XVIII. More on Proper Forcing

§0. Introduction

From the last eight chapters you may have gotten the impression that we are done with properness, but this is not so. First, we turn to the problem of not adding reals; remember that by V §7, VIII §4, for CS iterations of proper forcing notions not adding reals, the limit does not add reals, provided that two additional conditions hold: one is $\mathbb{D}$-completeness (for, say, a simple 2-completeness system) and the second is $(<\omega_1)$-properness (see V §2). Now, the first restriction is justified by the weak diamond (see V 5.1, 5.1A and AP §1); that is not to say that we have to demand exactly $\mathbb{D}$-completeness, but certainly we have to demand something in this direction. However, there was nothing there to justify the second demand: α-proper for every α. In the first section, (following [Sh:177]), we show that we cannot just omit it, even if we use an $\aleph_1$-completeness system. It is natural to hope that this counterexample will lead to a principle like the weak diamond (so provable from CH). Thus the construction of this counterexample leads to questions like: Assuming CH, can we find $\langle C_\delta : \delta < \omega_1\rangle$, C_δ an unbounded subset of δ, say of order type ω, such that for every club E of ω_1, for stationarily many limit $\delta < \omega_1$, $C_\delta \subseteq E$ or $\delta = \sup(C_\delta \cap E)$ or $(\forall \alpha \in \delta)[\min(E \setminus \alpha) < \min(C_\delta \setminus (\alpha + 1)]$? (They are kin to "the guessing clubs", the existence of which for, e.g. $\aleph_2$, follows from ZFC, see [Sh:g].) It interests us as the theorems (and proofs) from V, VIII §4 do not give us the consistency of their negation with CH. However, those statements do not follow from ZFC + CH; for this we prove in the second section a preservation theorem for CS iterations of proper forcing not adding reals. Again we have two conditions (called there $(4)_2$ or $(4)_{\aleph_0}$ and (5)). The first, $(4)_2$, is done "against" the weak diamond, and is weaker than the older $\mathbb{D}$-completeness, but this is just a side gain. The second condition says our forcing remains proper even if we force with "forcing notion from our family, in particular not adding reals". Note that for forcing notions of cardinality $\aleph_1$, this is a very mild condition. So, the results of §1 remain the only restrictions on theorems on preservation by CS iteration, and there is a gap between them and the results of §2.

Then, in the third section we turn to other preservation theorems, giving an alternative to the theorem from VI §1 – §3, and dealing with some examples. (For a simplification of possibility A in 3.3, see [Go]).

Finally, in the fourth section we turn to the problem of a unique P-point. In VI §4 we have proved that there may be no P-point; remember, a P-point F is a nonprincipal ultrafilter on ω such that if $A_n \in F$ for $n < \omega$ then for some $A \in F$ we have $\bigwedge_n A \subseteq_{\mathrm{ae}} A_n$ ($A \subseteq_{\mathrm{ae}} A_n$ means $A \setminus A_n$ is finite). Now, to prove the consistency of "there is a unique object" is typically harder then proving there is no one. Unique here means up to permutations of ω. In VI §5 we have proved a weaker result: there may be a unique Ramsey ultrafilter, but there could have been many P-points above it which were not isomorphic. We continue this and prove the consistency of "there is a unique P-point".

§1. No New Reals: A Counterexample and New Questions

1.1 Lemma. Suppose V satisfies $2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2$, and for some $A \subseteq \omega_1$, every $B \subseteq \omega_1$ belongs to $L[A]$ and for limit $\delta < \omega_1$,

$$L_\delta[A \cap \delta] \models \text{"}\delta \text{ is countable"}.$$

Then we can define a countable support iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < i^* \rangle$ such that the following conditions hold:

(a) Each $\underset{\sim}{Q}_i$ is proper and $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ has power $\aleph_1$".

(b) Each $\underset{\sim}{Q}_i$ is $\mathbb{D}$-complete for some simple $\aleph_1$-completeness system $\mathbb{D}$ (hence does not add reals).

(c) Forcing with $P_{i^*} = \mathrm{Lim}\bar{Q}$ adds reals.

Proof. We shall define $\underset{\sim}{Q}_i$ by induction on $i < i^*$, $i^* \leq \omega^2$, so that conditions (a) and (b) are satisfied, and $\underset{\sim}{C}_i$ is a $\underset{\sim}{Q}_i$-name of a closed unbounded subset of ω_1. Let $\langle f^*_\xi : \xi < \omega_1 \rangle \in L[A]$ be a list of all functions f which are from δ to δ for some limit $\delta < \omega_1$, and let $h : \omega_1 \to \omega_1$, $h \in L[A]$, be defined by $h(\alpha) = \mathrm{Min}\{\beta : \beta > \alpha \text{ and } L_\beta[A \cap \alpha] \models \text{"}|\alpha| = \aleph_0\text{"}\}$.

Suppose we have defined $\underset{\sim}{Q}_j$ for every $j < i$; then P_i is defined, is proper (as each $\underset{\sim}{Q}_j$, $j < i$, is proper, and III 3.2) and has a dense subset of power $\aleph_1$ (by III 4.1). Let $G_i \subseteq P_i$ be generic, so, clearly, there is a $B_i \subseteq \omega_1$ such that in $V[G_i]$, every subset of ω_1 belongs to $L[A, B_i]$. The following now follows:

1.1A Fact. In $V[G_i]$, every countable $N \prec (H(\aleph_2), \in, A, B_i)$ is isomorphic to $L_\beta[A \cap \delta, B_i \cap \delta]$ for some $\beta < h(\delta)$, where $\delta = \delta(N) \stackrel{\text{def}}{=} \omega_1 \cap N$.

We shall assume also that $V[G_i]$ has the same reals as V (otherwise we already have an example).

We now define by induction on $\alpha < \omega_1$, a set $T_\alpha = T^i_\alpha$ such that the following conditions are satisfied:

(i) Each $f \in T_\alpha$ is the characteristic function of a closed subset of some successor ordinal $\beta < \alpha$, i.e., $\mathrm{Dom}(f) = \beta$, and $f^{-1}(\{1\})$ is a closed subset of β and is included in the set of accumulation points of $\bigcap_{j<i} C_j$. If $\gamma < \alpha$, then $T_\gamma \subseteq T_\alpha$.

(ii) If $f \in T_\alpha, \gamma + 1 \leq \mathrm{Dom}(f)$, then $f{\restriction}(\gamma+1) \in T_\alpha$, and even $f{\restriction}(\gamma+1) \in T_\beta$ for $\gamma + 1 < \beta \leq \alpha$.

(iii) If $f \in T_\alpha$, $\mathrm{Dom}(f) = \beta$, $\beta < \gamma < \alpha$, γ a successor, then $f' = f \cup 0_{[\beta,\gamma)} \in T_\alpha$, i.e., $\mathrm{Dom}(f') = \gamma$, and

$$f'(\xi) = \begin{cases} f(\xi), & \textit{if } i < \beta, \\ 0, & \textit{if } \beta \leq \xi < \gamma. \end{cases}$$

(iv) If $f, g \in T_\alpha$, $f(\beta) \neq g(\beta)$, then $f^{-1}(\{1\}) \cap g^{-1}(\{1\}) \setminus \beta$ is finite.

(v) If $f \in T_\alpha$, $\gamma \geq \beta = \mathrm{Dom}(f)$, $\gamma + 1 < \alpha$, γ is an accumulation point of $\bigcap_{j<i} C_j$ and the order type of $f^{-1}(\{1\})$ has the form $\xi + 2$ or is > 0 and $< \omega$, then $f' = f \cup 0_{[\beta,\gamma)} \cup \{\langle \gamma, 1 \rangle\} \in T_\alpha$.

(vi) If $f \in T_\alpha$, $\delta + 1 = \mathrm{Dom}(f)$, δ limit, and $f(\beta) = 1$ for arbitrarily large $\beta < \delta$, then $\mathrm{Min}\{\xi : f{\restriction}\delta = f^*_\xi\}$ is larger than $\mathrm{Min}\{\xi : \delta \leq \xi \in C_j\}$ (for $j < i$).

(vii) If $\delta + 1 < \alpha$, δ is an accumulation point of $\bigcap_{j<i} C_j$, $\xi^* < \omega_1$, and $f \in T_\delta \cap L_\delta[A \cap \delta]$, *then* there is a $g \in T_\alpha$, $\delta + 1 = \mathrm{Dom}(g)$, such that for every $\mathcal{J} \in L_{h(\delta)}[A \cap \delta, B_i \cap \delta]$ (an open dense subset of $T_\delta \cap L_\delta[A \cap \delta]$ (ordered by inclusion)), for some $\gamma < \delta$ we have $g{\restriction}\gamma \in \mathcal{J}$ and $g{\restriction}\delta \notin \{f^*_\xi : \xi < \xi^*\}$ and $f = g{\restriction}\mathrm{Dom}(f)$.

(viii) For $f \in T_\alpha$, if $\delta = \sup(\delta \cap f^{-1}(\{1\}))$ (hence $f(\delta) = 1$), $\delta < \beta$, and $f(\beta) = 1$, then for every $j < i$, for some $\gamma < \beta$, the characteristic function of C_j restricted to δ is f^*_γ; and if δ, $f{\restriction}\delta$ and β satisfy this then $f{\restriction}(\delta+1) \cup 0_{[\delta+1,\beta)} \cup 1_{[\beta,\beta+1)}$ belongs to $T_{\beta+2}$.

Let us carry the induction.

Case A. α is limit, or $\alpha = \gamma + 1$, γ limit. Let $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$ or $T_\alpha = \bigcup_{\beta<\gamma} T_\beta$.

Case B. $\alpha < \omega$. Let $T_\alpha = \{f : f$ a function from $\beta < \alpha$ to $\{0\}\}$.

Case C. $\alpha = \beta + 3 > \omega$. Let $T_\alpha = T_{\beta+2} \cup \{f : \mathrm{Dom}(f) = \beta + 2, f{\restriction}(\beta+1) \in T_{\beta+2}$, provided that (viii) is satisfied $\}$.

Case D. $\alpha = \delta + 2$, δ limit, $\delta \in \mathrm{acc}\left(\bigcap_{j<i} C_j\right)$ (acc- denotes the set of accumulation points). This is the main case. Let $\{f^\delta_\ell : \ell < \omega\}$ be a list of $T_\delta \cap L_\delta[A \cap \delta]$, each appearing $\aleph_0$ times, and $\{\mathcal{J}^{\delta,i}_\ell : \ell < \omega\}$ be a list of all open dense subsets $\mathcal{J}$ of $(T_\delta \cap L_\delta[A \cap \delta], \subseteq)$ which satisfy: $\mathcal{J}$ belongs to

$L_{h(\delta)}[A \cap \delta, B_i \cap \delta]$ or $\mathcal{J} = \{f \in T_\delta \cap L_\delta[A \cap \delta] : f \not\subset f^*_\xi\}$ for some $\xi < h(\delta)$. We now define by induction on $n < \omega$, an ordinal $\beta_n = \beta_n^{\delta,\alpha} < \delta$ and a finite set $F_n = F_n^{\delta,\alpha} \subseteq \{f \in T_\delta \cap L_\delta[A \cap \delta] : \beta_n = \mathrm{Dom}(f)\}$ such that

$$(*) \qquad (\forall f \in F_n)(\exists g \in F_{n+1})(f \subseteq g) \text{ and}$$

$$\text{if } n \geq 1,\ (\forall f, g \in F_n)\left(f{\restriction}\beta_{n-1} \neq g{\restriction}\beta_{n-1} \Rightarrow f^{-1}(\{1\}) \cap g^{-1}(\{1\}) \subseteq \beta_{n-1}\right).$$

Subcase α. If $n = 0 \bmod 3$, then $\beta_{n+1} = \beta_n + 1$ and $F_{n+1} = \{f \cup \{\langle \beta_n, 0\rangle\} : f \in F_n\}$; and if $n = 0$, then $F_n = \emptyset$ and $\beta_n = 0$.

Subcase β. If $n = 1 \bmod 3$, then $\beta_{n+1} = \beta_n + 1$; let $f'_n = f^\delta_{(n-1)/3}$ and $\beta^*_n = \mathrm{Dom}(f^\delta_{(n-1)/3})$ if $\mathrm{Dom}(f^\delta_{(n-1)/3})$ is $< \beta_n$, and let $f'_n = \emptyset, \beta^*_n = 0$ otherwise; now let

$$F_{n+1} = \{f \cup 0_{[\beta_n, \beta_{n+1})} : f \in F_n\} \cup \{f'_n \cup 0_{[\beta^*_n, \beta_{n+1})}\}.$$

Subcase γ. If $n = 2 \bmod 3$, $(n-2)/3 = m^2 + k$, $k \leq 2m$, then every $f \in F_{n+1}$ belongs to $\mathcal{J}_k = \mathcal{J}_k^{\delta,i}$. Note that we have to take care to satisfy $(*)$; hence let[†] $F_n = \{f^n_\ell : \ell < |F_n|\}$, and define β^n_ℓ for $\ell \leq |F_n|$ and g^n_ℓ for $\ell < |F_n|$ by induction on ℓ : $\beta^n_0 = \beta_n$; if β^n_ℓ is defined, choose g^n_ℓ, $f^n_\ell \cup 0_{[\beta_n, \beta^n_\ell]} \subseteq g^n_\ell \in \mathcal{J}_k$, and $\beta^n_{\ell+1} = \mathrm{Dom}(g^n_\ell)$. Now let

$$\beta_{n+1} = \beta^n_{|F_n|} \text{ and } F_{n+1} = \{g^n_\ell \cup 0_{[\beta^n_{\ell+1}, \beta_{n+1})} : \ell < |F_n|\}.$$

Note that only in Subcase γ, do we have a free choice, and we eliminate it by choosing the first candidate for F_{n+1} by the canonical well-ordering of $L[A, B_i]$, and we also require that $\langle \mathcal{J}_\ell^{\delta,i} : \ell < \omega\rangle$ be the first such sequence in the canonical well ordering of $L_\delta[A \cap \delta, B_i \cap \delta]$. So we have finished defining the F_n's and we let

$$\begin{aligned} T_{\delta+2} = T_\delta \cup \big\{ f : &\mathrm{Dom}(f) = \delta + 1 \text{ and : either } f = f' \cup 0_{[\gamma, \delta+1)}, \\ &\text{where } f' \in T_\delta,\ \gamma = \mathrm{Dom}(f') \textit{ or } \text{for some } k < \omega, \\ &(\forall n > k)[f{\restriction}\beta_n \in F_n] \text{ and} \\ &f(\delta) = 1 \Leftrightarrow \delta = \sup f^{-1}(\{1\})\big\}. \end{aligned}$$

† Of course, we suppress the dependency of $\beta_n, f^n_\ell, \alpha^n_\ell, g^n_\ell$ on δ and i.

It is easy to check that that $T_{\delta+2}$ is as required. (Case β in the definition of F_n enables us to satisfy demand (vii)).

Case E. $\alpha = \delta + 2$, δ limit, $\delta \notin \operatorname{acc}\left(\bigcap_{j<i} C_j\right)$. Let $T_\alpha = T_\delta \cup \{f : \operatorname{Dom}(f) = \delta + 1, (\exists g \in T_\delta)[g \subseteq f \;\&\; f{\restriction}((\delta+1) \setminus \operatorname{Dom}(g)) \text{ is zero }]\}$.

So we have defined $T_\alpha = T^i_\alpha$ for $\alpha < \omega_1$, and let $Q_i \in V[G_i]$ be $\bigcup_{\alpha<\omega_1} T^i_\alpha$ ordered by inclusion; and it is easy to see that $\underset{\sim}{Q}_i$ is as required (in (a) and (b) of 1.1). Let $\underset{\sim}{C}_i = \bigcup\{f^{-1}(\{1\}) : f \in \underset{\sim}{G}_{Q_i}\}$, so $\Vdash_{Q_i}$ "$\underset{\sim}{C}_i$ is a club of ω_1".

So $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \omega^2\rangle$ is defined, and it is easy to see that we can replace (in $V[G_i]$) B_i by $\bar{C}^i = \langle C_j : j < i\rangle$. Let $G \subseteq P_{\omega^2}$ be generic, and C_i the interpretation of $\underset{\sim}{C}_i$. Let f_i be the characteristic function of C_i, and $C \overset{\text{def}}{=} \bigcap_{i<\omega^2} C_i$, and $\{\alpha_\zeta : \zeta < \omega_1\}$ an enumeration of C (in increasing order). We shall suppose that forcing by P_{ω^2} does not add reals, and shall deduce that $\langle f_i : i < \omega^2\rangle \in V$, which is clearly false, as $\Vdash_{Q_0}$ "$C_0 \notin V$".

By the assumption the sequence $\langle f_i{\restriction}\alpha_0 : i < \omega^2\rangle$ belongs to V, and we shall show how to compute $\langle f_i{\restriction}\alpha_\zeta : i < \omega^2\rangle$ for every ζ, by induction on ζ; as the computation is done in V we get the desired contradiction. More formally, there is a function F in V such that

$$\langle f_i{\restriction}\alpha_{\zeta+1} : i < \omega^2\rangle = F\big[\langle f_i{\restriction}\alpha_\zeta : i < \omega^2\rangle\big].$$

So suppose $\langle f_i{\restriction}\alpha_\zeta : i < \omega^2\rangle$ is given, and let, for $i < \omega^2$:

$$\beta_i \overset{\text{def}}{=} \operatorname{Min}\,(C_i \setminus (\alpha_\zeta + 1)), \quad \xi_i \overset{\text{def}}{=} \operatorname{Min}\{\xi : f_i{\restriction}\alpha_\zeta = f^*_\xi\}.$$

By demand (i) in the definition of the T^i_α's $C_i \subseteq \operatorname{acc}\left(\bigcap_{j<i} C_j\right)$. So clearly $\beta_j < \beta_i$, for $j < i$ and $\beta_i \in C_j$ for $j \le i$. Also by demand (vi) on the T^i_α's, $\beta_j < \xi_i$ for $j < i$, and by demand (viii) on the T^i_α's $\xi_j < \beta_i$ for $j < i$. We can conclude that $\operatorname{Sup}\{\beta_i : i < \omega n\} = \operatorname{Sup}\{\xi_i : i < \omega n\}$ for all $n \in \omega$; but from $\langle f_i{\restriction}\alpha_\zeta : i < \omega^2\rangle$ we can compute $\gamma_n \overset{\text{def}}{=} \operatorname{Sup}\{\xi_i : i < \omega n\}$. As $\beta_i \in C_j$ for $j < i$, $\gamma_n \in C_j$ when $j < \omega n$, and clearly $\gamma_n < \gamma_{n+1}$, so we have $\gamma \overset{\text{def}}{=} \bigcup_{n<\omega} \gamma_n \in \bigcap_{j<\omega^2} C_j$. By the definition of the α_ζ's, $\gamma = \alpha_{\zeta+1}$. As we know $T^0_\gamma \cap L_\delta[A]$, and we know $\{\gamma_n : n < \omega\} \subseteq C_0$; $f_0{\restriction}\gamma$ is uniquely determined (by demand (iv)). Similarly we continue to reconstruct $f_i\restriction\gamma$ by induction on $i<\omega^2$ (see end of Case D in the construction - the canonical choice), thus finishing the proof. $\square_{1.1}$

1.2 Remark. The ω^2 in 1.1. is best possible.

1.3 Lemma. (1) Fixing $\langle f^*_\alpha : \alpha<\omega_1\rangle$, a list of $F=\{f : f : (\beta+1)\to\{0,1\}, f^{-1}(\{1\})$, closed, $\beta<\omega_1 (f\in V$, of course) $\}$ (so we are assuming CH) and $h:\omega_1\to\omega_1$, we can repeat the construction in the proof of 1.1 (omitting the assumption on A), and its conclusion holds provided that

$(*)_1$ (α) if χ is large enough, $T\subseteq\{f^*_\alpha\restriction\gamma\colon \gamma<\omega_1,\alpha<\omega_1\}$, $T\in N\prec(H(\chi),\in,<^*_\chi)$, N countable, $\mathcal{J}\in N$ a dense subset of $(T,\subseteq)$, *then* $\mathcal{J}\cap N\in\{\mathcal{J}^\delta_\ell : \ell<h(\delta)\}$ where $\delta=\omega_1\cap N$, and $\{J^\delta_\ell : \ell<\omega_1\}$ is a list of all subsets of $\{\gamma^*_\alpha\restriction\gamma : \gamma,\alpha<\delta\}$.

(β) moreover, after CS iteration of length $i<\omega^2$ of forcing notions of this form $((T,\subseteq))$, giving generic sets $G_j (j<i)$, (α) continues to hold with $\{\mathcal{J}^\delta_\ell : \ell<h(\delta)\}$ replaced with the family of subsets of $\{f^*_\alpha\restriction\gamma : \gamma,\alpha<\delta\}$ definable in $(\delta\cup\{f^*_\alpha\restriction\gamma : \gamma,\alpha<\delta\}\cup\{\langle\gamma,\alpha,\mathcal{J}^\gamma_\alpha\rangle\colon \gamma\le\delta,\alpha\le h(\gamma)\}, G_j)_{j<i}$,

or, at least

$(*)_2$ for each $\delta<\omega_1$, we have $\bar g^\delta=\langle g^\delta_\eta : \eta$ a sequence of successor length $<\omega^2$, each $\eta(i)$ is in ${}^\omega 2\rangle$ such that:

(α) $g^\delta_{\eta(i)} : \delta\to\{0,1\}, (g^\delta_{\eta(i)})^{-1}(\{1\})$ is an unbounded subset of δ (and if η,ν have length $i+1$, $\eta\restriction i\ne\nu\restriction i$, then $(g^\delta_{\eta(i)})^{-1}(\{1\})\cap(g^\delta_{\nu(i)})^{-1}(\{1\})$ is bounded in δ).

(β) Suppose $i<\omega^2, \chi$ large enough, $N\prec(H(\chi),\in,<^*_\chi)$, N countable, $\delta=N\cap\omega_1, \bar Q^i=\langle P_j,\underset{\sim}{Q}_j : j\le i\rangle\in N$ defined as in 1.1, is a CS iteration, each $\underset{\sim}{Q}_j$ proper satisfying (i) - (viii) from the proof of 1.1 (with (vii) rephrased in terms of $\{J^\delta_\ell : \ell<\kappa(\delta)\}$), P_i adds no reals, and $\langle g^\delta_{\eta(j)} : j<i\rangle$ is generic for $P_i\cap N$, then for at least one $\nu\in{}^\omega 2, g^\delta_{\eta\frown\nu}$ is $(N[g^\delta_{\eta(j)} : j<i], Q_i)$-generic.

(2) If Q^* is adding $\aleph_1$ Cohen reals, and $V\models$CH then $V^{Q^*}\models(*)_2$.

Proof of 1.3 (1) Same proof as 1.1.

(2) Left to the reader. Note: let $Q=\{f : f$ a finite function from ω_1 to $\{0,1\}\}$,

let f^* be the generic function; now in $V[f^*]$, if

$$N \prec (H(\chi)^{V[f^*]}, \in, V \cap H(\chi), f^*, <^*_\chi) \text{ countable,}$$

then $N \in V[f^*\restriction(N \cap \omega_1)]$. So for δ, defining $\bar{g}^\delta$, we have to consider only $N \in V[f^*\restriction\delta]$, so $f^*\restriction[\delta,\omega_1)$ is "free" to be used for defining $\bar{g}^\delta$.

1.4 Lemma. (1) We could weaken the demands on V (in 1.1) to $V \models$CH, provided that we also waive the requirement $\Vdash_{P_i}$ "$|\underset{\sim}{Q}_i| = \aleph_1$".

(2) Assume CH and

$(*)_3$ there are $\bar{C} = \langle C_\delta : \delta < \omega_1 \text{ limit}\rangle$ and $h : \omega \to \omega$ such that:

(α) C_δ is an unbounded subset of δ of order type ω

(β) for every club E of $\omega_1, S_h(E,\bar{C}) \overset{\text{def}}{=} \{\delta < \omega_1 : C_\delta \cap E$ is unbounded in δ, and, moreover, for arbitrarily large $\alpha \in C_\delta, |C_\delta \cap \text{Min}(C_\delta \cap E \setminus \alpha)| \geq h(|\alpha \cap C_\delta|)\}$ contains a club of ω_1†,

(γ) h diverges to infinity.

Then the conclusion of 1.1 holds except that we weaken condition (a) to: Q_i satisfies the $\aleph_2$-pic and is proper.

(3) Assume CH (for clarity). There is a forcing notion Q, $|Q| = 2^{\aleph_1}$, Q is $\aleph_1$-complete satisfying $\aleph_2$-pic, and $\Vdash_Q$ "$(*)_3$"

Proof of 1.4(1): By (2) and (3).

Proof of 1.4(2): The proof is similar to the proof of 1.1. The main difference is that defining $T^i_{\delta+2}$ for $\delta \in \text{acc}\left(\bigcap_{j<i} C_j\right)$, we do not choose the members f of $T_{\delta+1}$ such that $\delta = \sup f^{-1}(\{1\})$ by "inverse limit construction" i.e., by constructing the F_n's, but by induction on $\zeta < \omega_1$. W.l.o.g. h is non decreasing Choose $\langle h_i : i < \omega_1\rangle$, $h_i : \omega \to \omega$ diverging to ∞, non decreasing, $[i < j \Rightarrow$ for every k large enough, $h_j(k) < h_i(k) < h(k)]$ and $[i, \omega_1, k < \omega \Rightarrow h_{i+1}(\ell)/h_i(\ell+k)$ goes to infinity]; why can we? choose h_i by induction on $i < \omega_1$, for each i we diagonalize. Defining Q_i, we shall assume that in

† So if $\bar{C} \in N \prec (H(\chi), E, <^*_\chi)$, $\delta = N \cap \omega_1 < \omega_1$, and $\delta \in S_h(E,\bar{C})$, *then*: if $E \in N$ is a club of ω_1 then $C_\delta \cap E$ is unbounded below δ.

$V[G_{P_i}], (\bar{C}, h_i)$ still exemplify $(*)_3$. So, for limit i, we have to repeat the proof of preservation of properness and preserve $(*)_3$.

