c/P$ are not exact forms. Indeed to prove that differential form Φ is not exact it is necessary to show that its convolution with the section $s \in H^0(X_P, E_\lambda|_{X_P})$ is not equal to zero:

$$\int\limits_{Z(P)} f(z(p), \bar{z}(p)) s(z(p)) \, \mathrm{d}z(p) \wedge \mathrm{d}\bar{z}(p) \neq 0 \, .$$

Taking into account explicit form of $A_w \Phi$ we shall have to prove that

$$\int\limits_{Z} s(z(p)) \, \mathrm{e}^{\lambda+\rho-\rho_w}(z_P w) \phi(z) \, \mathrm{d}\bar{z} \wedge \mathrm{d}z \neq 0 \, .$$

But this follows from the fact that $s(z(p)) \, \mathrm{e}^{\lambda+\rho-\rho_w}(z_P w) \in H^0(X, E_\lambda)$ and the differential form $\Phi \in H^n(X, E_{-\lambda-\rho})$ is not exact by the condition of the theorem. That completes the proof of the theorem.

Remark 1. As it follows from the proof of the theorem the integral form of the representatives of the cohomology classes could be written as follows:

$$\Psi = A_w \Phi = \Big(\int\limits_{Z_P} \mathrm{e}^{\lambda+\rho-\rho_w}(yw) \phi(yz) \, \mathrm{d}y \wedge \mathrm{d}\bar{y} \Big) \, \mathrm{d}\bar{z}(p) \, .$$

Here $\mathrm{d}\bar{z}(p)$ is antiholomorphic part of the quasi-invariant measure on the subgroup $Z(P) = Z_P \backslash Z$.

Harmonic Forms — Integral Representation

Let the higher order forms, used in the theorem to construct representatives of the cohomology classes $H^{n-l(w)}(X, E_{-w(\lambda+\rho)-\rho_w})$ belong to the group $\mathcal{H}^n(E_{-\lambda-\rho})$. Then representatives of the cohomology classes $H^{n-l(w)}(X, E_{-w(\lambda+\rho)-\rho_w})$ will transform according to the subrepresentation of the group G. But only harmonic representatives of the group $H^{n-l(w)}(X, E_{-w(\lambda+\rho)-\rho_w})$ form subrepresentation of the group G. Hence, we obtain the following important corollary:

Corollary. *Let G be a compact Lie group, λ be a dominant integer weight, e^λ be a character of the maximal torus extended to the function on G^c : $\mathrm{e}^\lambda(g) = \mathrm{e}^\lambda(h(g))$. Here $g = n(h)h(g)z(g)$ is a Hauss decomposition of the element g. Let w be such an element of the Weyl group, that $w^{-1}Nw = Z_P$, where Z_P is a maximal nilpotent subgroup of the Levi factor G_P of the parabolic subgroup P. Then the group of harmonic forms $\mathcal{H}^{n-l(w)}(E_{-w(\lambda+\rho)} - \rho_w)$ in a local trivialization on the subgroup Z consists of the following differential forms:*

$$\Phi = \int\limits_{Z_P} \mathrm{e}^{\lambda+\rho-\rho_w}(yw)\frac{\bar{p}(yz)}{\mathrm{e}^{\lambda+\rho}(yzz^\sigma y^\sigma)}\,\mathrm{d}y \wedge \mathrm{d}\overline{yz}$$

where $z \in Z$, $y \in Z_P$, $p(z) \in H^0(X, E_\lambda)$.

Acknowledgements

The research described in this publication was possible in part by the Award number UP1-309 of the U.S. Civilian Research and Development Foundation (CRDF) for independent states of the former Soviet Union.

References

[1] Borel A., Weil A., *Representations linearies et spaces homogeneous Kählerians des groups de Lie compacts*, seminar Bourbaki, May 1954.

[2] Bott R., *Homogeneous Vector Bundles*, Ann. Math. **66** (1957) 203–248.

[3] Kostant B., *Quantization and Unitary Representations*, Lecture Notes in Mathematics **170** (1970) 87–208.

[4] Souriau J.-M., *Structure des systemes dynamiques*, Dunod, Paris 1970.

[5] Sniatycki J., *On Cohomology Groups Appearing in Geometric Quantization*, Calgari 1975 (preprint).

[6] Rawnsley J. H., *Polarization and Geometric Quantization*, Transaction of the AMS **230** (1977) 235–255.

[7] Hurt N., *Geometric Quantization in Action*, D. Reidel Publishing Company, London 1983.

[8] Schiffmann G., *Intégrales d'entrelacement*, C. R. Acad. Sci. Paris **266** (1968) 47–49 and 859–861.

[9] Zhelobenko D., *Symmetry in the Class of Elementary Representations*, Func. an. i jego pril. **1** (1967) 15–39.

[10] Knapp A. W., Stein E. M., *Intertwining Operators for Semisimple Groups*, Ann. Math. **93** (1971) 489–578.

[11] Kunze R., Stein E. M., *Uniformly Bounded Representations III*, Amer. J. Math. **89** (1967) 385–442.

[12] Kodaira K., *On a Cohomology Groups of a Compact Analytic Varieties with the Coeficients in Some Faisceaux*, Proc. Nat. Acad. Sci. **31** (1953) 865–868.

[13] Rawnsley J., Shmidt W., Wolf J., *Singular Unitary Representations and Indefinite Harmonic Theory*, J. Functional Analisys **51** (1983) 1-114.

[14] Shmidt W., Wolf J., *Geometric Quantization and Derived Functor Modules for Semisimple Lie Groups*, J. Functional Analisys **90** (1990) 48–113.

[15] Holod P., Skrypnyk T., *Explicit Realization of the Irreducible Unitary Representations of Compact Lie Groups in the Space of Sections of Linear Fiber Bundle*, Ukrainian Mathematical Journal **50** (1998) 1316–1324.

[16] Warner G., *Harmonic Analysis on Semisimple Lie Groups*, Springer-Verlag, Berlin, Heidelberg, New York 1972.