We now define the Q_i's. First, we define Q_i^0: initiating the construction in the proof of 1.1, in Case D we have to change somewhat (to guarantee that forcing with Q_i preserves "$\bar{C}$ exemplifies $(*)_3$"). We choose by induction on $\zeta < \omega_1$, a function $f_\zeta^{\delta,i} : \delta \to \delta$ such that: (letting $C_\delta = \{\beta_n^\delta : n < \omega\}$, increasing in n)

(α) for each $\gamma < \delta$ we have

$$f_\zeta^{\delta,i} \restriction \gamma \in T_\delta^i \cap \{f_\xi^* \restriction \gamma : \xi < \delta\}.$$

(β) The set $Y_\zeta^{\delta,i} = \{n : f_\zeta^{\delta,i} \restriction [\beta_n^\delta, \beta_{n+1}^\delta) \neq 0_{[\beta_n^\delta, \beta_{n+1}^\delta)}\}$ satisfies: for n large enough if $n < m$ are successive members of $Y_\zeta^{\delta,i}$ then $n + h_{2i}(n) < m < n + h_{2i+1}(n)$.

(γ) if $\xi < \zeta$ then $Y_\zeta^{\delta,i} \cap Y_\xi^{\delta,i}$ is finite.

(δ) if $\langle \mathcal{J}_\ell^* : \ell < \omega \rangle$ is a list of dense subsets of $T_\delta^i \cap \{f_\xi^* \restriction \gamma : \xi < \delta, \gamma < \delta\}$, each satisfying $\otimes_{\mathcal{J}_\ell^*}$ (see below), then for some ζ, for every $n \in Y_\zeta^{\delta,i}$: if $m \leq n$ and there is g, $f_\zeta^{\delta,i} \restriction \beta_m^\delta \subseteq g \in T_\delta^i \cap \{f_\xi^* \restriction \gamma : \xi, \gamma < \delta\}$, $g \in \bigcap_{\ell < n} \mathcal{J}_\ell^*$, then $f_\zeta^{\delta,i}$ satisfies this for such maximal $m (\leq n)$:

$$|(f_\xi^{\delta,i})^{-1}(\{1\}) \cap (\beta_{n+1}^\delta \setminus \beta_n^\delta)|.$$

Where

$\otimes_{\mathcal{J}}$ if $f \in T_\delta^i \cap \{f_\xi^* \restriction \gamma : \xi < \delta, \gamma < \delta\}$ and

$$\begin{aligned} A_{f,\mathcal{J}} = \{\alpha \in C_\delta : \ & \text{if } f \subseteq f' \in T_\delta^i \cap \{f_\xi^* \restriction \gamma : \xi < \alpha, \gamma < \alpha\} \text{ then for} \\ & \text{some } f'' \in \mathcal{J} \text{ we have } f' \subseteq f'' \in T_\delta^i \cap \{f_\xi^* \restriction \gamma : \xi < \alpha, \gamma < \delta\} \end{aligned}$$

then for infinitely many $\alpha \in C_\delta$ we have

$$|\{\beta \in C_\delta : \alpha \leq \beta \text{ and } (\alpha, \beta) \subseteq A_{f,\mathcal{J}}\}| \geq h(|C_\delta \cap n|)$$

How? First choose inductively m_i such that: $m_i + i + 1 < m_{i+1} < \omega$, and for i large enough $m_i + i + 1 + h_{2i}(m_i + i + 1) < m_{i+1}$, $m_{i+1} + i + 2 < m_i + h_{2i+1}(m_i)$ (this is possible as $k < \omega \Rightarrow \langle h_{2i+1}(m)/h_{2i}(m + k) : m < \omega\rangle$ goes to infinity). Second list $\{j : j < i\}$ as $\langle j_k : k < \omega\rangle$, and diagonally choose $Y^{\delta,i}_\zeta \cap [m_i, m_i + i + 1)$, a singleton. Now, for $\alpha \in Y^{\delta,i}_\zeta$ we deal with the $\mathcal{J}^*_{g_\zeta(i)}$ where: for each ℓ, for some k no two successive members of $g^{-1}_\zeta\{\ell\}$ are of difference $> k$.

Now, Q^0_i is defined analogously to the Case D in the proof of Lemma 1.1. Then Q_i is the result of CS iteration starting with Q^0_i, and continuing with shooting club through $S_{h_{i+1}}[E, \bar{C}]$ for every club $E \subseteq \omega_1$, (by initial segments).

(3) Q is forcing $\bar{C}$ by initial segments and then CS iteration (of length $2^{\aleph_1}$) of shooting club through $S_h(E, \bar{C})$ for every club $E \subseteq \omega_1$ (by initial segments).

$\square_{1.4}$

1.5 Claim. Under the assumptions of 1.1 for $\varepsilon < \omega_1$ additively indecomposable, we can add to the conclusion: for $\zeta < \varepsilon$ and $i < i^*$, the forcing notion Q_i is ζ-proper.

Proof. We again assume $G_i \subseteq P_i$ generic is given; hence $\langle C_j : j < i\rangle$ (which serve as B_i too) is also given and by induction on α we define T^i_α, so that in the definition of T^i_α we use A and $\langle C_j \cap \alpha : j < i\rangle$ only (and the list $\{f^*_\xi : \xi < \omega_1\} \in L[A]$), so that a variant of (i) - (viii) holds. The changes are:

(iv)′ if $f, g \in T^i_\alpha$, $f(i) \neq g(i)$, then $f^{-1}(\{1\}) \cap g^{-1}(\{1\}) \setminus i$ has order-type $< \varepsilon$.

(vii)′ In addition to (vii), if $\langle \delta_\zeta : \zeta \leq \zeta^*\rangle$ is an increasing sequence of accumulation points of $\bigcap_{j<i} C_j$, $\langle \delta_\xi : \xi \leq \zeta\rangle \in L_{\delta_{\zeta+1}}[A \cap \delta_{\zeta+1}]$, for $\zeta < \zeta^*$, $f \in T^i_{\delta_0} \cap L_{\delta_0}[A \cap \delta_0]$, $f_m \in T^i_{\zeta^*+1}$ for $m < m^*$ and $m^* < \omega$, $\zeta^* < \varepsilon$, then there is $g \in T^i_{\zeta^*+2}$, $f \subseteq g$, $\mathrm{Dom}(g) = \zeta^* + 1$, such that the following conditions hold:

(α) For every $\mathcal{J} \in L_{h(\delta)}[A \cap \delta, B_i \cap \delta]$ (an open dense subset of $T^i_\delta \cap L_\delta[A \cap \delta]$ (ordered by inclusion)) for some $\gamma < \delta$, $g{\restriction}\gamma \in \mathcal{J}$, where $\delta \in \{\delta_\zeta : \zeta \leq \zeta^*\}$.

(β) $g^{-1}(\{1\}) \setminus \{\delta_\zeta : \zeta \leq \zeta^*\}$ is a bounded subset of δ_{ζ^*}.

(γ) For every $m < m^*, g^{-1}(\{1\}) \cap f_m^{-1}(\{1\}) \setminus \{\delta_\zeta : \zeta \leq \zeta^*\} \subseteq \mathrm{Dom}(f)$.

In the proof of Case D, we use the canonical well-ordering of $H(\aleph_1)^{L[A]}$ on our assignments (for the existence of $g \in T^i_{\delta+2}$, $\mathrm{Dom}(g) = \delta + 1$), and construct a witness, preserving and using (vii)$'$. $\square_{1.5}$

1.6 Discussion. 1) Also 1.3, 1.4 can be generalized to this context.

2) We have shown that just excluding the forcing notions like the one from Example V.5.1 (by demanding $\mathbb{D}$-completeness for a simple 2-completeness system) is not enough to ensure that CS iteration of proper forcing does not add reals. In VIII §4, on the other hand, we have quite weak restrictions on such $\underset{\sim}{Q}_i$ ensuring $\mathrm{Lim}\langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ does not add reals. However, the examples above (1.1-1.4) lead naturally to forcing notions which fall in between (and the corresponding consistency problems), which we now proceed to represent.

1.7 Problem. Let $f_\delta : \delta \to \delta$ for any limit $\delta < \omega_1$. Is there $f : \omega_1 \to \omega_1$ such that for every $\delta < \omega_1$, for arbitrarily large $\alpha < \delta$, $f_\delta(\alpha) < f(\alpha)$? I.e., the problem is, assuming CH, whether it is possible that for every such $\langle f_\delta : \delta < \omega_1\rangle$ there is a suitable f [negative answer follows from $\diamondsuit_{\aleph_1}$, and c.c.c. forcing preserves a negative answer].

1.7A Definition. For any sequence $\bar{f} = \langle f_\delta : \delta < \omega_1\rangle$, $f_\delta : \delta \to \delta$, let $P^0_{\bar{f}} = \{g : g$ a function from some $\alpha < \omega_1$ into ω_1, such that for every (limit) $\delta \leq \alpha$, for arbitrarily large $\beta < \delta$, $f_\delta(\beta) < g(\beta)\}$;

ordered by inclusion.

1.7B Discussion. Now if CH + Ax[forcing notions of the form $P^0_{\bar{f}}$] holds in some universe, the answer to 1.7 is yes (in that universe). So it is enough to show that if we iterate, with countable support, such forcing notions, then no real is added. A positive answer follows by 1.8 below and next section. A negative answer could have been viewed as proving a very weak form of diamond. The situation is similar for the other problems here.

1.8 Problem. Let $C_\delta \subseteq \delta$ be an unbounded subset of δ, for $\delta < \omega_1$. Is there a closed unbounded $C \subseteq \omega_1$ such that for no δ, $C_\delta \subseteq C$? Consider in particular the cases when we restrict ourselves to:

(a) C_δ has order-type ω, $\delta = \text{Sup } C_\delta$,

(b)$_\xi$ C_δ has order-type ξ, $\delta = \text{Sup } C_\delta$ and ξ limit,

(c) C_δ has order-type $< \delta$, $\delta = \text{Sup } C_\delta$,

(d) $C_\delta \equiv \emptyset \text{ mod } D_\delta$, D_δ a filter on δ, $\delta = \text{Sup } C_\delta$, for a given $\bar{D} = \langle D_\delta : \delta < \omega_1 \rangle$.

1.8A Definition. For $\bar{C} = \langle C_\delta : \delta < \omega_1 \rangle, C_\delta \subseteq \delta$, let $P^1_{\bar{C}} = \{f : f$ a function from some $\alpha + 1 < \omega_1$ to $\{0,1\}, f^{-1}(\{1\})$ is closed and for no $\delta \leq \alpha$, $C_\delta \subseteq f^{-1}(\{1\})\}$.

Order: inclusion.

We may consider also

1.8B Definition. For $\bar{D} = \langle D_\delta : \delta < \omega_1 \rangle$, D_δ a filter on δ, let

$$\begin{aligned} P^1_{\bar{D}} = \{f : f \text{ a function from some } \alpha + 1 < \omega_1 \text{ to } \{0,1\} \\ \text{such that } f^{-1}(\{1\}) \text{ is closed and for no } \delta \leq \alpha, \\ f^{-1}(\{1\}) \cap \delta = \delta \text{ mod } D_\delta\} \end{aligned}$$

Order: inclusion.

1.9 Problem. Let C_δ be an unbounded subset of δ, for $\delta < \omega_1$. Is there a closed unbounded $C \subseteq \omega_1$ such that for every δ, $C \cap C_\delta$ is a bounded subset of δ, when we restrict ourselves as in 1.8?

1.9A Definition. For a sequence $\bar{C} = \langle C_\delta : \delta < \omega_1 \rangle$, C_δ an unbounded subset of δ, let

$$\begin{aligned} P^2_{\bar{C}} = \{g : g \text{ a function from some } \alpha + 1 < \omega_1 \text{ to } \{0,1\}, \text{ such that} \\ g^{-1}(\{1\}) \text{ is closed and for every } \delta \leq \alpha, \text{Sup}[C_\delta \cap g^{-1}(\{1\})] < \delta\}. \end{aligned}$$

Order: inclusion.

1.9B Definition. For a sequence $\bar{D} = \langle D_\delta : \delta < \omega_1 \rangle$, D_δ a filter on δ, let

$P^2_{\bar{D}} = \{g : g$ a function from some $\alpha + 1 < \omega_1$ to $\{0,1\}$ such that $g^{-1}(\{1\})$ is closed and for every $\delta \le \alpha$, $g^{-1}(\{1\}) \cap \delta \equiv \emptyset \bmod D_\delta\}$

1.10 Claim. 1) $P^0_{\bar{f}}, P^1_{\bar{C}}$ and $P^2_{\bar{C}}$ (when one of the Cases (a)-(d) from 1.8 holds) are proper and $\mathbb{D}$-complete for some simple $\aleph_1$-completeness systems and
2) $P^0_{\bar{f}}, P^1_{\bar{C}}$ is strongly proper.
3) $P^2_{\bar{C}}$ is proper (and does not add reals) even in V^Q if forcing by Q adds no reals (for $P^0_{\bar{f}}, P^1_{\bar{C}}$ this follows by part 2).

Proof. Left to the reader.

1.11 Definition. For each $\delta < \omega_1$, let F_δ be a function, from $\mathrm{Dom}(F_\delta) = \{f : f$ a function from some $\alpha + 1 < \delta$ to $\{0,1\}$ such that $f^{-1}(\{1\})$ is closed $\}$ to ω. Let $C_\delta \subseteq \delta$ be an unbounded subset of δ of order type ω and $\bar{C} = \langle C_\delta : \delta < \omega_1 \rangle$.
Let

$P_{\bar{C},F} = \{g : g$ a function from some $\alpha + 1 < \omega_1$ to $\{0,1\}$
$g^{-1}(\{1\})$ is closed and for every $\delta \le \alpha$ for
some n_δ : if $\beta \in C_\delta, |C_\delta \cap \beta| > n_\delta$, and
$\mathrm{Min}(C_\delta \setminus (\beta+1)) > \mathrm{Min}\ (g^{-1}(\{1\}) \setminus (\beta + 1))$ and
$\beta < \gamma \in C_\delta, \mathrm{Min}\ (C_\delta \setminus (\gamma+1)) > \mathrm{Min}\ (g^{-1}(\{1\}) \setminus (\gamma+1))$, *then*
$|\gamma \cap C_\delta| > F_\delta(g \restriction (\mathrm{Min}\ (C_\delta \setminus (\beta+1))))\}$.

1.11A Claim. $P_{F,\bar{C}}$ (for $F, \bar{C}$ as in definition 1.11) is proper, $\mathbb{D}$-complete for some simple $\aleph_1$-completeness system and

$\otimes$ it is proper not adding reals even after forcing by any proper forcing notion not adding reals.

(see 2.13(2)).

Proof. Left to the reader.

Remark. In 1.11 (and 1.11A) demand

$$|\gamma \cap C_\delta| > F_\delta(|C_\delta \cap (\beta+1)|, g{\restriction}(\min(C_\delta \setminus (\beta+1)))).$$

§2. Not Adding Reals

We prove here that we can iterate (CS iterations) the forcing notions introduced at the end of the previous section, and not add reals. The real work is in Definition 2.2 and Lemma 2.8, but the reader may look at Conclusion 2.12 (or at 2.16). For our aim , naturally, we phrased a condition NNR_2, on CS iterations of proper forcing, saying we add no reals (condition (3)), a quite weak condition for avoiding "collision" with the weak diamond, and another condition, (5), intended to avoid collision with the counterexample of §1. It says each Q_i stays proper even if we force with forcing notions of the kind we iterate. Having phrased the condition, the main point is proving it is preserved by CS iteration, mainly in limit stages.

So, suppose $\bar{Q}$ has length δ, and for each $\alpha < \delta$, $\bar{Q}{\restriction}\alpha$ is as required. Assume for simplicity we do not try to kill $\Phi^{\aleph_0}_{\aleph_1}$; say, using $\aleph_1$-completeness systems. As seems natural, we start with a countable $N \prec (H(\chi), \in, <^*_\chi)$ and $p \in P_\delta \cap N$ and try to find q, $p \le q \in P_\delta$, q is (N, P_δ)-generic and determine $\underset{\sim}{G}_{P_\delta} \cap N$, which should be an old set if we succeed. So, if $\sup(\delta \cap N) = \bigcup_{n<\omega} \alpha_n$, $\alpha_n < \alpha_{n+1}$, $\alpha_n \in N$ we should try to choose approximations $q_n \in P_{\alpha_n}$, $q_{n+1}{\restriction}\alpha_n = q_n$. But, as we do not have $\aleph_1$-completeness we cannot do this per se. In V§7 a major point in the proof is that we have "above" N a sequence $\bar{N} = \langle N_i : 1 \le i \le \zeta \rangle$, ζ and each N_i countable (letting $N_0 = N$), $\bar{N}$ quite "long" in suitable sense, and we demand q_n is (N_i, P_{α_n})-generic for "many" i's. So if e.g. $\alpha_{n+1} = \alpha_n + 1$, $i < \zeta$ such that q_n is (N_i, P_{α_n})-generic, we have only $\aleph_0$ candidates for members of the relevant completeness system. However "using N_i is destroying it", "it is consumed ", as q_{n+1} is not $(N_i, P_{\alpha_{n+1}})$-generic. Why is this so? In the first step, say choosing $G_{Q_0} \cap N_0$, we have no problem; for $\underset{\sim}{G}_{\underset{\sim}{Q}_1} \cap N_0[\underset{\sim}{G}_{Q_0}]$ we have to choose a common member from all the candidates A to be "a subset of $Q_1 \cap N_0[\underset{\sim}{G}_{Q_0}]$ in the appropriate family $\mathbb{D}_x$". Now the common member is naturally not in N_1. We can use stronger induction hypothesis, then use only $\aleph_0$-completeness system or even 2-completeness system, so we have for $\underset{\sim}{G}_{Q_0} \cap N_1$ only finitely many (or two) candidates, so a common member exists in N_1. But after ω steps it is not clear how to guarantee $G_{P_\omega} \cap N_0 \in N_1$.

One approach, suggested in [Sh:177], is to weaken "Q is α-proper" to "for $p \in Q \cap N_0$, $\bar{N} = \langle N_i : i \leq \alpha \rangle$ as usual, there is $q, p \leq q \in Q$, q is (N_i, Q)-generic for "many" $i \leq \alpha$"; this work for "easy" cases like interpreting "many" as "having the same order type". While this work for e.g. $P^1_{\underset{\sim}{C}}$ (from 1.8(a), 1.8A), this does not seem strong enough , but it covers the forcing notion of V for specializing Aronszajn tree, which the present condition do not. Here we rather say: having two candidates for $\underset{\sim}{G} \cap N_1$, we demand they are a subset of $(Q_0 \cap N) \times (Q_0 \cap N)$ generic over N_0; for making this work we are carried to the following.

Here we have N_1, $N_0 \prec N_1$, q_n is demanded to be (N_1, P_{α_n})-generic too; We have several possibilities, we actually have a finite tree of possibilities for $\underset{\sim}{G}_{P_{\alpha_n}} \cap N_1$ which is generic for an appropriate product of finitely many copies of $P_i^\prime s$, $i \leq \alpha_n$. But to proceed with this we have N_2, where again we have a finite tree of possibilities for $\underset{\sim}{G}_{P_{\alpha_n}} \cap N_2$. But only each one is generic over N_2. Above this we imitate V 4.4. Now q_n is (N_ℓ, P_{α_n})-generic for $\ell = 3, 4, 5$ and for each P_{α_n}-name $\underset{\sim}{\tau} \in N_4$ of an ordinal it allows only finitely many possibilities, but unlike V 4.4 we do not use ω-properness. So we have for each n a "tower" of six models. For higher $\ell \leq 5$, q_n "knows" less on N_ℓ, but our knowledge goes down "slowly" so moving from n to $n+1$, taking care of ℓ, the knowledge on $\underset{\sim}{G}_{P_{\alpha_n}} \cap N_{\ell+1}$ is enough to move ahead. Probably this explanation is meaningless for many readers, but will be helpful if read in the end or middle of reading the proof.

Note that §1 (particularly 1.5) show the impossibility of too good iteration theorems (say CS, of proper forcing not adding reals) but do not block consistency of appropriate forcing axiom with CH as we may instead of forcing with candidates, spoil their being candidates (as in III, V).

2.1 Definition. 1) For a finite tree t (i.e. $t = (|t|, <^t), |t|$ a finite set, $<^t$ a partial order on $|t|$ such that for $x \in t$, $\{y : y <^t x\}$ is linearly ordered), let

$$\begin{aligned} \mathrm{trind}_{\alpha}(t) = \{\bar\alpha : &\bar\alpha = \langle \alpha_\eta : \eta \in t\rangle, \text{ each } \alpha_\eta \text{ is an ordinal} \le \alpha \\ &\text{and } \eta < \nu \text{ in } t \text{ implies } \alpha_\eta \le \alpha_\nu\} \end{aligned}$$

$$\mathrm{trind}_{<\alpha}(t) = \bigcup_{\beta<\alpha} \mathrm{trind}_{\beta}(t).$$

2) For a given iteration $\bar Q = \langle P_i, \underset{\sim}{Q}_j : i \le \alpha, j < \alpha\rangle$, a finite tree t and $\bar\alpha \in \mathrm{trind}_\alpha(t)$ let

$P_{\bar\alpha} = \big\{\bar p : \bar p = \langle p_\eta : \eta \in t\rangle$, and for $\eta \in t$ we have $p_\eta \in P_{\alpha_\eta}$ and if $t \models \eta < \nu$ then $p_\eta = p_\nu \restriction \alpha_\eta\big\}$

ordered by

$\bar p \le \bar q$ *iff* for every $\eta \in t$, $P_{\alpha_\eta} \models p_\eta \le q_\eta$.

3) $t \subseteq_{\text{end}} s$ *if* t is s restricted to the set of members of t and: $s \models$ ["$\eta < \nu$", $\nu \in t$] implies [$\eta \in t$].

4) We write $\langle i\rangle$ for $\bar\alpha = \langle \alpha_\eta : \eta \in t\rangle$, when t has one element, say $<>$ and $\alpha_{<>} = i$.

5) For $\bar Q$ an iteration of length α, t a finite tree, $\bar\alpha \in \mathrm{trind}_\alpha(t)$, and model N:

(a) $a\mathrm{Gen}^{\bar\alpha}_{\bar Q}(N) = \big\{G : G$ a subset of $P_{\bar\alpha} \cap N$ generic over N such that for each $\eta \in t, G_\eta \overset{\text{def}}{=} \{p_\eta : \bar p \in G\}$ has an upper bound in $P_{\alpha_\eta}\big\}$

[Note: G can essentially be identified with $\langle G_\eta : \eta \in t\rangle$].

(b) $s\mathrm{Gen}^{\bar\alpha}_{\bar Q}(N) = \big\{G : G$ a subset of $P_{\bar\alpha} \cap N$ generic over N which has an upper bound in $P_{\bar\alpha}\big\}$.

(c) $\mathrm{Gen}^{\bar\alpha}_{\bar Q}(N) = \big\{G : G$ a subset of $P_{\bar\alpha} \cap N$ generic over $N\big\}$

6) If $t_1 \subseteq t_2$, $\bar\alpha^\ell \in \mathrm{trind}_{\alpha_\ell}(t_\ell)$, and $\bar p^\ell \in P_{\bar\alpha^\ell}$ for $\ell \in \{1,2\}$, and $\bigwedge_{\eta\in t_1} \alpha^1_\eta \le \alpha^2_\eta$ then $\bar p^1 \le \bar p^2$ means $\bigwedge_{\eta\in t_1} p^1_\eta \le p^2_\eta \restriction \alpha^1_\eta$.

2.2 Definition. $\bar Q$ is an NNR_2-iteration for $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ means:

(1) $\bar{Q}$ is a CS iteration of $\mathcal{E}_0$-proper forcing (see V2.2).

(2) For $\ell = 0, 1, 2, \mathcal{E}_\ell$ is a stationary subset of $\mathcal{S}_{<\aleph_1}(\lambda_\ell)$ for some uncountable λ_ℓ.

(3) Forcing with P_α adds no reals for $\alpha \leq \ell g(\bar{Q})$.

$(4)_2$ *If*:

(a) $N \prec (H(\chi), \in, <^*_\chi)$ is countable, (χ regular large enough),

(b) $N \cap \lambda_1 \in \mathcal{E}_1$,

(c) $\langle \bar{Q}, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \rangle$ belongs to N,

(d) $i^* \leq i \leq j \leq \alpha \leq \ell g(\bar{Q})$, $i^* \in N$, $i \in N$, $j \in N$ and i^*, i are non-limit,

(e) $G^a \in \ \mathrm{Gen}^{<i>}_{\bar{Q}}(N)$,

(f) $p \in N \cap P_j$, $p \restriction i \in G^a$, and

(g) q_0, q_1 are upper bounds of G^a in P_i, and $q_0 \restriction i^* = q_1 \restriction i^*$,

then there is $G' \in \mathrm{Gen}^{<j>}_{\bar{Q}}(N)$, extending G^a, $p \restriction j \in G'$ and q_0^+, $q_1^+ \in P_j$ such that: $q_0 \leq q_0^+ \restriction i$, $q_1 \leq q_1^+ \restriction i$, and for $\ell = 0, 1$, q_ℓ^+ is an upper bound to G'. Let G' extends G^a, mean that $[p' \in G' \Rightarrow p' \restriction i \in G^a]$ and $q_0^+ \restriction i^* = q_0^+ \restriction i^*$.

(5) Assume i are not limit ordinals, $i < j \leq \alpha$, $\bar{Q}'$ a CS iteration of length β satisfying (1) - (4), $\alpha \leq \beta, \bar{Q} = \bar{Q}' \restriction \alpha$, $\bar{Q}'$ satisfies (1)-(4), t a finite tree, $\eta^* \in t$, $\bar{\alpha} \in \mathrm{trind}_\beta(t)$, $\alpha_{\eta^*} = i$, $s = t \restriction \{\eta : \eta \leq \eta^*\}$, (so $P'_{\bar{\alpha} \restriction s}$ is essentially P_i), and let $\underset{\sim}{R} \stackrel{\mathrm{def}}{=} P'_{\bar{\alpha}} / P_i$ (a P_i-name).

If $\underset{\sim}{R}$ is an $\mathcal{E}_2$ - proper forcing not adding reals, *then*

$\Vdash_{P_i * \underset{\sim}{R}}$ "P_j / P_i is a $\mathcal{E}_2$−proper forcing not adding reals"†.

2.2A Remark. Note that for $i < j \leq \ell g(\bar{Q})$, i non-limit, we have: P_j / P_i is $(\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2)$-proper.

2.3 Definition. $\bar{Q}$ is an $NNR_{\aleph_0}$ - iteration for $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ is defined similarly, replacing $(4)_2$ by $(4)_{\aleph_0}$ below (so, in clause (5) now we mean this (4)) where:

† Actually, e.g. if Q_0, Q_1 are proper forcings not adding reals and $Q_0 \times Q_1$ is proper, then $Q_0 \times Q_1$ does not add reals; in fact by 2.5 the "not adding reals" is redundant.

$(4)_{\aleph_0}$ *if*:

(a) $N \prec (H(\chi), \in, <^*_\chi)$ is countable, (χ regular large enough),

(b) $N \cap \lambda_1 \in \mathcal{E}_1$,

(c) $\langle \bar{Q}, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \rangle$ belongs to N,

(d) $i < j \leq \alpha$, i non-limit, $i \in N$ and $j \in N$,

(e) $G^a \in \text{Gen}^{<i>}_{\bar{Q}}(N)$,

(f) $p \in N \cap P_j$, $p{\restriction}i \in G^a$,

(g) $t \in N$ a finite tree, $\bar{\alpha} \in \text{trind}_i(t)$, each α_η non-limit,

(h) $\bar{q} \in P_{\bar{\alpha}}$, and if $\eta \in t, \alpha_\eta = i$, then q_η is a bound to G^a, and

(i) $\bar{\beta} = \langle \beta_\eta : \eta \in t \rangle$ where for $\eta \in t$:

$$\beta_\eta = \begin{cases} \alpha_\eta & \text{if } \alpha_\eta < i \\ j & \text{if } \alpha_\eta = i; \end{cases}$$

Then there are $G' \in \text{Gen}^{<j>}_{\bar{Q}}(N)$ extending G^a and $\bar{r} \in P_{\bar{\beta}}$, such that:

(α) each $r_\eta (\eta \in t, \alpha_\eta = i)$ is a bound of G'

(β) $\bar{q} \leq \bar{r}$.

Before we state and prove the main lemma, we prove a few claims.

2.4 Claim. 1) Suppose $x \in \{2, \aleph_0\}$, $\bar{Q}'$ is an iteration satisfying (1) - (3),$(4)_x$ of Definition 2.2 or 2.3, $\bar{Q} = \bar{Q}'{\restriction}\alpha$, $\beta = \ell g(\bar{Q}')$, $\bar{Q}$ is an NNR_x-iteration for $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$, χ is regular large enough, $N \prec (H(\chi), \in, <^*_\chi)$ is countable, $\langle \bar{Q}', \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \alpha \rangle$ belongs to N, $N \cap \lambda_2 \in \mathcal{E}_2$, $t \subseteq_{\text{end}} s$ are finite trees, $\bar{\beta} \in N$ is in $\text{trind}_\beta(s)$, $\bar{\alpha} = \bar{\beta}{\restriction}t$, $\text{Rang}[\bar{\beta}{\restriction}(s \setminus t)] \subseteq \alpha$, $G_{\bar{\alpha}} \in \text{Gen}^{\bar{\alpha}}_{\bar{Q}}(N)$ and $\bar{q} = \langle q_\eta : \eta \in t \rangle \in P_{\bar{\alpha}}$ is above $G_{\bar{\alpha}}$, $\bar{p} \in P_{\bar{\beta}} \cap N$, and $\bar{p}{\restriction}t \in G_{\bar{\alpha}}$.

Assume in addition:

$(*)_t$ for each $\eta \in t$, the forcing notion $P_{\bar{\alpha}}/P_{\bar{\alpha}{\restriction}\{\nu:\nu \leq \eta\}}$ is $\mathcal{E}_2$-proper not adding reals.

Then there is a $G_{\bar{\beta}} \in \text{Gen}^{\bar{\beta}}_{\bar{Q}}(N)$ extending $G_{\bar{\alpha}}$ (recall, $G_{\bar{\beta}}$ extending $G_{\bar{\alpha}}$ means $[\bar{\sigma} \in G_{\bar{\beta}} \Rightarrow \bar{\sigma}{\restriction}t \in G_{\bar{\alpha}}]$) to which $\bar{p}$ belongs and $\bar{r} \in P_{\bar{\beta}}$ above $G_{\bar{\beta}}$, $\bar{q} \leq \bar{r}{\restriction}t$, (and it follows $\bar{p} \leq \bar{r}$).

2) Moreover, if $\eta \in t$, and $\nu \leq^t \eta$ is maximal such that $(\exists \rho \in s \setminus t)(\nu < \rho)$, *then* $r_\eta{\restriction}[\alpha_\nu, \alpha_\eta) = q_\eta{\restriction}[\alpha_\nu, \alpha_\eta)$.

Proof. We prove it by induction on the number of elements of s.

By the induction hypothesis, we can show that if $t \subseteq t_1 \subseteq s \,\&\, t_1 \neq s$, then $(*)_{t_1}$ holds. Hence, it is easy to reduce the claim to the case $s \setminus t$ has a unique element, say η. Assume first that there is a maximal $\nu \in t$ with $\nu <^s \eta$ and let $t^* = t\restriction\{\rho : \rho \in t,\, \rho \leq \nu\}$; so by $(*)_t$ we know $P_{\bar{\alpha}}/P_{\bar{\alpha}\restriction t^*}$ is a $\mathcal{E}_2$-proper forcing not adding reals. Let R be the $P_{\bar{\alpha}\restriction t^*}$-name $P_{\bar{\alpha}}/P_{\bar{\alpha}\restriction t^*}$. Note that $P_{\bar{\alpha}\restriction t^*}$ is isomorphic to P_{α_ν}. Let $i = \alpha_\nu$, $j = \alpha_\eta$ and apply (5) of Definition 2.2 (or of 2.3, of course). We obtain $r = \langle r_\rho : \rho \in s\rangle$.

But why is r_ρ a condition in P_{α_η} and not a $P_{\bar{\alpha}}$-name of such a condition? As all the influence of $P_{\bar{\alpha}}/P_{\bar{\alpha}\restriction t^*}$ is on the set of dense subsets of P_{α_η} in $N[G_{P_{\bar{\alpha}}}]$, which we know (and it is in V) so as we know there is r_ρ we can inspect each candidate (not i), we use our demanding $q \leq \bar{r}\restriction t$ rather than $\bar{q} = \bar{r}\restriction t$. $\square_{2.4}$

2.5 Claim. If $\bar{Q}$ satisfies (1), (2), (3) of Definition 2.2, $\alpha = \ell g(\bar{Q})$, t is a finite tree, $\bar{\beta} \in \mathrm{trind}_\alpha(t)$ and $P_{\bar{\beta}}$ is $\mathcal{E}_2$-proper, *then* it does not add reals. Also, if $G_{\bar{\alpha}} \subseteq P_{\bar{\alpha}}$ is generic over V, we let $G_\eta = \{q_\eta : \bar{q} \in G_{\bar{\alpha}}\}$, then $\langle G_\eta : \eta \in \mathrm{Dom}(\bar{\alpha})\rangle$ determines $G_{\bar{\alpha}}$ (hence we do not distinguish strictly). Similarly, for "$G \subseteq P_{\bar{\alpha}} \cap N$ generic over N".

Proof. Immediate.

2.6 Claim. Let (i) $\bar{Q}$ be a CS iteration of ${}^\omega\omega$-bounding proper forcing notions.

(ii) $i < j \leq \ell g(\bar{Q}), N_0 \prec N_1$ are countable elementary submodels of $(H(\chi), \in, <^*_\chi)$, $\langle \bar{Q}, i, j\rangle \in N_0$, χ regular large enough, and $N_0 \in N_1$.

(iii) $p \in P_j \cap N_0$, $q \in P_i$, $p\restriction i \leq q$, q is (N_1, P_i)-generic, and (N_0, P_i)-generic.

(iv) for every pre-dense $\mathcal{I} \subseteq P_i$ from N_0, some finite $\mathcal{J} \subseteq \mathcal{I} \cap N_0$ is pre-dense above q.

Then there is $r \in P_j$, $r\restriction i = q$, $p \leq r$, r is (N_1, P_j)-generic and (N_0, P_j)-generic such that for every pre-dense $\mathcal{I} \subseteq P_j$ from N_0 some finite $\mathcal{J} \subseteq \mathcal{I} \cap N_0$ is pre-dense above r.

2.6A Remark. This claim is from [Sh:177] and is implicit in VI §1.

Proof of 2.6. By VI Theorem 0.A, P_i, P_j/P_i are ${}^\omega\omega$-bounding. Let $\langle \underset{\sim}{\tau}_n : n < \omega\rangle \in N_1$ list all P_j-names of ordinals which belong to N_0.

We can find a functions $F, H \in N_0$ such that: for every $\underset{\sim}{p}$, a P_i-name of a member of $P_j/\underset{\sim}{G}_{P_i}$, and $\underset{\sim}{\tau}$, a P_j-name of a ordinal, we have $\underset{\sim}{p}' = F(\underset{\sim}{p}, \underset{\sim}{\tau})$ is a P_i-name of a member of $P_j/\underset{\sim}{G}_{P_i}$ satisfying $\underset{\sim}{p}'[G_{P_i}]\restriction i = \underset{\sim}{p}[G_{P_i}]\restriction i$, $\underset{\sim}{p}[G_{P_i}] \leq^{P_j} \underset{\sim}{p}'[G_{P_i}]$ in P and $\underset{\sim}{\sigma} = H(\underset{\sim}{p}, \underset{\sim}{\tau})$ is a P_i-name of an ordinal such that

$$\underset{\sim}{p}'[G_{P_i}] \Vdash_{P_j/G_{P_i}} \text{“}\underset{\sim}{\tau} = \underset{\sim}{\sigma}[G_{P_i}]\text{”}.$$

Let $\underset{\sim}{p}_0 = p\restriction[i, j)$, $\underset{\sim}{p}_{n+1} = F(\underset{\sim}{p}_n, \underset{\sim}{\tau}_n)$, $\underset{\sim}{\sigma}_n = H(\underset{\sim}{p}_n, \underset{\sim}{\tau}_n)$, so $\underset{\sim}{p}_n$, $\underset{\sim}{\sigma}_n \in N_0$, and $\underset{\sim}{\sigma}_n$ is a P_i-name of an ordinal and $\langle(\underset{\sim}{p}_n, \underset{\sim}{\sigma}_n) : n < \omega\rangle \in N_1$. For each n we can find $(\underset{\sim}{p}^+_n, \underset{\sim}{\bar{v}}^n) \in N_1$, moreover $\langle(\underset{\sim}{p}^+_n, \underset{\sim}{\bar{v}}^n) : n < \omega\rangle$ belongs to N_1 such that $\underset{\sim}{\bar{v}}^n$ is a P_i-name of a sequence of finite sets of ordinals which belongs to N_1 (so $\Vdash_{P_i}$ “$\underset{\sim}{\bar{v}} \in V$”) $\underset{\sim}{p}^+_n$ a P_i-name of a member of $P_j/\underset{\sim}{G}_{P_i}$, and in $V[G_{P_i}]$ $\underset{\sim}{p}^+_n[G^+_{P_i}]$ is above $\underset{\sim}{p}_n[G_{P_i}]$ (in $P_j/\underset{\sim}{G}_{P_n}$) and is $(N[G_{P_i}], P_j/G_{P_i})$-generic and $\underset{\sim}{\bar{v}}[G_{P_i}] \in N_1[G_{P_i}] \cap V$ (for any $G_{P_i} \subseteq P_i$ generic over V). Let $\underset{\sim}{\bar{v}} = \langle \underset{\sim}{v}_m : m < \omega\rangle$, $\underset{\sim}{v}_m = \bigcup_{n \leq m} v^n_m$ so $\underset{\sim}{\bar{v}} \in N_1$ is a P_i-name, $\Vdash_{P_i}$ “$\underset{\sim}{\bar{v}}$ is an ω-sequence of finite sets of ordinals” (we can make $\Vdash_{P_i}$ “$\underset{\sim}{\bar{v}} \in V$” but we not use).

Let $\langle \bar{u}^n : n < \omega\rangle$ list all sequences, $\bar{u} = \langle u_m : m < \omega\rangle \in N_1$, u_m a finite set of ordinals. Let $\bar{u}^n = \langle u^n_m : m < \omega\rangle$. Choose $\langle u^*_n : n < \omega\rangle \in V$, a sequence of finite sets of ordinals, such that:

$(*)$ for each $n < \omega$ and for every large enough $m, u^n_m \subseteq u^*_m$.

As $\underset{\sim}{\sigma}_n \in N_0$ are P_i-names, by assumption (iv), we can find $\langle v^*_n : n < \omega\rangle \in V$ a sequence of finite sets of ordinals such that

$$q \Vdash_{P_i} \text{“}\underset{\sim}{\sigma}_n \in v^*_n\text{”}$$

Now, clearly it suffices to prove:

$\otimes$ $q \Vdash_{P_i}$ “there is a condition $r \in P_j, r\restriction i = q, [i, j) \cap \mathrm{Dom}(r) = N_1 \cap [i, j), r$ is (N_1, P_j)-generic, r is above some p_n, and $r \Vdash_{P_j} [\bigwedge_n \underset{\sim}{\tau}_n \in v^*_n \cup u^*_n]$”

[as there is a P_i-name of such a condition, and we know the domain, there exists an actual such $r \in P_j$].

Why $\otimes$ holds? Let $G_i \subseteq P_i$ be generic over V, $q \in G_i$. Then $\underset{\sim}{\bar{v}}$ being a P_i-name from N_1, satisfies $\underset{\sim}{\bar{v}}[G_i] = \langle v_n : n < \omega \rangle \in N_1[G_i]$ so $v_n \subseteq N_1 \cap \mathrm{Ord}$, hence for some $\langle v'_n : n < \omega \rangle \in V$, and ω-sequence of finite sets of ordinals, $\bigwedge_{n<\omega} v_n \subseteq v'_n$, so w.l.o.g. $\langle v'_n : n < \omega \rangle \in N_1$; so for some $n(*)$, $\bigwedge_{m \geq n(*)} v_n \subseteq v'_n \subseteq u^*_n$. Choose r as $(N_1[G_1], P_j/P_i)$-generic such that $\mathrm{Dom}(r) = N_1[G_1] \cap [i,j) = N_1 \cap [i,j)$, and $\underset{\sim}{p}^+_{n(*)}[G_i]$, $p_{n(*)} \leq r$, such r exists by the theorem of preservation of properness. Now r is as required in $\otimes$. As we have done it in any $V[G_i], G_i \subseteq P_i$ generic over $V, q \in G_i$, clearly q forces ($\Vdash_{P_i}$) there is such r. $\square_{2.6}$

2.7 Claim. Let

(i) $\bar{Q}$ be a semiproper iteration of ${}^\omega\omega$-bounding forcing notions

(ii) $i < j \leq \ell g(\bar{Q}), N_0 \prec N_1 \prec (H(\chi), \in, <^*_\chi), \langle \bar{Q}, i, j \rangle \in N_0$ and $N_0 \in N_1$ both countable, and χ regular large enough.

(iii) $p \in P_j \cap N_0, q \in P_i, p \restriction i \leq q$, q is (N_1, P_i)-semi generic and (N_0, P_i)-generic,

(iv) for every $\underset{\sim}{\tau} \in N_0$ a P_i-name of a countable ordinal for some finite u, $q \Vdash$ "$\underset{\sim}{\tau} \in u$".

Then, there is an $r \in P_j, r \restriction i = q,\ p \leq q$, r is (N_j, P_j)-semi generic and (N_0, P_j)-semi generic such that for every P_j-name $\underset{\sim}{\tau} \in N_0$ of a countable ordinal, for some finite u, we have $r \Vdash_{P_j}$ "$\underset{\sim}{\tau} \in u$".

Proof. Same as 2.6 except that: using RCS, the issue of the domain of r disappears, and the names we deal with are names of countable ordinals. $\square_{2.7}$

2.8 Main Lemma. If $x \in \{2, \aleph_0\}$, $\bar{Q}$ is a CS iteration of (limit) length δ and for every $\alpha < \delta$, $\bar{Q} \restriction \alpha$ is an NNR_x-iteration for $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$, *then* $\bar{Q}$ is an NNR_x-iteration for $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$.

2.9 Remark. 1) Our main object is usually to preserve clause (3) of Definition 2.2: adding no real.

2) Comparing this with the result is V §7 and in VIII §4, we gain in replacing the completeness system by condition $(4)_x$, which is weaker; but "$(< \omega_1)$-proper" seems incomparable with condition (5) which replaces it.

2.10 Proof of 2.8 . We have to prove the five conditions from Definition 2.2 (or 2.3).

Conditions (1) and (2) are easy (part (1) follows from part (2), for part (2) see 2.2A and V).

Condition (3) follows from condition (4) (use $i = 0, j = \delta$, $\underset{\sim}{\tau}$ a name of a real).

So, it is enough to prove:

(a) condition (4) and

(b) condition (5) assuming (3), (4) hold.

2.10A Proof of Condition (5), Assuming Conditions (3), (4). So, forcing with P_δ adds no reals and $\bar{Q}$ satisfies (1) — (4).

Let i, j be non limit ordinals, $i < j \leq \delta$, $\underset{\sim}{R}$, $\bar{Q}'$, s, t, $\bar{\alpha}$, η^* be as in the assumption of (5). So let χ be regular large enough, N a countable elementary submodel of $(H(\chi), \in, <^*_\chi)$, $N \cap \lambda_2 \in \mathcal{E}_2, i \in N, j \in N, \underset{\sim}{R} \in N, \bar{Q} \in N, (q_0, q_1) \in P_i * \underset{\sim}{R}$ is $(N, P_i * \underset{\sim}{R})$-generic and force $\underset{\sim}{G}_{P_i * \underset{\sim}{R}} \cap N$ to be $G^a, p \in P_j \cap N, p{\restriction}i \in G^a \cap P_i$ (equivalently $p{\restriction}i \leq q_0$). It suffices to find $r \in P_j$ above q_0 and above p, and r is $(N[G^a], P_j)$-generic, and r forces a value to $\underset{\sim}{G}_{P_j} \cap N$.

By the assumption, $\alpha < \delta \Rightarrow \bar{Q}{\restriction}\alpha$ is NNR_x-iteration for $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$;hence if $j < \delta$ the conclusion holds. So, assume $j = \delta$.

Let $N_0 = N$ and choose N_1 satisfying:

(α) N_1 is countable

(β) $N_0 \prec N_1 \prec (H(\chi), \in, <^*_\chi)$

(γ) $N_1 \cap \lambda_0 \in \mathcal{E}_0$

(δ) $N_0 \in N_1$

(ε) G^a, q_0, $q_1 \in N_1$.

Let $i = i_0 < i_1 < i_2 < \cdots < i_n < \cdots (n < \omega)$ be such that:

$$i_n \text{ is not limit, } i_n \in N_0 \cap \delta \text{ and } \sup[\delta \cap N_0] = \sup\{i_n : n < \omega\}.$$

Let $\langle \underset{\sim}{\mathcal{I}}_n : n < \omega \rangle \in N_1$ be a list of the $P_i * \underset{\sim}{R}$-names of dense subsets of P_j / P_i in N_0.

Now we choose by induction on n, p^n, q^n such that:

(A) (1) $q^n \in P_{i_n}$,

(2) q^n is (N_ℓ, P_{i_n})-generic for $\ell = 0, 1$,

(3) $\mathrm{Dom}(q_n) = i_n \cap N_1$

(4) $q_0 \leq q^0$

(5) $q^{n+1} \restriction i_n = q^n$

(B) (1) p^n is (a P_{i_n} - name of) a member of $P_\delta \cap N_0$

(2) $p^n \restriction i_n \leq q^n$

(3) $p^n \leq p^{n+1}$, $p^0 = p$

(4) $p^{n+1} \in \underset{\sim}{\mathcal{I}}_n$ (more exactly: $p^{n+1} \in \underset{\sim}{\mathcal{I}}_n[G^a]$ i.e. $q^{n+1} \Vdash_{P_{i_{n+1}}}$ "$p^{n+1} \in \underset{\sim}{\mathcal{I}}_n[G^a]$" where

$\underset{\sim}{\mathcal{I}}_n[G^a] = \{r \in N : \text{for some } p' \in G^a \; p' \Vdash_{P_i} \text{"}r \in \underset{\sim}{\mathcal{I}}_n\text{"}\})$.

(C) $q^n \Vdash$ "$\underset{\sim}{G}_{i_n} \cap N_0$ is generic for $\Big(N_0[G^a], (P_{i_n}/P_i) \cap N_0\Big)$".

Note in (B)(1) that p^n should not depend on $\underset{\sim}{G}_{\underset{\sim}{R}}$.

For $n = 0$ - easy.

For the induction step, defining for $n+1$, first note that

$$(*) \qquad \Vdash_{P_i * [\underset{\sim}{R} \times (P_{i_n}/P_i)]} \text{"}P_{i_{n+1}}/P_{i_n} \text{ is } \mathcal{E}_2 - \text{proper not adding reals"}.$$

We get $(*)$ by applying (5) of the Definition with $\bar{Q} \restriction i_{n+1}$ (which is NNR_x-iteration for $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$), $\bar{Q}$, $t^*, \bar{\alpha}^*$, η^{n+1}, i_n, i_{n+1} here standing for $\bar{Q}, \bar{Q}'$, t, $\bar{\alpha}$, η^*, i, j there, where: t^* is t when we add η^n just above η^* and η^{n+1} just above η^n and let $\bar{\alpha}^* \restriction t = \bar{\alpha}$, $\alpha^*_{\eta^n} = i_n$ and $\alpha^*_{\eta^{n+1}} = i_{n+1}$.

To apply condition (5) we have, however, to know that $P_{\bar{\alpha}^* \restriction (t \cup \{\eta^n\})}$ is $\mathcal{E}_2$-proper not adding reals; but this is guaranteed by Claim 2.4.

So, $(*)$ above holds; so, after forcing with $P_i * [\underset{\sim}{R} \times (P_{i_n}/P_i)]$ (with q^n, (q_0, q_1) in the generic set) we shall find a $q \in P_{i_{n+1}}/P_{i_n}$ generic for $\big(N_0\big[G^a, \underset{\sim}{G}_{P_{i_n}/P_i}\big], \big[P_{i_{n+1}}/P_{i_n}\big]\big)$.

(Note: $N_0[G^a]$ had no new members of V, so no new members of $P_{i_{n+1}}/P_{i_n}$). Now, the forcing with $\underset{\sim}{R}$ is irrelevant (except for the information in G^a and $G^a \in V$!) So, there is a P_{i_n}-name of such a q, and there is a P_{i_n}-name of a q', $q \le q' \in P_{i_{n+1}}/P_{i_n}$, q' forcing a value to $\underset{\sim}{G}_{P_{i_{n+1}}} \cap N_0$.

So, there is a $\underset{\sim}{q}' \in N_1$, a P_{i_n}-name of a condition from $P_{i_{n+1}}/P_{i_n}$ satisfying the above if there is such q' at all.

Now, as in the proof on the preservation of properness, we can choose q_{n+1}.

In the end, let $r' = q^0 \cup \bigcup_n q^{n+1} \restriction [i_n, i_{n+1})$. Now $\langle \underset{\sim}{\mathcal{I}}_n \cap \underset{\sim}{G}_{P_j} : n < \omega \rangle$ is clearly a P_j-name, so, by condition (3), there is some r, $r' \le r \in P_j$, and r forces an (old) value to it. Now, this r finishes the proof. $\square_{2.10A}$

2.10B Proof of condition (4)$_{\aleph_0}$ when we deal with $NNR_{\aleph_0}$:

So let N, i, j, $\bar{\alpha}$, $\bar{\beta}$, G^a, p, t, $\langle q_\eta : \eta \in t \rangle$ are as there. By the assumption w.l.o.g. $j = \delta$.

Choose N_ℓ (for $\ell = 0, 1, 2, 3, 4, 5$) such that:

(α) every N_ℓ is countable, $N_0 = N$

(β) $N_\ell \prec N_{\ell+1} \prec (H(\chi), \in, <^*_\chi)$ for $\ell = 0, 1, 2, 3, 4$

(γ) $N_1 \cap \lambda_1 \in \mathcal{E}_1$, $N_2 \cap \lambda_2 \in \mathcal{E}_2$, $N_3 \cap \lambda_0 \in \mathcal{E}_0$, $N_4 \cap \lambda_0 \in \mathcal{E}_0$, $N_5 \cap \lambda_1 \in \mathcal{E}_0$, (remember $N_0 \cap \lambda_1 \in \mathcal{E}_1$)

(δ) $N_\ell \in N_{\ell+1}$ for $\ell = 0, 1, 2, 3, 4$

(ε) $\bar{q} \in N_1$, $G^a \in N_1$.

Let $i = i_0 < i_1 < i_2 < \ldots < i_n < \ldots (n < \omega)$ be such that: each i_n is non-limit, belongs to $N_0 \cap \delta$, and

$$\sup(\delta \cap N_0) = \sup\{i_n : n < \omega\}.$$

Let $\langle \mathcal{I}_n : n < \omega \rangle \in N_1$ be a list of the dense subsets of P_δ which belong to N_0. For simplicity, w.l.o.g. we can assume t is a subset of ${}^{\omega>}\omega$ ordered by $\triangleleft$ (being an initial segment), $\alpha_\eta = i \Leftrightarrow \ell g(\eta) = m^*$ (remember only $\eta <^t \nu \Rightarrow \alpha_\eta \le \alpha_\nu$ was required). Let $t^* = t \cap {}^{m^*}\omega$ and stipulate $t_{-1} = t \setminus t^*$.

Now we define by induction on $n < \omega$, p_n, $q^n_\eta (\eta \in t^*)$, G^a_n, t_n, $\bar{\alpha}^n$, G^b_η, G^c_η (for $\eta \in t_n$) such that:

(A) (1) $q^n_\eta \in P_{i_n}$ (for $\eta \in t^*$)

(2) q^n_η is (N_ℓ, P_{i_n})-generic for $\ell = 0,1,2,3,4,5$

(3) For every pre-dense subset $\mathcal{I}$ of P_{i_n} from N_4 for some finite $\mathcal{J} \subseteq \mathcal{I} \cap N_4$, $\mathcal{J}$ is pre-dense in P_{i_n} above q^n_η (hence this holds for $\ell \leq 4$)

(4) $q_\eta \leq q^0_\eta$ for $\eta \in t^*$

(5) if $\nu \in t \setminus t^*, \nu \lhd \eta_1 \in t^*$, $\nu \lhd \eta_2 \in t^*$ then $q^0_{\eta_1} \restriction \alpha_\nu = q^0_{\eta_2} \restriction \alpha_\nu$

(6) $q^{n+1}_\eta \restriction i_n = q^n_\eta$ for $\eta \in t^*$

(7) $\mathrm{Dom}(q^n_\eta)$ is $i_n \cap N_5$

(B) (1) G^a_n is a generic subset of $P_{i_n} \cap N_0$ over N_0

(2) $G^a_{n+1} \cap P_{i_n} = G^a_n$

(3) $G^a_0 = G^a$

(4) $q^n_\eta \Vdash_{P_{i_n}}$ "$\underset{\sim}{G}_{P_{i_n}} \cap N_0 = G^a_n$" for $\eta \in t^*$

(C) (1) $p_n \in N_0 \cap P_\delta$

(2) $p \leq p_n \leq p_{n+1}$

(3) $p_{n+1} \in \mathcal{I}_n$

(4) $p_n \restriction i_n \in G^a_n$

(D) (1) t_n is a nonempty finite tree, $t_0 = t, t_n \subseteq_{\mathrm{end}} t_{n+1}$,

(2) $\bar{\alpha}^n = \langle \alpha^n_\eta : \eta \in t_n \rangle$,

(3) $\bar{\alpha}^n = \bar{\alpha}^{n+1} \restriction t_n, \bar{\alpha}^0 = \bar{\alpha}$, so we may write α_η for α^n_η when $\eta \in t_n$

(4) if $\eta \in t_{n+1} \setminus t_n$ then there is a $\nu_\eta \in t_n$ such that: η is an immediate successor of ν_η, $\alpha_\eta = i_{n+1}$, $\alpha_{\nu_\eta} = i_n$.

(E) (1) $\langle G^b_\eta : \eta \in t_n \rangle$ belongs to $s\mathrm{Gen}^{\bar{\alpha}^n}_{\underset{\sim}{Q} \restriction i_n}(N_1)$

(2) $G^b_\eta \in N_2$

(3) $G^c_\eta \in \mathrm{Gen}^{\langle \alpha_\eta \rangle}_{\underset{\sim}{Q} \restriction i_n}(N_2)$

(4) $t_n \models \eta < \nu$ implies $G^c_\eta \subseteq G^c_\nu$

(5) $G^c_\eta \in N_3$

(6) $G^a_{\ell g(\eta) - m^*} \subseteq G^b_\eta \subseteq G^c_\eta$ for $\eta \in (t_n \setminus t) \cup t^*$ so $\ell g(\eta) \geq m^*$, of course

(7) if $\eta \in t^*$ then $q^n_\eta \Vdash$ "for some $\rho \in t_n \setminus \bigcup\limits_{m<n} t_m$ we have: $\alpha^n_\rho = i_n, \eta \trianglelefteq \rho$ and $G^c_\rho \subseteq \underset{\sim}{G}_{P_{i_n}}$".

If we succeed, then let $r_\eta = q^0_\eta \cup \bigcup_{n<\omega} q^{n+1}_\eta \restriction [i_n, i_{n+1})$ (for $\eta \in t^*$, so $\alpha_\eta = i$ and $\beta_\eta = j = \delta$) and let $r_\eta = q_\eta$ for $\eta \in t \setminus t^*$, all are members of P_δ.

For $\eta \in t^*, r_\eta$ is (N_0, P_δ)-generic and forces $\underset{\sim}{G}_{P_\delta} \cap N = \underset{\sim}{G}_{P_\delta} \cap N_0 = \bigcup_{n<\omega} G^a_n$ and $\langle r_\eta : \eta \in t\rangle \in P_{\bar{\beta}}$. So, it is enough to carry the definition.

The case $n = 0$ is easy (better to define $\langle q^0_\eta : \eta \in t_0\rangle \in P_{\bar{\alpha}}$ by steps i.e. choose q^0_η by induction on $\ell g(\eta)$; remember $P_{\bar{\alpha}}$ is proper not adding reals as $\bar{Q}\restriction i_0$ is an $\mathrm{NNR}_{\aleph_0}$-iteration).

Let us do the induction step: defining for $n+1$.

First Step. Choose $p_{n+1} \in \mathcal{I}_n \cap N_0$ such that $p_n \le p_{n+1}$ and $p_{n+1}\restriction i_n \in G^a_n$. Straightforward.

Second Step.
First note:

$(*)_1$ the following set is a dense subset of $P_{\bar{\alpha}^n}$:

$$\begin{aligned}\mathcal{J} = \{\bar{q}' : \bar{q}' \in P_{\bar{\alpha}^n},\ &\textit{either} \text{ for some } \eta \in t_n, \alpha_\eta = i_n \text{ and}\\ &q'_\eta \Vdash_{P_{i_n}} \text{“}\underset{\sim}{G}_{P_{i_n}} \cap N_0 \ne G^a_n\text{”}\\ &\textit{or} \text{ there is a } G' \in \mathrm{Gen}_{P_{i_{n+1}}}(N_0) \text{ such that}\\ &p_{n+1}\restriction i_{n+1} \in G', G' \cap P_{i_n} = G^a_n \text{ and:}\\ &\eta \in t_n \,\&\, \alpha_\eta = i' \Rightarrow q'_\eta \Vdash_{P_{i_n}} \text{“in } P_{i_{n+1}}/P_{i_n} \text{ the set } G'\\ &\qquad\qquad \text{has an upper bound”}\}.\end{aligned}$$

This follows by $(4)_{\aleph_0}$ for $\bar{Q}\restriction i_{n+1}$ (which is an $NNR_{\aleph_0}$-iteration).
Also

$(*)_2$ there is a $\bar{q}' \in \mathcal{J}$ which belongs to $\langle G^b_\eta : \eta \in t_n\rangle$ (i.e. $\eta \in t_n \Rightarrow q'_\eta \in G^b_\eta$ and $\bar{q}' \in P_{\bar{\alpha}^n}$) such that $[\eta \in t_n \,\&\, \alpha_\eta = i_n \Rightarrow q'_\eta$ is above $G^a_n]$.
(this is as there is $\bar{q}^* \in \langle G^b_\eta : \eta \in t_n\rangle$ which belongs to $\mathcal{J}$ (as $\mathcal{J} \in N_1$ and is a dense subset of $P_{\bar{\alpha}^n}$ and $\langle G^b_\eta : \eta \in t_n\rangle$ is in $\mathrm{sGen}_{P_{\bar{\alpha}^n}}(N_1)$ and the first possibility in the definition of $\mathcal{J}$ cannot hold as $G^a_n \subseteq G^b_\eta$ whenever $\alpha_\eta = i_n$).
Now choose G^a_{n+1} satisfying: (B)(1), (B)(2) and for every $\eta \in t_n$ of length $n+m^*$ for some $q^*_\eta \in P_{i_{n+1}} \cap N_1 : q^*_\eta\restriction i_n \in G^b_\eta$ and $q^*_\eta \Vdash$ “$\underset{\sim}{G}_{P_{i_{n+1}}} \cap N_0 = G^a_{n+1}$”.
This is possible by $(*)_1 + (*)_2$.

Third Step. Let for each $\nu \in t_n$ with $\alpha_\nu = i_n$, $\langle G_{\nu,m} : m < m_\nu \rangle$ be such that: $q^n_{\nu\restriction m^*} \Vdash_{P_{i_n}}$ "*if* $G^c_\nu \subseteq \underset{\sim}{G}_{P_{i_n}}$ *then* $\underset{\sim}{G}_{P_{i_n}} \cap N_3 \in \{G_{\nu,m} : m < m_\nu\}$.
(This sequence exists and is finite by (A)(3), and as P_{i_n} adds no new reals). Now we let

$$t_{n+1} = t_n \cup \{\nu\hat{\ }\langle m\rangle : m < m_\nu, \nu \in t_n, \alpha_\nu = i_n\}.$$

and choose $\bar{\alpha}^{n+1}$ by $(D)(3)$ and $(D)(4)$.

Fourth Step. Repeating the proof of 2.4 (but choosing the appropriate forcing conditions from $G^c_\eta(\eta \in t_n, \alpha_\eta = i_n)$), we choose $\langle G^b_\eta : \eta \in t_{n+1} \setminus t_n\rangle$ and $\langle r^b_\eta : \eta \in t_{n+1} \setminus t_n\rangle$ such that: $\langle G^b_\eta : \eta \in t_{n+1}\rangle \in s\mathrm{Gen}^{\bar{\alpha}^{n+1}}_{\bar{Q}\restriction i_{n+1}}(N_1)$ and $q^*_{\eta\restriction(m^*+n)} \in G^b_\eta$ ($q^*_{\eta\restriction(m^*+n)}$ is from the end of the second step) and $r^b_\eta \in P_{i_{n+1}} \cap N_2$, which is an upper bound to G^b_η and $r^b_n\restriction i_n \in G^c_{\eta\restriction(m^*+n)}$ (just order $t_{n+1} \setminus t_n$, and then choose (G^b_η, r^b_η) by induction on η, see 2.4(2)).

Fifth Step. We choose $\langle G^c_\eta : \eta \in t_{n+1} \setminus t_n\rangle$, $\langle r^c_\nu : \nu \in t_{n+1} \setminus t_n\rangle$ satisfying (E) and $\left[\nu \in t_{n+1} \,\&\, \alpha_\nu = i_{n+1} \Rightarrow r^b_\nu \in G^c_\nu\right]$ and for $\eta \in t_{n+1} \setminus t_n$, $\eta = \nu\hat{\ }\langle m\rangle$, we have $r^c_\eta \in P_{i_{n+1}} \cap N_3$, $r^c_\eta\restriction i_n \in G_{\nu,m}$, r^c_η a bound of G^c_η; this is possible as in the proof of the preservation of properness.

Sixth Step. We choose $\langle q^{n+1}_\eta : \eta \in t \setminus t_{-1}$ so $\alpha_\eta = i\rangle$ by Claim 2.6 making sure that $\{r^c_\nu : \nu \in t_{n+1} \setminus t_n, \eta \lhd \nu\}$ is pre-dense above q^{n+1}_η, this to guarantee (E)(7); do it for each such η separately.

So, we have finished the induction step, hence the proof of $(4)_{\aleph_0}$. Hence, the proof of the Main Lemma. $\square_{2.10B}$

2.10C Proof of Condition $(4)_2$ When we are Dealing with NNR_2

We mix the proof of VIII, §4 and the previous proof †.

So let $\chi, N, i^*, i, j, G^a, p, q_0, q_1$ are as there. By the assumption w.l.o.g. $j = \delta$. Let $\chi_1 = (2^\chi)^+, t = \{\langle\rangle, \langle 0\rangle, \langle 1\rangle\} \subseteq {}^{\omega>}\omega, \alpha_{<>} = i^*, \alpha_{<0>} = \alpha_{<1>} =$

† The readers who are happy to have the details should thank Lee Stanley for his advice.

$i, q_{<>} = q_0 \restriction i^* (= q_1 \restriction i^*)$ and $q_{<0>} = q_0, q_{<1>} = q_1$, and $\bar{q} = \langle q_\eta : \eta \in t \rangle$, stipulate $t_{-1} = \{\langle\rangle\}$.

Choose $N_\ell (\ell = 0, 1, 2, 3, 4, 5)$ such that:

(α) every N_ℓ is countable, $N_0 = N$,

(β) $N_\ell \subseteq N_{\ell+1} \prec (H(\chi_1), \in, <^*_{\chi_1})$ for $\ell = 0, 1, 2, 3, 4$,

(γ) $N_1 \cap \lambda_1 \in \mathcal{E}_2$, $N_2 \cap \lambda_2 \in \mathcal{E}_2$, $N_3 \cap \lambda_0 \in \mathcal{E}_0$, $N_4 \cap \lambda_0 \in \mathcal{E}_0$, $N_5 \cap \lambda_0 \in \mathcal{E}_0$,

(δ) $N_\ell \in N_{\ell+1}$ for $\ell = 0, 1, 2, 3, 4$,

(ε) $\bar{q} \in N_1$, $G^a \in N_1$.

Let $i = i_0 < i_1 < i_2 < \ldots < i_n < \ldots (n < \omega)$ be such that: each i_n belong to $N_0 \cap \delta$, is a non-limit ordinal and

$$\sup(\delta \cap N_0) = \sup\{i_n : n < \omega\}.$$

Let $\langle \mathcal{I}_n : n < \omega \rangle \in N_1$ be a list of the dense subsets of P_δ which belong to N_0. Let $t^* = t \cap {}^1\omega$.

Now we define by induction on $n < \omega$, $k_n \in \omega, \langle M_k : k \leq k_n \rangle$, p_n, $q^n_\eta (\eta \in t^*)$, $G^a_n, t_n, \bar{\alpha}^n, s_k, \bar{\beta}^{n,k}, h_k, h^n_k (k \leq k_n), G^b_\eta, G^c_\eta$ (for $\eta \in t_n$) such that:

(A) (1) $q^n_\eta \in P_{i_n} (\eta \in t^*)$

(2) q^n_η is (N_ℓ, P_{i_n})-generic for $\ell = 0, 1, 2, 3, 4, 5$

(3) For every pre-dense subset $\mathcal{I}$ of P_{i_n} from N_4, for some finite $\mathcal{J} \subseteq \mathcal{I} \cap N_4, \mathcal{J}$ is pre-dense in P_{i_n} over q^n_η (hence this holds for $\ell \leq 4$)

(4) $q_\eta \leq q^0_\eta$ for $\eta \in t^*$

(5) $q^0_{\langle 0 \rangle} \restriction \alpha_{\langle\rangle} = q^0_{\langle 1 \rangle} \restriction \alpha_{\langle\rangle}$

(6) $q^{n+1}_\eta \restriction i_n = q^n_\eta$

(7) $\mathrm{Dom}(q^n_\eta)$ is $i_n \cap N_5$

(B) (1) G^a_n is a generic subset of $P_{i_n} \cap N_0$ over N_0

(2) $G^a_{n+1} \cap P_{i_n} = G^a_n$

(3) $G^a_0 = G^a$

(4) $q^n_\eta \Vdash_{P_{i_n}}$ "$\underset{\sim}{G}_{P_{i_n}} \cap N_0 = G^a_n$" (for $\eta \in t^*$).

(C) (1) $p_n \in N_0 \cap P_\delta$

(2) $p \leq p_n \leq p_{n+1}$

(3) $p_{n+1} \in \mathcal{I}_n$ for $\eta \in t^*$.

(4) $p_n \restriction i_n \in G^a_n$

(D) (1) t_n is a nonempty finite tree, $t_0 = t, t_n \subseteq_{\text{end}} t_{n+1}$,

(2) $\bar{\alpha}^n = \langle \alpha^n_\eta : \eta \in t_n \rangle$,

(3) $\bar{\alpha}^n = \bar{\alpha}^{n+1} \restriction t_n, \bar{\alpha}^0 = \bar{\alpha}$, so we may write α_η for α^n_η

(4) if $\eta \in t_{n+1} \setminus t_n$ then there is $\nu_\eta \in t_n$ such that: η is an immediate successor of ν_η, $\alpha_\eta = i_{n+1}, \alpha_{\nu_\eta} = i_n$

(E) (1) $\langle G^b_\eta : \eta \in t_n \rangle$ belongs to $\mathrm{sGen}^{\bar{\alpha}^n}_{\bar{Q} \restriction i_n}(N_1)$

(2) $G^b_\eta \in N_2$

(3) $G^c_\eta \in \mathrm{Gen}^{<\alpha_\eta>}_{\bar{Q} \restriction i_n}(N_2)$ for $\eta \in t^*$

(4) $t_n \models \eta < \nu$ implies $G^c_\eta \subseteq G^c_\nu$

(5) $G^c_\eta \in N_3$

(6) $G^a_{\ell g(\eta)-1} \subseteq G^b_\eta \subseteq G^c_\eta$ for $\eta \in t_n(\ell g(\eta) \geq 1$, of course)

(7) if $\eta \in t^*$ then

$$q^n_\eta \Vdash_{P_{i_n}} \text{“for some } \rho \in t_n \setminus \{\langle\rangle\} \text{ we have: } \alpha^n_\rho = i_n \text{ and } G^c_\rho \subseteq \mathop{G}\limits_{\sim}{}_{P_{i_n}}\text{”}$$

(we can demand it is a P_{i_n}-name $\underset{\sim}{\rho}_n$ and $\underset{\sim}{\rho}_n \lhd \underset{\sim}{\rho}_{n+1}$).

(F) (1) $M_0 = N_0$

(2) $M_k \prec M_{k+1} \prec (H(\chi), \in, <^*_\chi)$ for $k < k_n$

(3) M_k is countable,

(4) $M_k \in M_{k+1}$

(5) $M_k \in N_1$

(6) $k_n < k_{n+1} < \omega, k_0 = 1$ (stipulate $k_{-1} = -1$)

(G)(1) $s_0 = \{<>\}, s_1 = t$

(2) if $k_n < k \leq k_{n+1}$ then $s_k = s_{k_n} \cup \{\nu\hat{\ }\langle \ell \rangle : \ell < 2^{k-k_n}, \nu \in s_{k_n}$, $\ell\mathrm{g}(\nu) = n+1\}$

(3) for $k_n \leq k < k_{n+1}$, we define h_k, a function with domain s_{k+1} and range s_k: $h_k \restriction s_{k_n}$ = identity, and for $\nu\hat{\ }\langle \ell \rangle \in s_{k+1} \setminus s_{k_n}$ $h_k(\nu\hat{\ }\langle \ell \rangle) = \nu\hat{\ }\langle [\ell/2] \rangle$

(4) $\bar{\beta}^{n,k} = \langle \beta^{n,k}_\nu : \nu \in s_{k_n}, k \leq k_n \rangle$ is defined as follows: $\beta^{n,0}_{<>} = i_n$ (remember $s_0 = \{\langle\rangle\}$) and if $k > 0, \beta^{n,k}_\nu$ is i^* if $\ell\mathrm{g}(\nu) = 0, i_{\ell\mathrm{g}(\nu)-1}$

if $0 < \ell g(\nu)$ but ν is not maximal in s_k, and finally i_n if ν is maximal in s_{k_n}

(5) $s_{k_n} = t_n$

(6) For k such that $k_n \le k \le k_{n+1}$, h^n_k is the function with domain $s_{k_{n+1}} = t_{n+1}$ to s_k, $h^n_k(\eta) = h_k \circ h_{k+1} \circ h_{k_{n+1}-1}(\eta)$ (and $h^n_{k_{n+1}} = \mathrm{id}_{t_{n+1}}$), also if $k \le k_n$, h^n_k is defined by the downward induction on m as $h^m_k \circ h^n_{k_{m+1}}$ where $k_m \le k \le k_{m+1}$ (no incompatibility).

(H)(1) if $\nu, \eta \in t_n = s_{k_n}$, $k \le k_{n+1}$ and $h^n_k(\eta) = h^n_k(\nu)$ (both well defined) then $G^b_\nu \cap M_k = G^b_\eta \cap M_k$, and we denote this value by $G^{b,n,k}_{h^n_k(\nu)}$,

(2) $\langle G^{b,n,k}_\rho : \rho \in s_k \rangle \in s\mathrm{Gen}^{\bar\beta^{n,k}}_{\bar Q}(M_k)$ and it belongs to M_{k+1} (and to N_1).

If we succeed, then let $r_\eta = q^0_\eta \cup \bigcup_{n<\omega} q^{n+1}_\eta {\restriction} [i_n, i_{n+1})$ (for $\eta \in t^*$, so $\alpha_\eta = i$) and let $r_\eta = q_\eta$ for $\eta \in t \setminus t^*$, they are members of P_δ. For $\eta \in t^*$, r_η is (N_0, P_δ)-generic and forces $\underset{\sim}{G}_{P_\delta} \cap N = \underset{\sim}{G}_{P_\delta} \cap N_0 = \bigcup_{n<\omega} G^a_n$ (remember (C)(1)–(4)), and $\langle r_\eta : \eta \in t \rangle \in P_{\bar\beta}$. Here, $\bar\beta$ is as in Definition 2.3(i). So, it is enough to carry the definition.

The case $n = 0$ is easy (better to define $\langle q^n_\eta : \eta \in t_0 \rangle \in P_{\bar\alpha}$ by steps).

Let us do the induction step: defining for $n + 1$.

First Step. Choose $p_{n+1} \in \mathcal{I}_n \cap N_0$ such that $p_n \le p_{n+1}$ and $p_{n+1} {\restriction} i_n \in G^a_n$.

Second Step.

First Note:

$(*)_1$ the following set is a dense subset of $P_{\bar\beta^{n,1}}$:

$$\begin{aligned}\mathcal{J} = \{\bar q' : \ & \bar q' \in P_{\bar\beta^{n,1}} \text{ and } \textit{either} \text{ for some } \eta \in s_1, \beta^{n,1}_\eta = i_n \text{ and} \\ & q'_\eta \Vdash_{P_{i_n}} \text{“}\underset{\sim}{G}_{P_{i_n}} \cap N_0 \ne G^a_n\text{”} \\ & \textit{or there} \text{ is a } G' \in \mathrm{Gen}_{P_{i_{n+1}}}(N_0) \text{ such that:} \\ & p_{n+1} {\restriction} i_{n+1} \in G' \cap P_{i_n} = G^a_n \text{ and:} \\ & \eta \in s_1 \,\&\, \beta^{n,1}_\eta = i_n \Rightarrow q'_\eta \Vdash_{P_{i_n}} \text{“ in } P_{i_{n+1}}/P_{i_n} \text{ the set } G' \\ & \text{has an upper bound”}\}.\end{aligned}$$

This follows by $(4)_2$ for $\bar{Q}\restriction i_{n+1}$ (which is an NNR_2-iteration).
Also

$(*)_2$ there is a $\bar{q}' \in \langle G^b_\eta : \eta \in s_1\rangle$ (i.e. $\bigwedge_{\eta\in s_1} q'_\eta \in G^{b,n,1}_\eta$) such that $[\eta \in s_1 \,\&\, \beta^{n,1}_\eta = i_n \Rightarrow q'_\eta$ is above $G^a_n]$ and $\bar{q}' \in \mathcal{J}$.
(This is as $\langle G^{b,n,1}_\eta : \eta \in s_1\rangle$ is in $\mathrm{sGen}_{P_{\bar{\beta}^{n,1}}}(M_1)$ and $G^a_n \subseteq G^{b,n,1}_\eta$ whenever $\beta^{n,1}_\eta = i_n$).

Now choose G^a_{n+1} satisfying: (B)(1), (B)(2) and for every $\eta \in s_1$ with $\beta^{n,1}_\eta = i_n$ for some $q^{n,0}_\eta \in P_{i_{n+1}} \cap M_1$ we have: $q^{n,0}_\eta\restriction i_n \in G^{b,n,1}_\eta$ and $q^{n,0}_\eta \Vdash$ "$\underset{\sim}{G}_{P_{i_{n+1}}} \cap N_0 = G^a_{n+1}$".

This is possible by $(*)_1 + (*)_2$.

Third Step. Let for each $\nu \in t_n (= s_{k_n})$ with $\beta^{n,k_n}_\nu = i_n$, $\langle G_{\nu,m} : m < m_\nu\rangle$ be such that (on q^n_ν see (A)(1), (2), (3)):
$q^n_{\nu\restriction 1} \Vdash_{P_{i_n}}$ "*if* $G^c_\nu \subseteq \underset{\sim}{G}_{P_{i_n}}$ then $\underset{\sim}{G}_{P_{i_n}} \cap N_3 \in \{G_{\nu,m} : m < m_\nu\}$".
(This sequence exists and is finite by (A)(3) and as P_{i_n} adds no new reals). W.l.o.g. m_ν is a power of 2, $m_\nu = 2^{n_\nu}$, and does not depend on ν, and let k_{n+1} be such that $k_{n+1} - k_n = 2^{m_\nu}$ for any such ν. So $s_k, k_n < k \le k_{n+1}$ and t_{n+1} are well defined. Now we can choose appropriate $M_k (k_n < k \le k_{n+1})$ such that†: $M_k \prec N_1\restriction H(\chi), M_k \in N_1$, $M_{k-1} \prec M_k$, $M_{k-1} \in M_k$, $M_k[\langle G^b_\eta : \eta \in t_n\rangle] \prec (N_1\restriction H(\chi))[\langle G^b_\eta : \eta \in t_n\rangle]$. Why can we choose such M_k's? By (E)(1), $\langle G^b_\eta : \eta \in t_n\rangle \in \mathrm{sGen}^{\bar{\alpha}^n}_{\bar{Q}\restriction i_n}(N_1)$, and $P_{\bar{\alpha}^n}$ is $\mathcal{E}_0$-proper. Let $G^{b,n,k}_\eta = G^b_\eta \cap M_k$ for $\eta \in t_n$. Also $\beta^{n+1,k}(k \le k_{n+1})$ and $s_k(k \le k_{n+1})$ are well defined now. Now we define by induction on $k = 0, \dots, k_{n+1}$, a condition $q^{n,k}_\eta(\eta \in s_k \,\&\, \beta^{n+1,k}_\eta = i_{n+1})$ and $\langle G^{b,n+1,k}_\eta : \eta \in s_k\rangle \in \mathrm{sGen}^{\bar{\beta}^{n+1,k}}_{\bar{Q}}(M_k)$ such

† Remember $\chi_1 = (2^\chi)^+$ and $N_\ell \prec (H(\chi_1), \in, <^*_{\chi_1})$ for $\ell = 1, 2, 3, 4, 5$.

that

$$(a)\ k \leq k_{n+1} \& \eta \in s_k \& \beta_\eta^{n,k} < i_{n+1} \Rightarrow G_\eta^{b,n+1,k} = G_\eta^{b,n,k}$$

$$(b)\ q_\eta^{n,k} \in M_{k+1} \text{ for } \eta \in s_{k_{n+1}} (\text{if } k = k_{n+1} \text{ then } q_n^{n,k} \in N_1)$$

$$(c)\ q_\eta^{n,k} \restriction i_n \in G_\eta^{b,n,k} \text{ when } \eta \in s_{k_{n+1}} \& \beta_\eta^{n,k} = i.$$

$$(d)\ q_\eta^{n,k} \in G_\eta^{b,n+1,k+1}$$

$$(e)\ G_{h_k^n(\eta)}^{b,n+1,k} \subseteq G_\eta^{b,n+1,k} \text{ for } \eta \in s_{n_{k+1}}.$$

For $k = 0$ $q_\eta^{n,0}$ was already defined and let $G_{\langle\rangle}^{b,n+1,0} = G_{n+1}^a$-see second step. For $k+1$ we repeat the proof.

Fourth Step. Repeating the proof of 2.4 (but, choosing the appropriate forcing conditions from $G_\eta^c(\eta \in t_n \setminus \{\langle\rangle\}, \alpha_\eta = i_n))$, we choose $\langle G_\eta^b : \eta \in t_{n+1} \setminus t_n \rangle$ and $\langle r_\eta^b : \eta \in t_{n+1} \setminus t_n \rangle$ such that: $q_{\eta\restriction(1+n)}^{n,k_{n+1}} \in G_\eta^b$ and $r_\eta^b \in P_{i_{n+1}} \cap N_2$, which is an upper bound to G_η^b and $r_n^b \restriction i_n \in G_{\eta\restriction(1+n)}^c$ (just order $t_{n+1} \setminus t_n$, and then choose (G_η^b, t_η^b) by induction on η see 2.4(2)).

Fifth Step. We choose $\langle G_\eta^c : \eta \in t_{n+1} \rangle$, $\langle r_{r_\nu}^c : \nu \in t_{n+1} \setminus t_n \rangle$ satisfying (E) and $[\nu \in t_{n+1} \& \alpha_\nu = i_{n+1} \Rightarrow r_\nu^b \in G_\nu^c]$ and $r_\eta^c \in P_{i_{n+1}} \cap N_3$, $r_\eta \restriction i_n \in G_{\eta\restriction(1+n), \eta(1+n)}, r_\eta^c$ a bound of G_η^c; this is possible as in the proof of the preservation of properness.

Sixth Step. We choose $\langle q_\eta : \eta \in t \setminus \{0\}, \alpha_\eta = i \rangle$ by Claim 2.6 for each such η separately taking care that $\{r_\nu^c : \nu \in t_{n+1} \setminus t_n, \eta \lhd \nu\}$ is pre-dense above q_η^{n+1} (this will guarantee (E)(7)).

So, we have finished the induction step hence the proof of $(4)_2$. Hence, the proof of the Main Lemma also for $x = 2$. $\square_{2.10C}$

2.11 Claim. If $\bar{Q}$ has length $\alpha + 1$, $\bar{Q} \restriction \alpha$ is an NNR_x-iteration for $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$, $\Vdash_{P_\alpha}$ "Q_α is strongly proper, and condition $(4)_x$ holds for $i = \alpha$, $j = \alpha + 1$" *then* $\bar{Q}$ is an NNR_x-iteration for $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$.

Proof. Straight.

Now we can phrase various conclusions on sufficient conditions for the limit of a CS iteration not to add reals.

2.12 Conclusion. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \le \alpha, j < \alpha \rangle$ is a countable support iteration of strongly proper forcing satisfying $(*)$ defined below. *Then* we can conclude that forcing with P_α adds no reals (hence, being proper, no new ω-sequences of ordinals, and in fact $\bar{Q}$ is an NNR_2-iteration) where

$(*)$ If $i_0 < i_1 < \alpha$ then $(*)^{i_0,i_1,i_1+1}_{\bar{Q}}$ holds, where we let

$(*)^{i_0,i_1,i_2}_{\bar{Q}}$ $i_0 < i_1 < i_2 \le \alpha = \ell g(\bar{Q})$ and in $V^{P_{i_0}}$: if $N \prec (H(\chi), \in, <^*_\chi)$ is countable, $\langle \bar{Q}, i_0, i_1, i_2 \rangle \in N$, $p \in [P_{i_2}/P_{i_0}] \cap N$, $q', q'' \in P_{i_1}/P_{i_0}$ are $(N[G_{P_{i_0}}], P_{i_1}/P_{i_0})$-generic, $p \restriction i_1 \le q'_1$, $p \restriction i_1 \le q''$ and q', q'' force $\underset{\sim}{G}_{P_{i_1}/P_{i_0}} \cap N = G^1$, *then* for some $(N[G_{P_{i_0}}], P_{i_2}/P_{i_0})$-generic r', $r'' \in P_{i_2}/P_{i_0}$ we have: $p \le r', p \le r''$, $q' \le r', q'' \le r''$ and r', r'' force $(\underset{\sim}{G}_{P_{i_2}}/P_{i_0}) \cap N = G^r$ for some G^r.

Proof. Straight.

2.13 Claim. 1) A sufficient condition for $(*)$ from 2.12 is that each $\underset{\sim}{Q}_i$ is $(\mathbb{D}, \mathcal{E})$-complete for some simple 2-completeness system (see VIII, 4.2, 4.4).

2) We can in 2.11, 2.12 replace strongly proper by:

$\otimes$ "proper not adding reals even after forcing by any proper forcing notion not adding reals."

(3) If $V \models$ CH, κ supercompact with Laver diamond *then* for some proper forcing P not adding reals, of cardinality κ, satisfying the κ-c.c., in V^{P_κ} we have $\aleph_1 = \aleph_1^V$, $\aleph_2 = \kappa$, $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ of course and:

$\mathrm{Ax}_{\omega_1}[\mathrm{Pr}(Q)]$ where $\mathrm{Pr}(Q)$ means:

(A) forcing with Q does not add reals

(B) part (A) holds even in a larger universe which has the same reals gotten by a proper forcing

(C) the forcing notion Q is proper and for some simple 2-completeness system $\mathbb{D}$ (or, even a $\aleph_1$-completeness system) Q is $\mathbb{D}$-complete.

2.14 Remark. 1) Part 3 is a specific case, of course.

We can now conclude the consistency of appropriate other axioms (see Ch. VIII).

2) We can now solve the problems from the end of §1.

2.15 Definition. 1) A finite tree t is simple if it has a root(= a minimal member) and all maximal $\eta \in t$ are from the same level (the level of η in t is $\ell g \eta \stackrel{\text{def}}{=} |\{\nu : \nu < \eta\}|$). t is called standard if $t \subseteq {}^{\omega >}\omega$ is closed under initial segments, the order being $\triangleleft$. Let $\max(t)$ be the set of maximal members of t.
2) If $\bar{\varepsilon}$ is a finite non-decreasing sequence of ordinals, $n = \ell g \bar{\varepsilon}$, t a simple finite tree with n levels then $\bar{\alpha}_{t,\bar{\varepsilon}} = \langle \alpha_\eta : \eta \in t \rangle$ where $\alpha_\eta = \varepsilon_{\ell g \eta}$.

2.16 Theorem. Suppose $\mathcal{E} \subseteq \mathcal{S}_{\leq \aleph_0}(\lambda)$ is stationary, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha^*, j < \alpha^* \rangle$ a CS iteration, and for each $\alpha < \alpha^*$, $(*)^{\alpha,\alpha+1}_{\bar{Q},\mathcal{E}}$ holds (see below), *then* forcing with P_{α^*} adds no reals, where for $\beta < \gamma \leq \alpha^*$ we define:

$(*)^{\beta,\gamma}_{\bar{Q},\mathcal{E}}$ Assume

(a) $k < \omega, n < \omega, \bar{\varepsilon} = \langle \varepsilon_0, \ldots, \varepsilon_{n-1} \rangle, \varepsilon_0 < \ldots < \varepsilon_{n-1} \leq \beta$,
$m_i < \omega$ for $i < n$,
t a standard simple tree with n levels,
$t^*_\ell = t \cup \{\eta \hat{\ } \langle i \rangle : i < 2^\ell, \eta \in \max(t)\}$

$$h_\ell : t_{\ell+1} \to t_\ell \text{ is } h(\nu) = \begin{cases} \nu & \text{if } \nu \in t \\ \eta \hat{\ } \langle [i/2] \rangle & \nu = \eta \hat{\ } \langle i \rangle, \eta \in \max(t) \end{cases}$$

and let $h = h_k$, $t_0 = t^*_k$, $t_1 = t^*_{k+1}$.
If $\bar{q} = \langle q_\eta : \eta \in t_0 \rangle$, let $\bar{q}^h = \langle q_{h(\eta)} : \eta \in t_1 \rangle$

(b) $N \prec (H(\chi), \in, <^*_\chi)$ is countable, $\bar{Q}, \lambda_0, \bar{\epsilon}, \beta, \gamma \in N$ and $N \cap \lambda_0 \in \mathcal{E}_0$, while $\beta \leq \gamma \leq \alpha^*$.

(c) $G_0 \subseteq P_{\bar{\alpha}_{t_0, \bar{\varepsilon} \hat{\ } \langle \beta \rangle}} \cap N$ is generic over N, (so we may write $G_0 = \langle G^0_\eta : \eta \in t_0 \rangle$).

(d) $\bar{p} \in N \cap P_{\bar{\alpha}_{t_0, \bar{\varepsilon} \hat{\ } \langle \gamma \rangle}}$ is compatible with G^0 (note $P_{\bar{\alpha}_{t_0, \bar{\varepsilon} \hat{\ } \langle \beta \rangle}} \subseteq P_{\alpha_{t_0, \bar{\varepsilon} \hat{\ } \langle \gamma \rangle}}$, so this means $\bigwedge_{\eta \in t_0} p_\eta \restriction \beta \in G^0_\eta$).

(e) $\bar{q} \in P_{\bar{\alpha}_{t_1, \bar{\varepsilon} \hat{\ } \langle \beta \rangle}}$ such that it is above G^h_0 i.e., $\bar{r} \in G_0 \Rightarrow \bar{r}^h \leq q$.

Then we can find $G_1, \bar{r}$, such that

(α) $G_1 \subseteq P_{\bar{\alpha}_{t_0}, \bar{\varepsilon} ^\frown \langle\gamma\rangle} \cap N$ is generic over N

(β) $\bar{p} \in G_1$.

(γ) $G_0 \subseteq G_1$ (see remark in (d)).

(δ) $\bar{r} \in P_{\bar{\alpha}_{t_1}, \bar{\varepsilon} ^\frown \langle\gamma\rangle}$, $q \leq \bar{r}$.

(ε) $\bar{r}$ is above G_1^h.

Proof. We prove by induction on $\alpha \leq \ell g(\bar{Q})$ that for every $\beta < \gamma \leq \alpha$,
$(*)^{\beta,\gamma}_{\bar{Q},\mathcal{E}}$
(in particular that $\bar{Q}\restriction\alpha$ is a CS iteration of $\mathcal{E}$-proper forcing). The main point is the case $\gamma = \alpha$ is a limit ordinal whose proof is similar to the proof in 2.10C. $\square_{2.16}$

§3. Other Preservations

A central theme in this book is that it is worthwhile to have general preservation theorems on iterated forcing. While it seems that this is reasonably accepted in the community for properness, this seemingly is not so for preservation theorems like "proper+${}^{\omega}\omega$-bounding" and even less for a general framework for them. So here we try another way to materialize the theme (in 3.1-3.6). We then present several applications (but, generally, we do not repeat VI). A simple case of our framework is [Sh:326, A 2.6(3), pp.397-9]

This section passed through several versions, e.g. in most of them the proof of 3.6 was left to the reader. Goldstern [Go] starts from an earlier one, he cuts the generality for the sake of completeness. Relative to the present version he restricts himself to the case A and $\alpha^* = \omega$, in Definition 3.4 omit demand (xi) ((x) irrelevant) and demand it adds reals. Also $R_n \subseteq R_{n+1}$ and he omits S and $\mathbf{g}$ (so uses $({}^{\omega}\omega)^V$ as a cover: a $\mathbf{g}_{a\cap V}$ is chosen in the proof.).
Lately we added the proof of 3.6 (and added 3.4B, 3.13) and in some of the cases (i.e. when $d[a] \in a$, $\alpha^* > 1$ and we are not in Possibility $C(C^*)$) we added the condition $\oplus_k$ (or $\oplus_1$).

3.1 Context. $S \subseteq \mathcal{S}_{<\aleph_1}(A)$ for some $A = \bigcup S$ (usually S is stationary). For $a \in S$ $c[a], d[a]$ are subsets of a, and there are $c'[a], d'[a]$ defined such that:

Case a: if $d[a] \in a$ then $c'[a] = c[a]$, $d'[a] = d[a]$.

Case b: if $d[a] \notin a$ then $c[a] \notin a$, $c[a] = c'[a] \cap a$, $d[a] = d'[a] \cap a$.

Advise to the reader: At first reading the reader may think of a typical case: $\chi_0 << \chi$, $A = H(\chi_0)$, and elements of S are of the form $N \cap H(\chi_0)$, for some $N \prec (H(\chi), \in, <^*_\chi)$ such that $\chi_0 \in N$, all in the original universe V_0. A typical case for $d[a] \notin a$ would be $d[a] = a$, or $d[a] = a \cap \omega_1$, and below (in Definition 3.2) choose one possibility, say (B).

In addition we have $\mathbf{g} = \langle \mathbf{g}_a : a \in S \rangle$ where $\mathbf{g}_a$ is a function from $d[a]$ to $c[a]$ and α^* is an ordinal > 0.

The set $\bigcup S$ is, for simplicity, transitive, $\bar{R}$ is a three place relation, (more exactly a definition of one) written as $fR_\alpha g$, and whenever $fR_\alpha g$, for some $a \in S$ we have: $\alpha \in \alpha^* \cap a$ and f, g are functions from $d[a]$ to $c[a]$; for notational simplicity $\Big[d[a] \in a \Leftrightarrow c[a] \in a\Big]$ and $(\forall a \in S)[d[a] \in a]$ or $(\forall a \in S)[d[a] \notin a]$; and $d'[a], c'[a] \in a$ (of course $d'[a] \cap a = d[a]$, $c'[a] \cap a = c[a]$), and $\pm R_\alpha$ is absolute (enough to restrict to extension by forcings e.g. by proper forcing). Generally, saying absolutely or in any generic extension V^Q, we may mean for generic extensions by proper forcing, or any other property preserved by the iterations to which we want to apply this section.

3.2 Definition. 1) We say $(\bar{R}, S, \mathbf{g})$ covers (in V) *if* for χ large enough, for every $x \in H(\chi)^V$ there is a countable $N \prec (H(\chi), \in, <^*_\chi)$ to which $(\bar{R}, S, \mathbf{g})$ and x belong, and N is $(\bar{R}, S, \mathbf{g})$-good, which means:
$a \stackrel{\text{def}}{=} N \cap (\bigcup S)$ belongs to S, (so $\{d'[a], c'[a]\} \in N$) and: *for every* function $f \in N$ such that f maps $d[a]$ into $c[a]$, (so $d[a] \subseteq \mathrm{Dom}(f)$ but not necessarily $\mathrm{Dom}(f) \subseteq d[a]$) *for some* $\beta \in \alpha^* \cap a$, we have $(f \restriction d[a]) R_\beta \mathbf{g}_a$, the most natural case is: f a function from $d'[a]$ to $c'[a]$.
2) We say $(\bar{R}, S, \mathbf{g})$ fully covers (in V) if: the above holds for every countable $N \prec (H(\chi), \in, <^*_\chi)$ to which $(\bar{R}, S, \mathbf{g})$ and x belong and $N \cap (\bigcup S) \in S$ and in addition S is stationary.

3) We say $(\bar{R}, S, \mathbf{g})$ weakly covers if $d'[a] = d$, $c'[a] = c$ for every $a \in S$ (so c, d are constants, for example ω) and for every $f \in {}^d c$ for some a, α we have $f R_\alpha \mathbf{g}_a$.

3.2A Remark.

1) Actually, if the function $a \mapsto \mathbf{g}_a$ is one to one, then we can omit α and write $f R \mathbf{g}_a$ where R is defined by $f R \mathbf{g}$ iff $(\exists a \in S)[\mathbf{g} = \mathbf{g}_a \ \& \ \bigvee_{\alpha \in a \cap \alpha^*} f R_\alpha \mathbf{g}_a]$; the notation above is just more natural in the applications we have in mind.
2) Of course, in Definition 3.2, x is not necessary.
3) If $V^1 \subseteq V^2 \subseteq V^3$ are universes, $(\bar{R}^1, S^1, \mathbf{g}^1) \in V^1$ weakly covers in V^2 and $(\bar{R}^2, S^2, \mathbf{g}^2) \in V^2$ weakly covers in $V^3, \ell g \bar{R}^1, \ell g \bar{R}^2 < \omega_1$ and $\bigvee_{\alpha \in a} R^\ell_\alpha$ have the same definition for all $\ell = 1, 2$ and $a \in S^\ell$ (which is absolute for the cases of extension) and are partial orders and S^1 is a stationary subset of $\mathcal{S}_{<\aleph_1}(\bigcup S^1)$ even in V^3 *then* $(\bar{R}, S^1, \mathbf{g}^1)$ weakly covers in V^3.
4) We can translate an instance of Case a (in 3.1) to an instance of Case b, by replacing $d[a]$ by a and replacing $f \in {}^{d[a]}c[a]$ by a function $f^{[a]}$ where the function $f^{[a]}$ is $f \cup 0_{a \setminus d[a]}$, for example. This may help to apply e.g. 3.3. Possibility A, the case $a \in S \Rightarrow d[a] \notin a$ *but* has a price: $d[a] \notin a$ makes Definition 3.4 stronger, as the assumption becomes weaker (see clauses (vii)+(ix)), though we add the assumption in clause (x) so really there is no clear order.

3.3 Definition.; We say $(\bar{R}, S, \mathbf{g})$ strongly covers if (it is as in 3.1 and) it covers (in V, see Definition 3.2(1)) and one of the following possibilities holds:
Possibility A: Each R_α is closed (2-place relation on ${}^{d[a]}c[a]$)† (note that if R_α is open then $R = \bigcup_{n<\omega} R_{\alpha,n}$ where each $R_{\alpha,n}$ is closed, hence this possibility applies replacing α^* by $\omega\alpha^*$, using $R'_{\omega\alpha+n} = R_{\alpha,n}$) and: $[a \in S \Rightarrow d[a] \notin a]$ or $\alpha^* = 1$ or $\oplus_k$ for every $k < \omega$, which means††

† It is enough that each $\{f : f R_\alpha \mathbf{g}_a\}$ is closed.
†† Instead of the forcing notion P we can just demand that this holds absolutely.

$\oplus_k$ *if*

(a) P is a proper forcing notion preserving "$(\bar{R}, S, \mathbf{g})$-covers" and in V^P, $\underset{\sim}{Q}$ is a proper forcing in V^P [or just P, $\underset{\sim}{Q}$ are P_i, $\underset{\sim}{Q}_i$ as we get in our iterations]

(b) in V^P, $N \prec (H(\chi), \in, <^*_\chi)^{V^P}$ is countable, and $(\bar{R}, S, \mathbf{g})$-good (so in particular $(\bar{R}, S, \mathbf{g}) \in N$, $a = N \cap \bigcup S \in S$) and $\underset{\sim}{Q} \in N$, $p \in \underset{\sim}{Q} \cap N$

(c) for each $\ell < k$ we have: $\underset{\sim}{f}_\ell \in N$ is a Q-name of a member of ${}^{d'[a]}c'[a]$,

(d) $\chi_1 < \chi$ (χ_1 large enough e.g. $(P, \underset{\sim}{Q}) \in H(\chi_1)$ but $2^{\chi_1} < \chi$), $N_1 \prec (H(\chi_1), \in, <^*_{\chi_1})$ is countable, $N_1 \in N$, $\{Q, p, \bar{R}, S, \mathbf{g}, \underset{\sim}{f}_\ell\} \in N_1$, $p \in G^1 \in \mathrm{Gen}(N_1, Q) \cap N$.

(e) $\beta_\ell \in a \cap \alpha^*$ and $\underset{\sim}{f}_\ell \restriction d[a_1][G^1] R_{\beta_\ell} \mathbf{g}_a$

then for any $y \in N \cap H(\chi_1)$ there are N_2, G_2 satisfying (the parallel of) clause (d), such that $y \in N_2$ and: for some $\gamma_\ell \in a$, $\gamma_\ell \leq \beta_\ell$ (for $\ell < k$) we have $\underset{\sim}{f}_\ell[G_2] R_{\gamma_\ell} \mathbf{g}_a$ (for $\ell < k$).

Also instead of $\oplus_k$ we can require:

$\oplus'_k$ *if* in some (e.g. proper) forcing extension, N is $(\bar{R}, S, \mathbf{g})$-good, $N \cap \bigcup S = a \in S$, $k < \omega$, for $\ell < k$ we have $f^*_\ell R_{\beta_\ell} \mathbf{g}_a$ (where $\beta_\ell \in a \cap \alpha^*$), $\langle f^*_{\ell,n} : n < \omega \rangle$ converge to f^*_ℓ (i.e. $f^*_{\ell,n} \in {}^{d[a]}c[a]$, $\forall x \in d[a] \exists m \forall n > m [f^*_{\ell,n}(x) = f^*_\ell(x)]$) and $\langle f^*_{\ell,n} : n < \omega \rangle$, $f^*_\ell \in N$ *then* for some $\gamma_\ell \leq \beta_\ell, \gamma_\ell \in a$ we have $\bigvee_{n<\omega} \bigwedge_{\ell<\kappa} f^*_{\ell,n} R_{\gamma_\ell} \mathbf{g}_a$

<u>Remark:</u>

1) we can specify how $f^*_\ell, f^*_{\ell,n}$ come from N_1, (see the proof of 3.7E) (possibly in some V^Q, Q $(\bar{R}, S, \mathbf{g})$-preserving). This is close to VI §1 (if $\eta \in {}^\omega\omega$, $\eta_n \in {}^\omega\omega$ for $n < \omega$ and $\eta_n \restriction n = \eta \restriction n$ and $x \in \mathrm{Dom}(R)$ then for some T, xRT and $\eta \in \lim T$, $(\exists^\infty n)(\eta_n \in \lim T)$). The original $\oplus_k$ is better when not all $\langle \langle p^n_m : m < \omega \rangle : n < \omega \rangle$ work, but some do.

2) So possibility A splits to four cases: $[a \in S \Rightarrow d[a] \notin S]$, $\alpha^* = 1$, $\bigwedge_k \oplus_k$ and $\bigwedge_k \oplus'_k$.

Possibility B: Here we assume $d[a] \notin a$ for $a \in S$ or $\alpha^* = 1$ or at least $\oplus_k$ for every $k < \omega$. Let χ be large enough. For each $a \in S$ if (Skolem hull of a in

$(H(\chi), \in, <^*_\chi)\Big) \cap \bigcup S = a$, then player II has an absolute winning strategy (i.e. an absolute definition of it) which works in any generic extension V^Q of V by a (proper) forcing notion $Q \in H(\chi)$; during the play, stipulating $b_{-1} = \emptyset$, in the n'th move player I chooses $f^n_0, \dots, f^n_n$ satisfying $f^n_\ell \restriction d[a] \in {}^{d[a]}c[a]$ (see clause (b) below) and $\alpha^n_0, \dots, \alpha^n_{n-1}, \alpha^n_n$,

such that:

(α) for $\ell < n$ $\alpha^n_\ell \in a \cap \alpha^*$, and $\alpha^n_\ell \leq \alpha^{n-1}_\ell$

(β) if $\ell < n, \alpha^n_\ell = \alpha^{n-1}_\ell$ then $f^n_\ell \restriction b_{n-1} = f^{n-1}_\ell \restriction b_{n-1}$

(γ) $f^n_\ell R_{\alpha^n_\ell} \mathbf{g}_a$ for $\ell \leq n$ (hence $\alpha^n_\ell \in a$)

Player II chooses finite b_n, $b_{n-1} \subseteq b_n \subseteq a$.

In the end player II wins if:

(a) letting $\alpha_\ell = \min\{\alpha^n_\ell : \ell < n < \omega\}$ and $n(\ell) = \min\{n : \alpha^n_\ell = \alpha_\ell\}$ and $f_\ell = \bigcup_{\substack{n<\omega \\ n \geq n(\ell)}} f^n_\ell \restriction b_n$, we have $f_\ell R_{\alpha_\ell} \mathbf{g}_a$

or

(b) $a \neq (\bigcup S) \cap$ (Skolem hull of $a \cup \{f^n_\ell : \ell \leq n < \omega\}$).

Possibility C: Let χ be large enough. For each $a \in S$ in any forcing extension of V (of our family) player II has a winning strategy in the following game. In the n'th move: player I chooses N_n, H_n such that:

(a) N_n is a countable model of ZFC$^-$ (so $\in^{N_n}$ is $\in \restriction N_n$ but N_n is not necessarily transitive), $N_n \cap (\bigcup S) = a, S \in N_n$, $\mathbf{g} \in N_n$, $\bar{R} \in N_n$ (and $d'[a] \in N_n, c'[a] \in N_n$) and $[\ell < n \Rightarrow N_\ell \subseteq N_n]$ and $N_n \models$ "$(\bar{R}, S, \mathbf{g})$ covers" and

$$[f \in {}^{d'[a]}c'[a] \ \&\ f \in N_n \Rightarrow (f \restriction d[a]) R_a \mathbf{g}_a]$$

(where $R_a = \bigvee_{\alpha \in a \cap \alpha^*} R_\alpha$)

(b) $H_n \subseteq \{\langle f_0, \dots, f_{n-1}\rangle$: for some finite $d \subseteq d'[a]$, each f_ℓ is a function from d to $c'[a]\}$ and $H_n \in N_n$ is not empty.

(c) if $\langle f_0, \dots, f_{n-1}\rangle \in H_n$ and $d \subseteq \mathrm{Dom}(f_0)$ is finite *then* $\langle f_0 \restriction d, \dots, f_{n-1} \restriction d\rangle \in H_n$.

(d) if $\langle f_0, \cdots, f_{n-1}\rangle \in H_n, \mathrm{Dom}(f_0) \subseteq d$, d finite $\subseteq d[a]$
then for some $\langle f'_0, \dots, f'_{n-1}\rangle \in H_n$ we have $\mathrm{Dom}(f'_\ell) = d$, and $f_\ell \subseteq f'_\ell$

(e) $m < n$ & $\langle f_0, \dots, f_{n-1}\rangle \in H_n \Rightarrow \langle f_0, \dots, f_{m-1}\rangle \in H^*_m$ (see below).

Player II chooses $\langle f_0^n, \dots, f_{n-1}^n \rangle \in H_n \cap N_n$ and let[†]
$H_n^* = \{\langle f_0, \dots, f_{n-1}\rangle$: for each ℓ the functions f_ℓ, f_ℓ^n are compatible[††] $\}$

In the end player II wins if: for every $m < \omega$, $\bigcup_{n \geq m} f_m^n$ is a function which has domain $d[a]$ and $(\bigcup_{n \geq m} f_m^n) R_a \mathbf{g}_a$ [note: if e.g. $\{f : f R_a \mathbf{g}_a\}$ is a Borel set, then the game is determined and a winning strategy is absolute].

Possibility A^*: Each R_α is closed and

$\otimes$ if $a_1, a_2 \in S$, $a_1 \in a_2$, *then* $(c'[a_1], d'[a_1]) = (c'[a_2], d'[a_2])$ and absolutely for every $f \in {}^{d'[a_2]}c'[a_2]$ we have: $(f \restriction d[a_1]) R_{a_1} \mathbf{g}_{a_1} \Rightarrow (f \restriction d[a_2]) R_{a_2} \mathbf{g}_{a_2}$

and : $(\forall a \in S)(d[a] \notin a)$ or $\alpha^* = 1$ or $\oplus_1$. Note that in cases A^*, B^*, C^*, for some (c', d') we have $(c'[a], d'[a]) = (c', d')$ for every $a \in S$ (as S is directed).

Possibility B^*: We assume

$\otimes$ if $a_1, a_2 \in S$, $a_1 \in a_2$ *then* $(c'[a_1], d'[a_1]) = (c'[a_2], d'[a_2])$ and absolutely for every $f \in {}^{d'[a_2]}c'[a_2]$ we have $(f \restriction d[a_1]) R_{a_1} \mathbf{g}_{a_1} \Rightarrow (f \restriction d[a_2]) R_{a_2} \mathbf{g}_{a_2}$,

and player II has an absolute winning strategy in a game similar to the one in Possibility B except that only f_0^n, α_0^n, b_n are chosen. And: $(\forall a \in S)(d[a] \notin a)$ or $\alpha^* = 1$ or $\oplus_1$.

Possibility C^*: We assume

$\otimes$ if $a_1, a_2 \in S$, $a_1 \in a_2$ *then* $(c'[a_1], d'[a_1]) = (c'[a_2], d'[a_2])$ and absolutely for every $f \in {}^{d'[a_2]}c'[a_2]$ we have $(f \restriction d[a_1]) R_{a_1} \mathbf{g}_{a_1} \Rightarrow (f \restriction d[a_2]) R_{a_2} \mathbf{g}_{a_2}$,

and player II has an absolute winning strategy in a game similar to the one in Possibility C

(a) as before

(b)* $H_n \subset \{f$: for some finite $d \subseteq d'[a]$, f_0 is a function from d to $c'[a]\}$

(c)* if $f \in H_n$, $d \subseteq \mathrm{Dom}(f)$ is finite *then* $f \restriction d \in H_n$

(d)* if $f \in H_n$, $d \subseteq \mathrm{Dom}(f)$, $d \subseteq d'[a]$ *then* for some $f' \in H_n$, we have $\mathrm{Dom}(f') = d$ and $f \subseteq f'$

(e)* $H_n \subseteq H_{n+1}$

† We could give the second player more influence, see proof of 3.6.

†† We could add $\ell < m < n \Rightarrow f_\ell^m \subseteq f_\ell^n$, no real difference.

3.3A Remark. 1) In Possibility B, we can restrict the forcing to a suitable family.

2) Below in the cases $d[a] \notin a$ we use (see Possibility C) $d'[a] = c'[a] = \bigcup S$. This is essentially a notational change.

3) In Possibility C^* we can weaken $\otimes_1$ to the weaker version

$\otimes_1^-$ *if* for some forcing notion P, in V^P, $(\bar{R}, S, \bar{\mathbf{g}})$ still covers, N is a countable elementary submodel of $(H(\chi)^{V^P}, \in)$ to which $(\bar{R}, S, \mathbf{g})$ belongs, and so is a model of ZFC$^-$, and $a \overset{\text{def}}{=} N \cap (\bigcup S) \in S$ and if $a_1 \in S \cap N$ and $f \in N \cap ({}^{d[a]}c[a])$ then for some $a_2, a_1 \in a_2 \in S \cap N$ and $f \restriction d[a_2] R_{a_2} \mathbf{g}_{a_2}$ *then* $f R_a \mathbf{g}_a$.

3.3B Observation. 1) In[†] Definition 3.3

(a) $(\forall a \in S)[d[a] \notin a]$ & Possibility B* implies Possibility B.

(b) $(\forall a \in S)[d[a] \notin a]$ & Possibility C* implies Possibility C.

(c) $(\forall a \in S)[d[a] \notin a]$ & Possibility A implies Possibility B.

(d) Possibility A^* implies Possibility B^*

2) If Possibilities A* or B* or C* of Definition 3.3 hold, (or just $\otimes$ from there), Q is a proper forcing and $\Vdash_Q$ "for every $f \in {}^{d'[a]}c'[a]$, for every $a_1 \in S \cap N$ for some a_2 satisfying $a_1 \in a_2 \in S \cap N$ we have $(f \restriction d[a_2]) R_{a_2} \mathbf{g}_{a_2}$" and $Q \in N \prec (H(\chi), \in, <^*_\chi)$, $N \cap (\bigcup S) \in S$, N countable and $q \in Q$ is (N, Q)-generic *then* $q \Vdash$ "$N[\underset{\sim}{G}_Q]$ is $(\bar{R}, S, \mathbf{g})$-good".

3) A sufficient condition for $\oplus_k$ of Definition 3.3 is[††]

$\oplus_k^*$ if (a),(b),(c), (d), (e) are as in $\oplus_k$ of Definition 3.3, then for some $p' \in G_1$, $\gamma_\ell \in (\beta_\ell + 1) \cap a$ and Borel set (even Σ_1 set over $\bigcup S$ i.e. quantifying over ${}^\omega(\bigcup S)$, with $\bar{R}$, S, $\mathbf{g}$ as parameters will do), $A_\ell \in N$ (for $\ell < k$) we have

(α) $p' \Vdash_Q$ "$\underset{\sim}{f}_\ell \in A_\ell$ for $\ell < k$"

(β) $(\forall f \in A_\ell)(\exists \gamma \leq \beta_\ell)(f R_\gamma \mathbf{g}_a)$

Proof. (1) Easy, For clause (a) note that:

[†] We can replace $(\forall a \in S)[d(a) \notin a]$ by $\alpha^* = 1$, and/or add $X \in \{A, B, C\}$ & $(\forall a \in S)[d(a) \notin a]$ implies Possibility X $\Leftrightarrow$ Possibility X^*. Note that for possibility C and C^*, w.l.o.g. $\alpha^* = 1$.

[††] Many times this is easy.

(i) increasing b_n may only help player II as it just strengthen the restrictions on player II,

(ii) having more f^n_ℓ may only help player II as it make the satisfaction of clause (b) of possibility B (or B*) more probable. So for player II, having a winning strategy in the two games are equivalent (but not so for the 'player I has no winning strategy; see hopefully in [Sh:311]). Similarly for clause (b).

(c) We should give a winning strategy for player II. Let $a = \{x_i : i < \omega\}$ and his strategy is to choose $b_n = \{x_\ell : \ell < n\}$.

2), 3) Left to the reader. $\square_{3.3B}$

3.4 Definition. We say that a forcing notion Q is $(\bar{R}, S, \mathbf{g})$-preserving for possibility X if (where $X \in \{A, B, C, A^*, B^*, C^*\}$, for Possibilities C, C* (in Def 3.3) we can omit (iv)-(xi) and conclusion (a) as they hold vacuously; if we omit "for possibility X" we mean $X = C$):

(∗) Assume (i) χ_1 is large enough, $\chi > 2^{\chi_1}$

(ii) $N \prec (H(\chi), \in, <^*_\chi), N$ countable, $N \cap (\bigcup S) = a \in S$ and $\langle Q, S, \mathbf{g}, \chi_1\rangle \in N$

(iii) N is $(\bar{R}, S, \mathbf{g})-$ good (see Definition 3.2(1)) and $p \in Q \cap N$.

(iv) In Possibilities A, B we have $k < \omega$ and for $\ell < k$ we have $\underset{\sim}{f}_\ell \in N$ is a Q-name of a function, $\Vdash_Q$ "$\mathrm{Dom}(\underset{\sim}{f}_\ell) = d'[a]$"; for Possibilities A*, B* the situation is similar but $k = 1$. For Possibilities C, C* we can let $k = 0$.

(v) if $\ell < k$, then f^*_ℓ is a function and $\mathrm{Dom}(f^*_\ell) = d[a]$

(vi) for $n < \omega$ we have: $p, p_n \in Q \cap N$, $p \leq p_n \leq p_{n+1}$

(vii) if $d[a] \in a$ then $\langle p_n : n < \omega\rangle \in N$ and $\langle f^*_\ell : \ell < k\rangle \in N$

(viii) for each $x \in \mathrm{Dom}(f^*_\ell)$ and $\ell < k$, for every n large enough $p_n \Vdash_Q$ "$\underset{\sim}{f}_\ell(x) = f^*_\ell(x)$"

(ix) for $\ell < k$ we have $f^*_\ell R_{\beta_\ell} \mathbf{g}_a$ where $\beta_\ell \in a \cap \alpha^*$.

(x) if $d[a] \notin a$, $\mathcal{I} \in N$ a dense open subset of Q then for some n, $p_n \in \mathcal{I}$

(xi) if $d[a] \in a$, then for some N_1 a countable elementary submodel of $(H(\chi_1), \in, <^*_{\chi_1})$ which belong to N and include

$d[a] \cup c[a] \cup \{d[a], c[a]\} \cup \{Q, S, \mathbf{g}\} \cup \{\underset{\sim}{f}_\ell : \ell < k\}$ we have*:

$$\bigwedge_n p_n \in N_1, \bigvee_n p_n \in \mathcal{I} \text{ for any } \mathcal{I} \in N_1, \text{ a dense subset of } Q.$$

Then there is a q, $p \leq q \in Q$ such that: q is (N, Q)-generic and

(a) $q \Vdash_Q$ "$(\underset{\sim}{f}_\ell \restriction d[a]) R_{\gamma_\ell} \mathbf{g}_a$ for some γ_ℓ, $\gamma_\ell \leq \beta_\ell \,\&\, \gamma_\ell \in a \cap \alpha^*$" for each $\ell < k$

(b) $q \Vdash_Q$ "$N[\underset{\sim}{G}_Q]$ is $(\bar{R}, S, \mathbf{g})$-good"

3.4A Claim. 1) If $\alpha^* = 1$ *then* "Q is $(\bar{R}, S, \mathbf{g})$-preserving" (see 3.4 above) is equivalent to : if $N \prec (H(\chi), \in, <^*_\chi)$, N countable, N is $(\bar{R}, S, \mathbf{g})$-good, $Q \in N$, $p \in N \cap Q$ *then* for some (N, Q)-generic $q \in Q$, $q \geq p$ we have $q \Vdash$ "$N[\underset{\sim}{G}_Q]$ is $(\bar{R}, S, \mathbf{g})$-good"†.

2) If $\otimes$ (of possibilities A^*, B^*, C^* of Definition 3.3) hold, Q proper and $\alpha^* = 1$ *then:* "Q is $(\bar{R}, S, \mathbf{g})$-preserving" is equivalent to : for every $f \in {}^{d'[a]}c'[a]$ from V^Q for some a_2 we have $a \in a_2 \in S$, $(f \restriction d[a_2]) R_{a_2} \mathbf{g}_{a_2}$.

3) If $(\bar{R}, S, \mathbf{g})$ is as in Possibility A* (of Definition 3.3) and $(\forall a \in S)([d[a] \in a])$ and $\otimes^+$ below holds then: for any proper forcing notion Q, if $\Vdash_Q$ "$(\bar{R}, S, \mathbf{g})$ covers" *then* Q is $(\bar{R}, S, \mathbf{g})$-preserving for possibility A* where

$\otimes^+$ Assume†† we have a countable $N \prec (H(\chi), \in, <^*_\chi)$ such that $(\bar{R}, S, \mathbf{g}) \in N$, $a_1 \in a_2 \cap S$, $a_2 = N \cap (\bigcup S) \in S$, $(c[a_1], d[a_1]) = (c[a_2], d[a_2])$ and $\{f, \langle f_n : n < \omega \rangle\} \in N$ and $f R_\alpha \mathbf{g}_{a_2}$, and $\{f, f_n : n < \omega\} \subseteq {}^{d[a_1]}c[a_1]$, $f_n R_{\alpha_n} \mathbf{g}_{a_1}$ and $(\forall x \in d[a])(\forall^* n)(f_n(x) = f(x))$ and α, $\alpha_n \in \alpha^* \cap a_1$. Then for some $n < \omega$ and finite $d \subseteq d[a_1]$ we have

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* We may add $a_1 \subseteq N_1$

$$N_1 \cap \bigcup S = a_1 \text{ and}$$

$$(c[a_1], d[a_1]) = (c[a], d[a])$$

and similarly add in $\oplus_k$ of Definition 3.2. Then in the proof of 3.5, 3.6 change somewhat (as in the proof of 3.4A), using some absoluteness for $xR\mathbf{g}_a$.

† This gives the results of VI §3.

†† We can add $N_1 \in N$, $N_1 \prec N$, $N_1 \cap (\bigcup S) = a_1$ and even more in this direction.

(∗) *if* $f' \in {}^{d[a_2]}c[a_2]$, $f' R_{\alpha_n} \mathbf{g}_{a_1}$ and $f' \restriction d = f_n \restriction d$ then $f' R_\alpha \mathbf{g}_{a_2}$

(we can look for $f' \in V$, or in $N_1[G_Q]$ for every $G_Q \subseteq Q$ generic over V where $Q \in N_1$ is proper, $N_1[G] \cap V = N_1$, $N[G] \cap V = N$, the second is more restrictive)

4) If Q is $(\bar{R}, S, \mathbf{g})$-preserving for possibility X for some X *then* Q is $(\bar{R}, S, \mathbf{g})$-preserving.

Proof. 1) Left to the reader.

2) Remember that (by ⊗ of case A*, B*) there is a pair (c', d') such that: $a \in S \Rightarrow (c'[a], d'[a]) = (c', d')$. Also note

$$a_1 \in S \;\&\; a_1 \in a_2 \in S \;\&\; a_2 = N \cap \bigcup S \;\&\; S \in N \prec (H(\chi), \in) \Rightarrow a_1 \subseteq a_2.$$

First we assume "Q is $(\bar{R}, S, \mathbf{g})$-preserving" and let $p \in Q$, $a \in S$ and $\underset{\sim}{f}$ be such that $p \Vdash_Q$ "$\underset{\sim}{f} \in {}^{d'[a]}c'[a]$" i.e. $p \Vdash_Q$ "$\underset{\sim}{f} \in {}^{d'}c'$". Take $N \prec (H(\chi), \in, <^*_\chi)$ such that a, $(\bar{R}, S, \mathbf{g})$, p, $f \in N$, and N is $(\bar{R}, S, \mathbf{g})$-good. So by the assumption, for some (N, Q)-generic q we have $p \leq q \in Q$ and $q \Vdash_Q$ "$N[\underset{\sim}{G}_Q]$ is $(\bar{R}, S, \mathbf{g})$-good". Let a_2 be $N \cap (\bigcup S)$, so q"$\Vdash \underset{\sim}{f} \restriction d[a_2] \in {}^{d[a_2]}c[a_2]$ satisfies $\underset{\sim}{f} \restriction d[a_2] R_{a_2} \mathbf{g}_{a_2}$", as required.

Second, to prove ⇒ i.e. the "if" direction, assume that in V^Q for every $f \in {}^{d'}c'$ from V^Q for some a_1 we have $a_1 \in S$ and $f \restriction d[a_1] R_{a_1} \mathbf{g}_{a_1}$. This means: for every $G_Q \subseteq Q$ generic over V the statement above holds. Now let, in V, $N \prec (H(\chi), \in, <^*_\chi)$ be $(\bar{R}, S, \mathbf{g})$-good and assume $q \in Q$ is (N, Q)-generic. Let $q \in G_Q \subseteq Q$, G_Q generic over V, so it suffices to prove $V[G_Q] \Vdash$ "$N[G_Q]$ is $(\bar{R}, S, \mathbf{g})$-good". So let $a_2 = N \cap (\bigcup S)$, and let $f \in N[G_Q]$, $f \in {}^{d'[a_2]}c'[a_2] = {}^{d'}c'$. So for some $a_1 \in S$ we have $f \restriction d[a_1] R_{a_1} \mathbf{g}_{a_1}$, but $N[G_Q] \prec (H(\chi)[G_Q], \in)$ hence w.l.o.g. $a_1 \in N[G_Q] \cap S = N \cap S$. Now apply ⊗ of Definition 3.3 possibility A* (or B*, or C*), which we are assuming, to deduce $f \restriction d[a_2] R_{a_2} \mathbf{g}_{a_2}$. As this holds for every such f really $V[G_Q] \models N[G_Q]$ is $(\bar{R}, S, \mathbf{g})$-good.

3) Let N, N_1, $\underset{\sim}{f}^*_0$, β_0, $\bar{p} = \langle p_n : n < \omega \rangle$ be as in Definition 3.4 for possibility A*. Let $a = N \cap (\bigcup S)$. See in particular clause (xi) there. We can find $M_2 \prec N_2 \prec (H(\chi_1), \in, <^*_{\chi_1})$, $\{N_1, \langle p_n : n < \omega \rangle, \underset{\sim}{f}_0\} \in M_2 \in N_2 \in N$ and $N_2 \cap \bigcup S = a_2 \in S$,

$M_2 \cap (|bigcup S) = b_2$ and $(c[b_2], d[b_2]) = (c[a_2], d[a_2]) = (c[a], d[a])$. Also we can find $\langle p_{n,m} : n, m < \omega \rangle$, $\langle f_n : n < \omega \rangle$ such that: $p_n \le p_{n,m} \le p_{n,m+1}$, $p_{n,0}$ is (M_2, Q)-generic, and $p_{n,0} \Vdash$ "$\underset{\sim}{f}_0 R_{\alpha_n} \mathbf{g}_{b_2}$" for some $\alpha_n \in b_2 \cap \alpha^*$ and $\langle p_{n,m} : m < \omega \rangle$ is a generic sequence for N_2 (i.e. if $\mathcal{I} \subseteq Q$ is dense, $\mathcal{I} \in N_2$ then $(\exists m)(\exists r \in \mathcal{I} \cap N_2)(r \le p_{n,m})$), $f_n \in {}^{d[a]}c[a]$, and

$$\forall x \in d[a] \forall n < \omega \forall^* m (p_{n,m} \Vdash \underset{\sim}{f}_0(x) = f_n(x)).$$

W.l.o.g. $\langle p_{n,m} : n, m < \omega \rangle$, $\langle \alpha_n : n < \omega \rangle$ and $\langle f_n : n < \omega \rangle$ belongs to N. Clearly $f_n R_{\alpha_n} \mathbf{g}_{b_2}$. (Here we used $\{f : f R_{\alpha_n} \mathbf{g}_{b_2}\}$ is closed and $\langle p_{n,m} : m < \omega \rangle$ is generic enough; Borel suffices. Why? Let $G_n = \{p \in Q \cap N_2 : (\exists m) p \le p_{n,m}\}$ be a subset of $Q \cap N_2$ generic over N_2, so $N_2[G_n] \models$ "$\underset{\sim}{f}_0[G_n] R_{\alpha_n} \mathbf{g}_{a_1}$" but $f_n = \underset{\sim}{f}_0[G_n]$.)

Now apply $\otimes^+$ with b_2, a, f_0^*, β_0, $\langle f_n : n < \omega \rangle$, $\langle \alpha_n : n < \omega \rangle$ here standing for a_1, a_2, f, α, $\langle f_n : n < \omega \rangle$, $\langle \alpha_n : n < \omega \rangle$ there, and get n and d_n as there. Let m be such that $p_{n,m}$ force a value to $\underset{\sim}{f}_0 \restriction d_n$, so it is $f_n \restriction d_n$. Let $q \in Q$ be (N, Q)-generic such that $p_{n,m} \le q$. Now suppose $q \in G_Q \subseteq Q$, G_Q generic over V; by the conclusion $(*)$ of $\otimes^+$ (i.e. the choice of n, d_n) we get $\underset{\sim}{f}_0[G_Q] R_{\beta_0} \mathbf{g}_a$. We still have to prove "$N[G_Q]$ is $(\bar{R}, S, \mathbf{g})$-good". But this holds by the proof of 3.4(2) above.

4) Easy. $\square_{3.4A}$

3.4B Claim. 1) Assume

(a) $(\bar{R}, S, \mathbf{g})$ is as in 3.1, $(\forall a \in S)(d[a] \in a)$,

(b) $(\bar{R}, S, \mathbf{g})$ covers,

(c) we have

$\oplus_1^+$ Assume we have a countable $N \prec (H(\chi), \in, <^*_\chi)$ such that $(\bar{R}, S, \mathbf{g}) \in N$, $a_1 \in a_2 \cap S$, $a_2 = N \cap (\bigcup S) \in S$, $(c[a_1], d[a_1]) = (c[a_2], d[a_2])$ and $\{f, \langle f_n : n < \omega \rangle\} \in N$ and $f R_\alpha \mathbf{g}_{a_2}$, and $\{f, f_n : n < \omega\} \subseteq {}^{d[a_1]}c[a_1]$, $f_n R_{\alpha_n} \mathbf{g}_{a_1}$ and $\forall x \in d[a_1] (\forall^* n)(f_n(x) = f(x))$ and α, $\alpha_n \in \alpha^* \cap a_1$. Then for some $n < \omega$ and finite $d \subseteq d[a_1]$ we have

$(*)$ *if* $f' \in {}^{d[a_2]}c[a_2]$, $f' R_{\alpha_n} \mathbf{g}_{a_1}$ and $f' \restriction d = f_n \restriction d$ then $f' R_\alpha \mathbf{g}_{a_2}$, moreover

(c)$^+$ for every proper forcing P preserving "$(\bar{R}, S, \mathbf{g})$-covers" we have $\oplus_1^+$ in V^P.

Then in the definition of "$(\bar{R}, S, \mathbf{g})$ strongly covers for possibility X" $X =$ A*, B^* we can omit $\oplus_1$ of Definition 3.2.

2) Assume (a), (b) as in (1) above and

(c) for each $k < \omega$ we have

$\oplus_k^{++}$ as in $\oplus_1^+$ but in the conclusion we replace "some n" by "for every n large enough"

or at least

$\oplus_k^+$ Assume we have a countable $N \prec (H(\chi), \in, <_\chi^*)$ such that $(\bar{R}, S, \mathbf{g}) \in N$, $a_1 \in a_2 \cap S$, $a_2 = N \cap (\bigcup S) \in S$ and $(c[a_1], d[a_1]) = (c[a_2], d[a_2])$ and $\{f_\ell : \ell < k\} \cup \{\langle f_n^\ell : n < \omega\rangle : \ell < k\} \in N$ and $f_\ell R_{\alpha(\ell)} \mathbf{g}_{a_1}$, and $\{f_\ell, f_n^\ell : \ell < k, n < \omega\} \subseteq {}^{d[a_1]}c[a_1]$ $f_n^\ell R_{\alpha_n(\ell)} \mathbf{g}_{a_1}$ and $\alpha(\ell)$, $\alpha_n(\ell) \in a_1 \cap \alpha^*$. *Then* for some $n < \omega$ and finite $d \subseteq d[a_1]$ we have

$(*)$ if $\ell < k$, $f'_\ell \in {}^{d[a_1]}c[a_1]$, $f'_\ell R_{\alpha_n(\ell)} \mathbf{g}_{a_1}$ and $f'_\ell \restriction d = f_n^\ell \restriction d$ *then* $f'_\ell R_\alpha \mathbf{g}_{a_2}$.

(c)$'$ Moreover (c) is preserved by proper forcing preserving "$(\bar{R}, S, \mathbf{g})$- covers".

Then in the definition of "$(\bar{R}, S, \mathbf{g})$ strongly cover for possibility X", when $\forall a \in S$ $(c[a], d[a]) = (c, d)$ $X =$ A, B we can omit $(\forall k)\oplus_k$.

Proof. Like the proof of 3.4A(3). $\square_{3.4B}$

3.5 Claim. 1) If $(\bar{R}, S, \mathbf{g})$ covers in V and Q is an $(\bar{R}, S, \mathbf{g})$-preserving forcing notion *then* in V^G, $(\bar{R}, S, \mathbf{g})$ still covers.

2) Assume $(\bar{R}, S, \mathbf{g})$ covers. The property "$(\bar{R}, S, \mathbf{g})$-preserving for possibility X" is preserved by composition (of forcing notions).

Proof. 1) Just read the definitions.

2) Each part has some versions, according to whether in Definition 3.4 we choose Possibility A, A*, B, B* or Possibility C, C* and whether $d[a] \in a$ or not.

Let $Q = Q_0 * \underset{\sim}{Q}_1$; let χ_1, χ, N, N_1, a, k, $\underset{\sim}{f}_\ell$, β_ℓ, f_ℓ^* (for $\ell < k$), p, p_n $(n < \omega)$ be as in Definition 3.4. Let $p = (q^0, \underset{\sim}{q}^1)$ and $p_\ell = (q_\ell^0, \underset{\sim}{q}_\ell^1)$. By condition (vi) of

Definition 3.4, for each $n < m < \omega$ we have $q^0_m \Vdash_{Q_0}$ "$\underset{\sim}{Q}_1 \models \underset{\sim}{q}^1 \le \underset{\sim}{q}^1_n \le \underset{\sim}{q}^1_m$", hence without loss of generality:

$(*)_1$ $\Vdash_{Q_0}$ "$\underset{\sim}{Q}_1 \models \underset{\sim}{q}^1 \le \underset{\sim}{q}^1_n \le \underset{\sim}{q}^1_m$ for $n < m < \omega$".

$(*)_2$ for every $x \in d[a]$ for every $n < \omega$ large enough, $(\emptyset, q^1_n)$ forces $\underset{\sim}{f}_\ell(x)$ to be equal to some (specific) Q_0-name, for each $\ell < k$.

[Why? By clause (x) or (xi) of Definition 3.4.]

Now we define $\underset{\sim}{f}'_\ell$, a Q_0-name of a member of ${}^{d[a]}c[a]$, such that $\Vdash_{Q_0}$ " for each $x \in d[a]$, for every n large enough $\underset{\sim}{q}^1_n \Vdash_{\underset{\sim}{Q}_1}$ '$[\underset{\sim}{f}'_\ell(x) = \underset{\sim}{f}_\ell(x)]$'". Easily: $d[a] \in a \Rightarrow \underset{\sim}{f}'_\ell \in N$.

By Definition 3.4 (and the assumption) there is $q_0 \in Q_0$ which is $(N, Q_0)-$generic, is above q^0 (in Q_0) and forces $N[\underset{\sim}{G}_{Q_0}]$ to be $(\bar{R}, S, \mathbf{g})$-good and for some $\gamma'_\ell \le \gamma_\ell$, $\gamma'_\ell \in N$ we have $q_0 \Vdash_{Q_0}$ "$\underset{\sim}{f}_\ell R_{\gamma'_\ell} \mathbf{g}_a$ for $\ell < k$".

Let $G_0 \subseteq Q_0$ be generic over V such that $q_0 \in G_0$. We want to apply Definition 3.4 with $N[G_0]$, $\underset{\sim}{q}^1[G_0]$, $\langle \underset{\sim}{q}^1_\ell[G_0] : \ell < \omega\rangle$, $\langle \underset{\sim}{f}_\ell[G_0] : \ell < k\rangle$, $\langle \underset{\sim}{f}'_\ell[G_0] : \ell < k\rangle$, $\langle \gamma'_\ell : \ell < k\rangle$, $\underset{\sim}{Q}_1[G_0]$ (and sometimes $N_1[G_0]$) here standing for N, p, $\langle p_\ell : \ell < \omega\rangle$, $\langle \underset{\sim}{f}_\ell : \ell < k\rangle$, $\langle f^*_\ell : \ell < k\rangle$, $\langle \beta_\ell : \ell < k\rangle$, Q there (and sometimes N_1) (and same $(\bar{R}, S, \mathbf{g})$).

So we have to check the assumptions of Definition 3.4; now we check all clauses of Definition 3.4.

clause (i): clear by the "old" (i).

clause (ii): holds as $q_0 \in G_0$ is (N, Q_0)-generic so $N[G_0] \cap (\bigcup S) = N \cap (\bigcup S)$ and the "old" (ii).

clause (iii): holds by the choice of $q_0 \in G_0$ that is $q_0 \Vdash$ "$N[G_0]$ is $(\bar{R}, S, \mathbf{g})$-good" by the choice of q_0 and clause (b) in the conclusion in Definition 3.4.

clause (iv): clear by the "old" (iv).

clause (v): If $x \in d[a]$, then ($x \in N$ or $\in N_1$ and) for some ℓ and Q_0-name $\underset{\sim}{\tau} \in N$ or $\in N_1$ we have $\Vdash_{Q_0}$ "$\left[\underset{\sim}{q}^1_\ell \Vdash_{\underset{\sim}{Q}_1} \text{"}\underset{\sim}{f}_\ell(x) = \underset{\sim}{\tau} \in c[a]\text{"}\right]$" (as the set of $(r_0, \underset{\sim}{r}_1) \in Q_0 \times \underset{\sim}{Q}_1$ such that

$$\Vdash_{Q_0} \text{"}\left[\underset{\sim}{r}_1 \Vdash_{\underset{\sim}{Q}_1} \text{"}\underset{\sim}{f}_\ell(x) = \underset{\sim}{\tau}\text{"}\right]\text{"}$$

for some Q_0-name $\underset{\sim}{\tau}$, is dense open subset of $(Q_0, \underset{\sim}{Q}_1)$ so some $(q^0_\ell, \underset{\sim}{q}^1_\ell)$ is in it, and there is such $\underset{\sim}{\tau}$ so w.l.o.g. $\underset{\sim}{\tau} \in N$ or $\in N_1$). So $\underset{\sim}{f}'_\ell[G_0](x) = \underset{\sim}{\tau}[G_0] \in c[a]$.

clause (vi): by $(*)_1$ above (in our proof) this holds.

clause (vii): Check (see $(*)_2$).

clause (viii): This is by the choice of $\underset{\sim}{f}'_\ell(x)$ and $\langle \underset{\sim}{q}^1_\ell : \ell < \omega \rangle$.

clause (ix): by the choice of q_0 (and as $q_0 \in G_0$) and the choice of γ'_ℓ (for $\ell < k$).

clause (x): by the "old" clause (x) and as in the proof of clause (v) above. In details, if $N[G_0] \models$ "$\mathcal{I} \subseteq \underset{\sim}{Q}_1[G_0]$ is dense open" so $\mathcal{I} \in N[G_0]$ then for some $\underset{\sim}{\mathcal{I}}' \in N$ we have $\Vdash_{Q_0}$ "$\underset{\sim}{\mathcal{I}}'$ is a dense open subset of $\underset{\sim}{Q}_1$" and $\mathcal{I} = \underset{\sim}{\mathcal{I}}'[G_0]$; let

$$\mathcal{J} = \{(r_0, \underset{\sim}{r}_1) \in Q_0 * \underset{\sim}{Q}_1 : \Vdash_{Q_0} \text{“}\underset{\sim}{r}_1 \in \underset{\sim}{\mathcal{I}}'\text{”}\},$$

clearly $\mathcal{J} \in N_1$ is a dense open subset of $Q_0 * \underset{\sim}{Q}_1$ hence for every large enough ℓ,

$$(q^0_\ell, \underset{\sim}{q}^1_\ell) \in \mathcal{J} \text{ hence } \underset{\sim}{q}^1_\ell[G_0] \in \underset{\sim}{\mathcal{I}}'[G_0] = \mathcal{I},$$

hence we finish.

clause (xi): Use a_1, $N_1[G_0]$. Note that we do not require $N_1[G_0] \cap V = N_1$, still $N_1[G_0] \prec N[G_0]$, $N_1[G_0] \in N[G_0]$ and $\langle \underset{\sim}{q}^1_\ell[G_0] : \ell < \omega \rangle$ is as required there.

So really we can apply 3.4 and get $q_1 \in \underset{\sim}{Q}_1[G_0]$ which is $(N[G_0], \underset{\sim}{Q}_1[G_0])$-generic, and $\underset{\sim}{Q}_1[G_0] \models$ "$\underset{\sim}{q}^1[G_0] \leq q_1$" and $\langle \gamma_\ell : \ell < k \rangle$, $\gamma_\ell \leq \gamma'_\ell$ such that $q_1 \Vdash_{\underset{\sim}{Q}_1[G_0]}$ "$\underset{\sim}{f}_\ell R_{\gamma_\ell} \mathbf{g}_a$". As G_0 was any generic subset of Q_0 to which q_0 belongs, for some Q_0-name $\underset{\sim}{q}_1$ we have $q_0 \Vdash_{Q_0}$ "$\underset{\sim}{q}_1$ is as above". Now $(q_0, \underset{\sim}{q}_1)$, $\langle \gamma_\ell : \ell < k \rangle$ are as required. If we do have the demands on a_1 in Definition 3.4, clause (xi) we should replace N_1 ny another model in the intermediate stage as done in the proof of 3.4A (but we use absoluteness of $xR\mathbf{g}_a$). $\square_{3.5}$

3.6 Theorem. 1) Suppose $X \in \{A, B, C, A^*, B^*, C^*\}$ and in V we have $(\bar{R}, S, \mathbf{g})$ strongly covers, $\langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ is a CS iteration of proper, $(\bar{R}, S, \mathbf{g})$-preserving for possibility X forcing notions, *then* P_α is a proper, $(\bar{R}, S, \mathbf{g})$-preserving for possibility X forcing notion.

2) This is true also for more general iterations, as in XV when $|\alpha^*| \le \aleph_1$ (in fact all cases in VI 0.1 apply) †.

Proof. 1) We prove by induction on $\zeta \le \alpha$, that: for every $\xi \le \zeta$, P_ζ/P_ξ is $(\bar{R}, S, \mathbf{g})$-preserving for possibility X (in V^{P_ξ}), moreover in Definition 3.4 we can get $\mathrm{Dom}(q) = (\zeta \setminus \xi) \cap N$. For ζ zero, there is nothing to prove, for ζ successor - use 3.5(2), so let ζ be limit, $\xi < \zeta$. Let $G_{P_\xi} \subseteq P_\xi$ be generic over V and $\chi, N, p, k, \underset{\sim}{f}_\ell, f^*_\ell, \beta_\ell$ (for $\ell < k$), and possibly p_n, χ_1, N_1 be as in $(*)$ of Definition 3.4 (with $P_\zeta/P_\xi, V[G_{P_\xi}]$ here standing for Q, V there); for $X =$ C, C* we have $k = 0$ so $\underset{\sim}{f}_\ell$, $\underset{\sim}{f}^*_\ell$, β_ℓ disappear and for cases $d[a] \notin a$ we have no N_1 and for $X = A^*$, B^* we have $k = 1$. Let $G_0 = \{p \in P_\zeta/G_{P_\xi} : p \in N_1$ when well defined and $p \in N$ otherwise and for some n, $p \le p_n\}$ (used in the proof of possibility B, $d[a] \in a$).) We can choose $\zeta_n, \zeta_0 = \xi, \zeta_n < \zeta_{n+1} \in N \cap \zeta$ and $\sup(N \cap \zeta) = \bigcup_{n<\omega} \zeta_n$. Let $q_0 \in G_{P_\xi}$ force all this (so we can work in V, so we have $\underset{\sim}{G}_0$).

The proofs are built after the proofs of preservation of properness and the proofs in VI §1, VI §3 (particularly the proof of Possibilities A, $d[a] \in a$).

The case when $\mathrm{cf}(\zeta) > \aleph_0$ is elaborated when possibility B, $d[a] \in a$, is considered (note that the arguments there apply to all Possibilities).

Possibility C: Let $\langle \underset{\sim}{f}_\ell : \ell < \omega \rangle$ list the P_ζ-names $\underset{\sim}{f} \in {}^{d'[a]}c'[d]$ satisfying $\underset{\sim}{f} \in N$. Let $\langle \underset{\sim}{\tau}_n : n < \omega \rangle$ list the P_ζ-names of ordinals which belong to N. We choose by induction on n, $q_n, f_n, \underset{\sim}{H}_n$ $\langle \underset{\sim}{f}^n_\ell : \ell \le n \rangle$ such that:

(a) $q_n \in P_{\zeta_n}$, $\mathrm{Dom}(q_n) \setminus \xi = N \cap \zeta_n$, $q_{n+1} \restriction \zeta_n = q_n$ (of course q_0 is given)

(b) q_n is $(N[G_{P_\xi}], P_{\zeta_n})$-generic

(c) $q_n \Vdash$ "$N[\underset{\sim}{G}_{P_{\zeta_n}}]$ is $(\bar{R}, S, \mathbf{g})$-good".

† But in the applications presented here we "forget" this. Of course if we consider forcing notions with an additional order $\le_{\mathrm{pr}}$ on them, and the corresponding iteration (see XV), then "pure $(\theta, 2)$-decidability" has to be added for appropriate θ (mainly $d[a] \in N, \theta = \aleph_0$).

(d) $\underset{\sim}{p}_n$ is a P_{ζ_n}-name of a member of $P_\zeta \cap N$ such that

$$q_n \Vdash_{P_{\zeta_n}} \text{“}\underset{\sim}{p}_n \restriction \zeta_n \in \underset{\sim}{G}_{P_{\zeta_n}}\text{”}$$

(e) $\underset{\sim}{H}_n$ is a P_{ζ_n}-name, $\underset{\sim}{H}_n = \{\langle f_0, \dots, f_{n-1}\rangle : d \subseteq d[a]$ is finite and $\underset{\sim}{p}_n \nVdash_{P_\zeta/\underset{\sim}{G}_{P_{\zeta_n}}}$ “$\langle \underset{\sim}{f}_0 \restriction d, \dots, \underset{\sim}{f}_{n-1} \restriction d\rangle \neq \langle f_0, \dots, f_n\rangle$”$\}$

(f) $\underset{\sim}{f}^n_\ell$ is a P_{ζ_n}-name such that

$$q_n \Vdash_{P_{\zeta_n}} \text{“}\langle \underset{\sim}{f}^n_\ell : \ell < n\rangle \in \underset{\sim}{H}_n \text{ and for every } m \le n \text{ we have}$$

$$\underset{\sim}{p}_{n+1}[\underset{\sim}{G}_{P_{\zeta_n}}] \nVdash_{P/P_{\zeta_n}} \text{“}\neg \bigwedge_{\ell<m} \underset{\sim}{f}_\ell \supseteq \underset{\sim}{f}^m_\ell\text{”''}.$$

(We can demand that q_n forces that $\underset{\sim}{p}_{n+1}[\underset{\sim}{G}_{P_{\zeta_n}}]$ forces $\ell < n \Rightarrow \underset{\sim}{f}^n_\ell[\underset{\sim}{G}_{P_{\zeta_n}}] \subseteq \underset{\sim}{f}_\ell$, minor difference.)

(g) $q_n \Vdash$ “$\underset{\sim}{p}_{n+1}$ forces a value to $\underset{\sim}{\tau}_n$”.

Now there is no problem to carry out the definition but still we have freedom to choose $\langle \underset{\sim}{f}^n_\ell : \ell < n\rangle$. For this we use the winning strategy from Possibility C of Definition 3.3; choosing there the nth move of player I as:

$$N_n \stackrel{\text{def}}{=} N[\underset{\sim}{G}_{P_{\zeta_n}}]$$

$$\begin{aligned}\underset{\sim}{H}_n[\underset{\sim}{G}_{P_{\zeta_n}}] \stackrel{\text{def}}{=} \{\langle g_0, \dots, g_{n-1}\rangle :\ & \text{for some finite } d \subseteq d[a] \text{ with have:}\\ & g_\ell \in {}^d c[a] \text{ for } \ell < n \text{ and}\\ & \underset{\sim}{p}_n[G_{P_{\zeta_n}}] \nVdash \text{“}\langle \underset{\sim}{f}_0 \restriction d, \dots, \underset{\sim}{f}_{n-1} \restriction d\rangle \neq \langle g_0, \dots, g_{n-1}\rangle\text{”}\}\end{aligned}$$

(so the nth move is defined in $V^{P_{\zeta_n}}$; we can work in $V^{\mathrm{Levy}(\aleph_0, (2^{|P_\alpha|})^+)}$). Now of course while playing, the universe changes but as the winning strategy is absolute there is no problem.

Possibility C^* By 3.3B(2) it is enough to show that for every P_ζ-name $\underset{\sim}{f}_0$ of a function from $d'[a]$ to $c'[a]$ for some $b \in S$, $((c'[b], d'[b]) = (c'[a], d'[a]))$ and $\underset{\sim}{f}_0 R_b \mathbf{g}_b$. This is proved as in the proof of Possibility C, dealing only with $\underset{\sim}{f}_0$ (and using Possibility C* of Definition 3.3 of course.)

Possibility A, $\underline{d[a] \notin a}$: Let $\{\underset{\sim}{\tau}_j : j < \omega\}$ list the P_ζ-names of ordinals which belong to N. We shall choose by induction on $j < \omega$, $n_j < \omega$ such that :

(A) $n_j < n_{j+1} < \omega$

(B) for some sequence $\langle \underset{\sim}{\tau}_{j,\ell} : \ell \le j\rangle \in N$, with $\underset{\sim}{\tau}_{j,\ell}$ a P_{ζ_ℓ}-name we have:

  (α) $p_{n_j} \restriction [\zeta_j, \zeta) \Vdash_{P_\zeta}$ "$\underset{\sim}{\tau}_j = \underset{\sim}{\tau}_{j,j}$"

  (β) for $\ell < j$ we have $p_{n_j} \restriction [\zeta_\ell, \zeta_{\ell+1}) \Vdash_{P_{\zeta_{\ell+1}}}$ "$\underset{\sim}{\tau}_{j,\ell+1} = \underset{\sim}{\tau}_{j,\ell}$"

(C) if $j = i+1$, $\ell \le i$ then $\Vdash_{P_{\zeta_{\ell+1}}}$ "$p_{n_i} \restriction [\zeta_\ell, \zeta_{\ell+1}) \le p_{n_j} \restriction [\zeta_\ell, \zeta_{\ell+1})$".

(D) if $j = i+1$ then $\Vdash_{P_\zeta}$ "$p_{n_i} \restriction [\zeta_i, \zeta) \le p_{n_j} \restriction [\zeta_i, \zeta)$"

[Why can we carry the induction? It is enough to prove for each j that, given $\langle n_\ell : \ell < j\rangle$ as required, the set of candidates for $p \in P_\zeta$ satisfying the requirements on p_{n_j} is dense which is easy by clause (x) of (*) of Definition 3.4.]

Let $\{\underset{\sim}{f}_j : j < \omega\}$ list the P_ζ-names of members of ${}^{d'[a]}c'[a]$ which belong to N (for $\ell < k$ we let $\underset{\sim}{f}_j$ be as given). Note also that we can replace $\langle p_n : n < \omega\rangle$ by $\langle p_{n_j} : j < \omega\rangle$.

Hence without loss of generality we have $\underset{\sim}{\tau}_{\ell,j} \in N$ for $j \le \ell < \omega$, $\underset{\sim}{\tau}_{\ell,j}$ a P_{ζ_j}-name such that $p_\ell \restriction [\zeta_\ell, \zeta) \Vdash_{P_{\zeta_\ell}}$ "$\underset{\sim}{\tau}_\ell = \underset{\sim}{\tau}_{\ell,\ell}$", $p_\ell \restriction [\zeta_j, \zeta_{j+1}) \Vdash_{P_{\zeta_{j+1}}}$ "$\underset{\sim}{\tau}_{\ell,j+1} = \underset{\sim}{\tau}_{\ell,j}$". Let $h(j,x) < \omega$ be such that $\underset{\sim}{\tau}_{h(j,x)} = \underset{\sim}{f}_j(x)$. We can now define for $n < \omega$, $j < \omega$, $\underset{\sim}{f}^*_{n,j}$ a P_{ζ_n}-name of a function from $d[a]$ to $c[a]$. Let $\underset{\sim}{f}^*_{n,j}(x)$ be $\underset{\sim}{\tau}_{h(j,x),n}$ if $h(j,x) \ge n$ and $\underset{\sim}{\tau}_{h(j,x),h(j,x)}$ if $h(j,x) < n$ so $\underset{\sim}{f}^*_{0,j} = f^*_j$ for $j < k$.

We choose by induction on n, $q_n, \underset{\sim}{k}_n$, $\underset{\sim}{\alpha}^n_\ell$ (for $\ell < k+n$) such that:

(a) $q_n \in P_{\zeta_n}$, $\mathrm{Dom}(q_n) \setminus \xi = N \cap \zeta_n$, $q_{n+1} \restriction \zeta_n = q_n$, ($q_0$ is given).

(b) q_n is (N, P_{ζ_n})-generic

(c) $q_n \Vdash_{P_{\zeta_n}}$ "$N[\underset{\sim}{G}_{P_{\zeta_n}}]$ is $(\bar{R}, S, \mathbf{g})$-good"

(d) $\underset{\sim}{k}_n$ is a P_{ζ_n}/G_{P_ξ}-name of a natural number, $\underset{\sim}{k}_n < \underset{\sim}{k}_{n+1}$ (for Possibility (A), with which we are dealing) $\underset{\sim}{k}_n = n+1$ is O.K).

(e) $p_{k_0} \restriction \zeta_0 \le q_0$ (in P_{ζ_0}).

(f) $q_n \restriction \zeta_n \Vdash_{P_{\zeta_n}}$ "$p_{\underset{\sim}{k}_{n+1}} \restriction [\zeta_n, \zeta_{n+1}) \le q_{n+1} \restriction [\zeta_n, \zeta_{n+1}]$ (in $P_{\zeta_{n+1}}/P_{\zeta_n}$)".

(g) for $\ell < k+n$, $\underset{\sim}{\alpha}^n_\ell$ is a P_{ζ_n}-name of an ordinal in $a \cap \alpha^*$, $\alpha^{n+1}_\ell \le \alpha^n_\ell$, $\alpha^0_\ell = \beta_\ell$

(h) for $\ell < k+n$ we have $q_n \Vdash_{P_{\zeta_n}}$ "$f^*_{n,\ell} R_{\alpha^n_\ell} \mathbf{g}_a$".

The induction step is by the induction hypothesis (and Definition of "$(\bar{R}, S, \mathbf{g})$-preserving" (see Definition 3.4)). In the end let $q = \bigcup_{n<\omega} q_n$. Now why q is (N, P_ζ)-generic? Clearly $q \in P_\zeta$ (by condition (a)); let $q \in G_\zeta \subseteq P_\zeta$ be generic over V, $G_\xi \subseteq G_\zeta$, and $G_{\zeta_n} \stackrel{\text{def}}{=} G_\zeta \cap P_{\zeta_n}$. Now for each P_ζ-name $\underset{\sim}{\tau}$ of an ordinal, for some $j < \omega$, $\underset{\sim}{\tau} = \underset{\sim}{\tau}_j$ necessarily $k(*) \stackrel{\text{def}}{=} \underset{\sim}{k}_j[\underset{\sim}{G}_{P_{\zeta_{j+1}}}] > j$ (see condition (d)) hence: q_j forces that $\underset{\sim}{\tau}_{j,j}[G_{\zeta_j}] \in N$. But for $\ell \geq j$ and $j_1 \in [j, \omega)$ we have $p_j \restriction [\zeta_\ell, \zeta_{\ell+1}) \leq p_{j_1} \restriction [\zeta_\ell, \zeta_{\ell+1})$ hence by (e)+(f), $p_j \restriction [\zeta_\ell, \zeta_{\ell+1}] \leq q_{\ell+1}$, so together $p_j \restriction [\zeta_j, \bigcup_{i<\omega} \zeta_i) \leq q$ so $p_j \restriction [\zeta_j, \zeta) \leq q$; hence also q forces $\underset{\sim}{\tau}_j = \underset{\sim}{\tau}_{j,j}$. By the last two sentences $q \Vdash_{P_\zeta}$ "$\underset{\sim}{\tau}_j[\underset{\sim}{G}_{P_\zeta}] \in N \cap \mathrm{Ord}$" so q is really (N, P_ζ)-generic. Now for each ℓ the sequence $\langle \underset{\sim}{\alpha}^n_\ell[G_\zeta] : \ell \leq n < \omega \rangle$ is non increasing (see condition (g)) hence eventually constant; say for $n \in [n_\ell, \omega)$ has value α^*_ℓ. Now if $x \in d[a]$, $j < \omega$ then for $n > h(j, x)$ clearly $k_n > h(j, x)$ so $\underset{\sim}{f}_j(x) = \underset{\sim}{f}^*_{j,n}(x)$, so for every finite $b \subseteq d[a]$, $\langle (\underset{\sim}{f}^*_{j,n} \restriction b)[\underset{\sim}{G}_{P_{\zeta_n}}] : n < \omega \rangle$ is eventually constant, equal to $(\underset{\sim}{f}_j \restriction b)[G_\zeta]$. So for n large enough, $(\underset{\sim}{f}_j \restriction b)[G_{P_\zeta}] = (\underset{\sim}{f}^*_{j,n} \restriction b)[\underset{\sim}{G}_{P_{\zeta_n}}]$ and $\underset{\sim}{f}^*_{j,n}[\underset{\sim}{G}_{P_{\zeta_n}}] R_{\alpha^*_j} \mathbf{g}_a$.

So in $V[G_\zeta]$, $[\underset{\sim}{f}_j][G_\zeta]$ satisfies

$\otimes$ for every finite $b \subseteq d[a]$ for some f', $\underset{\sim}{f}_j[G] \restriction b = f' \restriction b$ and $f' R_{\alpha^*_j} \mathbf{g}_a$.

But we are in Possibility A of Definition 3.3, so $R_{\alpha^*_j}$ is closed, so $\underset{\sim}{f}_j R_{\alpha^*_j} \mathbf{g}_a$. This finishes the proof that $q \Vdash$ "$N[G_\zeta]$ is $(\bar{R}, S, \mathbf{g})$-good". The last point is noting $\alpha^*_\ell \leq \underset{\sim}{\alpha}_{\ell,n}[G_\zeta] \leq \underset{\sim}{\alpha}_{\ell,0} = \beta_\ell$ for $\ell < k$, so we finish.

Possibility B, $\underline{d[a] \notin a}$: The proof is similar to the previous case, only the winning strategy in the game is described in Definition 3.3 (Possibility B) to make the $\underset{\sim}{k}_n$ large enough such that the part of the proof concerning $\underset{\sim}{f}_\ell[\underset{\sim}{G}_\zeta] R_{\alpha^*_\ell} \mathbf{g}_a$ works.

*Possibility A** $\underline{d[a] \notin a}$: By 3.2B(1), the next case implies it.

*Possibility B**, $\underline{d[a] \notin a}$: By 3.3B(2), we have to take care of $\underset{\sim}{f}_0$ only, and this is done as in Possibility B, $d[a] \notin a$, not increasing the set of f_ℓ's we consider.

Possibility B, $\underline{d[a] \in a:}$ We shall reason as in the proof of Possibility A, $d[a] \notin a$, for N_1 (which vary).

If $\mathrm{cf}(\zeta) > \aleph_0$ then the set $\mathcal{I} = \{p \in P_\zeta$: for some $\zeta' < \zeta$, and P_ζ-names $\underset{\sim}{g}_\ell$ we have $p \Vdash$ "$\underset{\sim}{f}_\ell = \underset{\sim}{g}_\ell$ for $\ell < k$"$\}$ is a dense subset of P_ζ and belongs to N_1. So for some n, $p_n \in \mathcal{I}$, by renaming without loss of generality $p \in \mathcal{I}$; w.l.o.g. $\zeta' \le \zeta_1$. We can easily find $\langle q_n : n < \omega\rangle$, $\langle \underset{\sim}{p}'_n : n < \omega\rangle$ such that $q_{n+1}\restriction\zeta_n = q_n$, $q_n \in P_{\zeta_n}$, $q_n \Vdash$ "$N[\underset{\sim}{G}_{P_{\zeta_n}}]$ is $(\bar{R}, S, \mathbf{g})$-good", $q_1 \Vdash$ "$\underset{\sim}{f}_\ell R_{\gamma_\ell} \mathbf{g}_a$" for some $\gamma_\ell \in a$, $\gamma_\ell \le \beta_\ell$ and $q_n \Vdash_{P_{\zeta_n}}$ "$\underset{\sim}{p}'_n \in P_\zeta \cap N$, $\underset{\sim}{p}'_n\restriction\zeta_n \in \underset{\sim}{G}_{P_{\zeta_n}}$ and $m < n \Rightarrow \underset{\sim}{p}'_m \le \underset{\sim}{p}'_n$", and $\underset{\sim}{p}'_0 = p$, and for every P_ζ-name of ordinal $\underset{\sim}{\tau} \in N$ for some n, $q_n \Vdash_{P_{\zeta_n}} [\underset{\sim}{p}'_n \Vdash_{P_\zeta}$ "$\underset{\sim}{\tau} = \underset{\sim}{\alpha}_\tau$", $\underset{\sim}{\alpha}_\tau \in N]$ where $\underset{\sim}{\alpha}_\tau$ is a P_{ζ_n}-name of an ordinal. Now $q_\omega = \bigcup\limits_{n<\omega} q_n$ is (N, P_ζ)-generic, and $p \le q_\omega$; so q_ω is as required, so in the case $\mathrm{cf}(\zeta) > \aleph_0$ we are done†.

So we are left with the case $\mathrm{cf}(\zeta) = \alpha_0$. We have $\aleph^* = 1$ or $\bigwedge\limits_k \oplus_k$; as the later case is harder we speak on it. This time we use the full version of clause (xi) of Definition 3.4. Let $\{\underset{\sim}{\tau}'_j : j < \omega\}$ list the P_ζ-names of ordinals from N and $\{\underset{\sim}{f}'_j : j < \omega\}$ list the P_ζ-names of functions $f \in {}^{d[a]}c[a]$ which belong to N with $\underset{\sim}{f}'_j = \underset{\sim}{f}_j$ for $j < k$ and $\{x_j : j < \omega\}$ list $d[a]$. We now define by induction on $n < \omega$, $\underset{\sim}{M}_n$, $\underset{\sim}{G}^n$, q_n, $\underset{\sim}{p}'_n$, $\underset{\sim}{b}_n$, $\underset{\sim}{\alpha}^n_\ell (\ell < k+n)$ (note that q_0 and also G^0 are already given):

(a) $q_n \in P_{\zeta_n}$, $\mathrm{Dom}(q_n) \setminus \xi = N \cap \zeta_n \setminus \xi$, $q_{n+1}\restriction\zeta_n = q_n$ (q_0 is given).

(b) q_n is (N, P_{ζ_n})-generic

(c) $q_n \Vdash_{P_{\zeta_n}}$ "$N[\underset{\sim}{G}_{P_{\zeta_n}}]$ is $(\bar{R}, S, \mathbf{g})$-good"

(d) $\underset{\sim}{M}_n$, $\underset{\sim}{G}^n$, $\underset{\sim}{p}'_n$, $\underset{\sim}{b}_n$, $\underset{\sim}{\alpha}^n_\ell$ (for $\ell < k+n$) are P_{ζ_n}-names

(e) $q_n \Vdash_{P_{\zeta_n}}$ "$\underset{\sim}{b}_n$ is a finite subset of $d[a]$, $\underset{\sim}{M}_n$ a countable elementary submodel of $(H(\chi_1)[\underset{\sim}{G}_{P_{\zeta_n}}], \in, <^*_{\chi_1})$ which belongs to $N[\underset{\sim}{G}_{P_{\zeta_n}}]$† and $\underset{\sim}{b}_n \subseteq \underset{\sim}{M}_n$"

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† This applies to all possibilities.

† And if we adopt the demand on a_1 in clause (xi) of Definition 3.4, we should add $\underset{\sim}{M}_n \cap (\bigcup S) \in S$

(f) $q_n \Vdash_{P_{\zeta_n}}$ "$\underset{\sim}{G}^n \subseteq P_\zeta/\underset{\sim}{G}_{\zeta_n} \cap \underset{\sim}{M}_n[\underset{\sim}{G}_{P_{\zeta_n}}]$ is generic over $\underset{\sim}{M}_n[\underset{\sim}{G}_{P_{\zeta_n}}]$, $\underset{\sim}{p}_n[\underset{\sim}{G}_{P_{\zeta_n}}] \in \underset{\sim}{G}^n$ so $\underset{\sim}{p}'_n \in P_{\zeta_n} \cap N$ (i.e. $\underset{\sim}{p}'_n[\underset{\sim}{G}_{P_{\zeta_n}}] \in P_{\zeta_n} \cap N$) and $\underset{\sim}{p}'_n \restriction \zeta_n \in \underset{\sim}{G}_{P_{\zeta_n}}$ and $\underset{\sim}{f}_j \in \underset{\sim}{M}_n$, $\underset{\sim}{f}_j[\underset{\sim}{G}^n] R_{\alpha^n_j} \mathbf{g}_\alpha$"

(g) $q_{n+1} \Vdash$ "$\underset{\sim}{p}'_n \le \underset{\sim}{p}'_{n+1}$, $\underset{\sim}{f}_j[\underset{\sim}{G}^n]\restriction \underset{\sim}{b}_n \subseteq \underset{\sim}{f}_j[\underset{\sim}{G}^{n+1}]\restriction \underset{\sim}{b}_n$ for $j < k+n$".

(h) $q_n \Vdash$ "$\underset{\sim}{p}'_{n+1}[\underset{\sim}{G}_{P_{\zeta_n}}]$ forces a value to $\underset{\sim}{\tau}'_n$ (in $P_\zeta/\underset{\sim}{G}_{P_{\zeta_n}}$)" and to $\underset{\sim}{f}_j \restriction \underset{\sim}{b}_n$ for $j < k+n$.

There is no problem to carry the definition using $\bigwedge_k \oplus_k$. Now we have some freedom: choosing the $\underset{\sim}{b}_n$. So actually this is a play of the game, where the choices made above are fixing the moves of player I (with some extras). It will suffice to have player II winning, which is O.K. (so less than "I wins the game" is used).

In the end we let $q_\omega = \bigcup_{n<\omega} q_n$ and continue as in Possibility A, $d[a] \notin a$.

Possibility B^*, $\underline{d[a] \in a}$: Combine the proofs for possibility B^* when $d[a] \notin a$ (i.e. use 3.3B(2)) but $\underset{\sim}{M}_n = N_1$ and the proof of possibility B when $d[a] \in a$.

2) Left to the reader $\square_{3.6}$

3.7 Application. Open dense subsets.

3.7A. Context and Definition. Let $\langle \eta^*_\ell : \ell < \omega \rangle$ enumerate ${}^{\omega>}\omega$ such that $\eta^*_m \restriction n \in \{\eta^*_i : i \le m\}$, let fR_ng mean

$f, g : {}^{\omega>}\omega \to {}^{\omega>}\omega$ and $\eta \in {}^{\omega>}\omega \setminus \{\eta^*_\ell : \ell < n\}$ implies that there is ν such that $\eta \trianglelefteq \nu \triangleleft \nu\char`^ f(\nu) \trianglelefteq \eta\char`^ g(\eta)$.

Note that if fR_ng and $g' : {}^{\omega>}\omega \to {}^{\omega>}\omega$ and $(\forall \eta)(g(\eta) \trianglelefteq g'(\eta))$ then fR_ng'.
Let, for some subuniverse V', $S \subseteq \mathcal{S}_{<\aleph_1}(H(\aleph_1)^{V'})$, and for $a \in S$, $\mathbf{g}_a \in \bigcup S$ be such that $(\forall f)(f \in a \;\&\; f$ is a function from ${}^{\omega>}\omega$ to ${}^{\omega>}\omega \Rightarrow \bigvee_n fR_n\mathbf{g}_a)$. Clearly such $\mathbf{g} = \langle \mathbf{g}_a : a \in S \rangle \in V'$ exists.

Let $R = \bigvee_{n<\omega} R_n$, and let F^* be the family of functions from ${}^{\omega>}\omega$ to ${}^{\omega>}\omega$.

3.7B Claim. 1) $(\bar{R}, S, \mathbf{g})$ covers *iff* S is stationary and

$$(\forall f \in V)[f \in F^* \to (\exists g \in \bigcup S)[fRg]]$$

2) If $(\bar{R}, S, \mathbf{g})$ covers *then* it strongly covers (for possibility A*) and

$$(\forall f \in {}^{\omega}\omega)(\exists g \in \bigcup S)(\bigwedge_n f(n) < g(n)).$$

3) If $N \prec (H(\chi), \in, <^*_\chi)$ is countable, $(\bar{R}, S, \mathbf{g})$ covers and $N \cap (\bigcup S) = a \in S$ *then* N is $(\bar{R}, S, \mathbf{g})$-good.
4) Each R_n and $R = \bigvee_{m<\omega} R_m$ are transitive.

Proof. Straightforward. E.g.
(2) First let us show that $\oplus^+_1$ of 3.4B(1) hold. So suppose that N, a_1, a_2, f, $\langle f_n : n < \omega \rangle$ and α, α_n are as assumptions of $\oplus^+_1$. For $n < \omega$ we define

$$d^0_n = \{\eta^*_\ell : \ell < n \text{ and } (\forall m \leq \ell)(\forall \eta \trianglelefteq \eta^*_m{}^\frown \mathbf{g}_{a_1}(\eta^*_m))(f_n(\eta) = f(\eta))\},$$

$$d^1_n = \{\eta^*_\ell : (\exists \nu \in d^0_n)(\eta^*_\ell \trianglelefteq \nu^\frown \mathbf{g}_{a_1}(\nu))\},$$

$$d^2_n = d^1_n \cup \{\eta^*_\ell : \ell < \alpha_n\}.$$

Note that
$(*)_1$ $d^0_n \subseteq d^1_n \subseteq d^2_n$ are finite subsets of ${}^{<\omega}\omega$,
$(*)_2$ each d^i_n $(i < 3)$ is closed under initial segments,
$(*)_3$ $(\forall \nu \in {}^{\omega>}\omega)(\forall^* n)(\nu \in d^0_n)$.
Using $(*)_3$ one easily constructs a function $f^* \in F^* \cap N$ such that
$(*)_4$ $(\forall k < \omega)(\eta^*_k \neq d^0_n \Rightarrow f_n R_k f^*)$
(note that the sequence $\langle d^0_n : n < \omega \rangle$ is in N). Then for some $\beta < \omega$ we have

$$f^* R_\beta \mathbf{g}_{a_2} \qquad \text{and} \qquad \mathbf{g}_{a_1} R_\beta \mathbf{g}_{a_2}.$$

Take n such that
$(*)_5$ $(\forall m \leq \beta)(\eta^*_m \in d^)_n)$
and put $d = \{\nu : (\exists \eta \in d^2_n)(\nu \trianglelefteq \eta^\frown \mathbf{g}_{a_2}(\eta)\}$.

Suppose that $f' \in F^*$ is such that $f' R_{\alpha_n} \mathbf{g}_{a_1}$ and $f' \restriction d = f_n \restriction d$. We are going to show that $f_n R_\alpha \mathbf{g}_{a_2}$. To this end suppose that $\ell \leq \alpha$ and consider the following three cases.

Case 1: $\eta^*_\ell \notin d^2_n$

Then $\eta^*_\ell \notin d^0_n$ and hence (by $(*)_5$) $\ell > \beta$. Since $\mathbf{g}_{a_1} R_\beta \mathbf{g}_{a_2}$ we find $\nu \in {}^{\omega>}\omega$ such that

$$\eta^*_\ell \trianglelefteq \nu \triangleleft \nu\char`^\mathbf{g}_{a_1}(\nu) \trianglelefteq \eta^*_\ell\char`^\mathbf{g}_{a_2}(\eta^*_\ell).$$

It follows from $(*)_2$ that $\nu \notin d^2_n$, so $\nu = \eta^*_k$ for some $k \geq \alpha_n$. Since $f' R_{\alpha_n} \mathbf{g}_{a_1}$ we find η such that

$$\eta^*_\ell \trianglelefteq \nu \trianglelefteq \eta \triangleleft \eta\char`^f'(\eta) \trianglelefteq \nu^*\char`^\mathbf{g}_{a_1}(\nu) \trianglelefteq \eta^*_\ell\char`^\mathbf{g}_{a_2}(\eta^*_\ell),$$

as required.

Case 2: $\eta^*_\ell \in d^2_n \setminus d^0_n$

Since $\eta^*_\ell \notin d^0_n$ we know $\ell > \beta$. As $f^* R_\beta \mathbf{g}_{a_2}$, we find k such that

$$\eta^*_\ell \trianglelefteq \eta^*_k \triangleleft \eta^*_k\char`^f^*(\eta^*_k) \trianglelefteq \eta^*_\ell\char`^\mathbf{g}_{a_2}(\eta^*_\ell).$$

Necessarily $\eta^*_k \notin d^0_n$ and therefore $f_n R_k f^*$. Consequently we find ν such that

$$\eta^*_\ell \trianglelefteq \eta^*_k \trianglelefteq \nu \triangleleft \nu\char`^f_n(\nu) \trianglelefteq \eta^*_k\char`^f^*(\eta^*_k) \trianglelefteq \eta^*_\ell\char`^\mathbf{g}_{a_2}(\eta^*_\ell).$$

Plainly $\nu \in d$ (as $\eta^*_\ell \in d^2_n$) and therefore $f_n(\nu) = f'(\nu)$, so we get what is required.

Case 3: $\eta^*_\ell \in d^0_n$

Since $f R_\alpha \mathbf{g}_{a_2}$ we find ν such that

$$\eta^*_\ell \trianglelefteq \nu \triangleleft \nu\char`^f(\nu) \trianglelefteq \eta^*_\ell\char`^\mathbf{g}_{a_2}(\eta^*_\ell).$$

As $\eta^*_\ell \in d^0_n$ we know that $f_n(\nu) = f(\nu)$ so we conclude

$$\eta^*_\ell \trianglelefteq \nu \triangleleft \nu\char`^f_n(\nu) \trianglelefteq \eta^*_\ell\char`^\mathbf{g}_{a_2}(\eta^*_\ell).$$

This finishes verifying the clause $\oplus^+_1$. Now we may apply 3.4B(1) and easily check that $(\bar{R}, S, \mathbf{g})$ strongly cover for possibility A* (i.e. this claim gives $\oplus_1$ of Definition 3.3).

The other parts should be clear. $\square_{3.7B}$

3.7C Claim. Suppose in V, $(\bar{R}, S, \mathbf{g})$ covers, Q is a proper forcing notion, *then*: Q is $(\bar{R}, S, \mathbf{g})$-preserving for possibility A* *iff*

$$V^Q \models (\forall f \in F^*)(\exists g \in \bigcup S)[fRg]$$

(so also $\Vdash_Q$ "$(\bar{R}, S, \mathbf{g})$ covers" is equivalent to them).

Proof. The "only if" part is straightforward.

The converse implication follows from 3.4A(3) (note that the demand $\otimes^+$ was proved in the proof of 3.7B(2)). $\square_{3.7C}$

3.7D Claim. If $(\bar{R}, S, \mathbf{g})$ covers then "proper+ $(\bar{R}, S, \mathbf{g})$ - preserving" is preserved by composition, and more generally by CS iteration.

3.7E Claim. 1) Suppose $(\bar{R}, S, \mathbf{g})$ covers, *then* for every dense open $A \subseteq {}^{\omega>}\omega$ there is a dense open $B \subseteq {}^{\omega>}\omega, B \in \bigcup S$ and $B \subseteq A$.
2) If F^V is the family of functions from ${}^{\omega>}\omega$ to ${}^{\omega>}\omega$ and $F \subseteq F^V$ is such that $\forall g \exists f\ [gRf]$ and $S \subseteq S_{<\aleph_1}(H(\chi_1))$ is stationary then we can find $\mathbf{g} = \langle \mathbf{g}_a : a \in S \rangle$, $\mathbf{g}_a \in F$ such that $(\bar{R}, S, \mathbf{g})$ covers.

Proof. 1) For a dense open set $A \subseteq {}^{\omega>}\omega$ define $f_A \in F$ by

$$f_A(\eta) \text{ is such that } \eta\hat{\ } f_A(\eta) \in A.$$

Let $n < \omega$, $g \in \bigcup S$ be such that $f_A R_n g$ and define

$$B \stackrel{\text{def}}{=} \{\eta \in {}^{\omega>}\omega : \text{ for some } \nu \in {}^{\omega>}\omega \setminus \{\eta^*_\ell : \ell < n\} \text{ we have } \nu\hat{\ }g(\nu) \trianglelefteq \eta\},$$

Clearly B is open dense, $B \in \bigcup S$, and $B \subseteq A$.

2) Straightforward. $\square_{3.7E}$

3.7F Remark. 1) In 3.7A we could have weakened fR_ng to: $\eta \notin \{\eta^*_\ell : \ell < n\}$ implies that for some $\nu, \nu \triangleleft \nu\hat{\ }f(\nu) \trianglelefteq \eta\hat{\ }g(\eta)$, call it R^w_n (and $\bar{R}^w, R^w, (S, \bar{R}^w, \mathbf{g})$ are defined accordingly). So we can demand $\langle\rangle \notin \text{Rang}(g)$. Then 3.7 B-E holds for this version too. (For 3.7B(2) second clause: for every

$f \in {}^\omega\omega$ let $f' : {}^{\omega>}\omega \to {}^{\omega>}\omega$ be such that $f'(\langle n\rangle) = \langle n, n+1, \dots, n+f(n)\rangle$, and $f'(\langle\rangle) = f'(\langle 0\rangle)$. So there are $g' \in \bigcup S$ and n such that $f' R_n g'$. Let $g(n) = \min\{\ell g(\nu \hat{} g'(\nu)) : \nu \hat{} g'(\nu) \trianglelefteq \langle n\rangle \hat{} f'(\langle n\rangle)\}$; easily $f^* <^* g \in \bigcup S$.)

2) Assume $\bar{R}$ is as in 3.7A and S, $\mathbf{g}$ as in 3.1. Then also the inverse of 3.7E(1) holds, see 3.7H.

3.7H Claim. Suppose $(\bar{R}, S^1, \mathbf{g}^1), (\bar{R}^w, S^2, \mathbf{g}^2)$ is as in 3.7A for the same V' (for $\bar{R}^w$ defined in 3.7F) and S^1, $S^2 \subseteq S_{\leq\aleph_0}(\aleph_1)^{V'}$ are stationary even in V, *then*: $(\bar{R}, S^1, \mathbf{g}^1)$ covers

iff for every dense open $A \subseteq {}^{\omega>}\omega$ there is a dense open $B \subseteq {}^{\omega>}\omega$ such that $B \in \bigcup S^2 = \bigcup S^1$ and $B \subseteq A$

iff $(\bar{R}^w, S^2, \mathbf{g}^2)$ covers.

Proof. first $\Rightarrow$ *second*: this is 3.7E(1).

second $\Rightarrow$ *third*:

Let $f(\in V)$ be a function from ${}^{\omega>}\omega$ to ${}^{\omega>}\omega$; we define $A_f = \{\rho : \rho \in {}^{\omega>}\omega$ and $(\exists \nu)(\nu \hat{} f(\nu) \trianglelefteq \rho)\}$. Clearly $A_f \in V$ is a dense open subset of ${}^{\omega>}\omega$. So by the assumption there is a dense open $B \subseteq {}^{\omega>}\omega$ which belongs to $\bigcup S^2$ and $B \subseteq A_f$. So, working in V' there is $g \in \bigcup S^2$ such that: g is a function from ${}^{\omega>}\omega$ to ${}^{\omega>}\omega$ and for every $\eta \in {}^{\omega>}\omega$ we have $\eta \hat{} g(\eta) \in B$. It suffices to prove that $f R_0 g$ (as $f R_0 g \Rightarrow f R g$ and R is a partial order). Now for every $\eta \in {}^{\omega>}\omega$, we know $\eta \hat{} g(\eta) \in B$ hence $\eta \hat{} g(\eta) \in A_f$, but by its definition this implies the existence of $\nu \in {}^{\omega>}\omega$ such that $\nu \hat{} f(\nu) \trianglelefteq \eta \hat{} g(\eta)$. So ν is as required.

third $\Rightarrow$ *first:*

Let f be a function from ${}^{\omega>}\omega$ to ${}^{\omega>}\omega$. Let us define a function f' from ${}^{\omega>}\omega$ to ${}^{\omega>}\omega$ as follows. For $\eta \in {}^{\omega>}\omega$, let $\langle \rho^k_\eta : k < k_\eta\rangle$ be a list of $\{\rho : \rho \in {}^{\omega>}\omega, \ell g(\rho) = \ell g(\eta)$ and $\bigwedge_{\ell < \ell g(\eta)} \rho(\ell) \leq \eta(\ell)\}$, so η appears in it and $1 \leq k_\eta < \omega$. W.l.o.g. $\eta = \rho^{(k_\eta)-1}_\eta$. We now choose by induction on $k \leq k_\eta$, a sequence $\nu^k_\eta \in {}^{\omega>}\omega$. Let $\nu^0_\eta = \eta$, and ν^{k+1}_η be:

$$(\rho^k_\eta \cup \nu^k_\eta \restriction [\ell g\eta, \ell g\nu^k_\eta)) \hat{} f[\rho^k_\eta \cup (\nu^k_\eta \restriction [\ell g\eta, \ell g\nu^k_n))] \hat{} \langle 0\rangle.$$

Finally $f'(\eta)$ is defined by $\eta \hat{} f'(\eta) = \nu^{(k_\eta)}_\eta$, remember $\eta = \rho^{k_\eta - 1}_\eta$.

So by the assumption there is $g' \in \bigcup S^2$ such that g' is a function from ${}^{\omega>}\omega$ to ${}^{\omega>}\omega$ and $f' R^w g'$. As $\bigcup S^2$ includes the set of functions from ${}^{\omega>}\omega$ to ${}^{\omega>}\omega$ in V', without loss of generality $f' R^w_0 g'$, and as $\langle\rangle \notin \mathrm{Rang}(f')$, by 3.7F we know $\forall\eta[\ell g(f'(\eta)) > 0)$. We now define a function g from ${}^{\omega>}\omega$ to ${}^{\omega>}\omega$; we define $g(\eta)$ by induction on $k = \ell g(\eta)$; given η of length k, we choose by induction on $\ell < k$ natural numbers $i_\ell \in \{\eta(\ell), \eta(\ell)+1\}$ such that for $m \le k$ we have i_ℓ is *not* the first element of $f'(\langle i_0, \dots, i_{\ell-1}\rangle)$ (possible as $f'(\langle i_0, \dots, i_{\ell-1}\rangle)$ has length > 0).

Let $\eta' = \langle i_0, \dots, i_{k-1}\rangle$ and $g(\eta) = g'(\eta')$. Note: η' is well defined and for every $\ell < k$ the sequence η' (and even $\eta' \restriction (\ell+1)$) is not an initial segment of $(\eta'\restriction\ell)\hat{\ } f'(\eta'\restriction\ell)$. By the choice of g' and definition of R^w_0 we know that there is $\nu^0 \in {}^{\omega>}\omega$ such that $\nu^0 \trianglelefteq \nu^0 \hat{\ } f'(\nu^0) \trianglelefteq \eta' \hat{\ } g'(\eta')$. By the choice of $\eta', \neg(\nu^0 \triangleleft \eta')$ so necessarily $\eta' \trianglelefteq \nu^0$. Let $\nu^1 = \eta \cup (\nu^0\restriction[k, \ell g\nu^0))$, so $\eta \trianglelefteq \nu^1$, $\ell g(\nu^1) = \ell g(\nu^0)$ and $(\forall\ell)[\nu^1(\ell) \le \nu^0(\ell)]$. Hence by the choice of $f'(\nu^0)$ there is $\nu^2, \nu^1 \trianglelefteq \nu^2 \trianglelefteq \nu^2 \hat{\ } f(\nu^2) \triangleleft \nu^2 \hat{\ } f(\nu^2) \hat{\ } \langle 0\rangle \trianglelefteq \nu^1 \hat{\ } f'(\nu^0)$, just choose m such that $\nu^1 = \rho^m_{\nu^0}$ and put $\nu^2 \stackrel{\text{def}}{=} \rho^m_{\nu^0} \hat{\ } (\nu^m_{\nu^0}\restriction[\ell g\nu^0, \ell g\nu^m_{\nu^0}))$. Note that $\nu^1 \hat{\ } f'(\nu^0) \trianglelefteq \eta \hat{\ } g'(\eta') = \eta \hat{\ } g(\eta)$ and hence So $f R_0 g$. As g was defined from f' alone; and R is a partial order so we may easily finish. $\square_{3.7H}$

3.8 Application. Old reals of positive measure:

This is closely related to Judah Shelah [JdSh:308, §1].

3.8A Context and Definition.

Let $S \subseteq \mathcal{S}_{<\aleph_1}(H(\aleph_1))^{V^1}$, $A \stackrel{\text{def}}{=} \bigcup S$ transitive model of ZFC$^-$ and S a stationary subset of $\mathcal{S}_{<\aleph_1}(\bigcup S)$. For $a \in S$ let $\mathbf{g}_a \in {}^\omega 2$ be random over a, for simplicity: $\mathbf{g}_a \in \bigcup S$ and $\alpha^* = \omega$. For $n < \alpha^*$ we define relation R_n by $f R_n g$ iff: $g \in {}^\omega 2$, f a sequence of nonempty rational intervals (in our context means $I_\rho = \{\eta \in {}^\omega 2 : \rho \triangleleft \eta\}$ for some $\rho \in {}^{\omega>}2$) and† $\sum_{\ell<\omega} \mathrm{Lb}\,(f(\ell)) \le 1$ (where $\mathrm{Lb}(\{\eta \in {}^\omega 2 : \rho \triangleleft \eta\}) \stackrel{\text{def}}{=} 2^{-\ell g(\rho)}$), and $m \ge n \Rightarrow g \notin f(m)$.

† Lb stands for Lebesgue measure.

3.8B. Claim.

1) If $(\bar{R}, S, \mathbf{g})$ covers *then* it strongly covers (for possibility A).
2) If $(\bar{R}, S, \mathbf{g})$ covers *then* $S \cap {}^\omega 2$ does not have measure zero (equivalently, it has positive outer measure).
3) If $(\bar{R}, S, \mathbf{g})$ covers *then* "proper + $(\bar{R}, S, \mathbf{g})$ - preserving" is preserved by composition and more generally by CS iteration.
4) If in V, $A \subseteq {}^\omega 2$ is not null (i.e. does not have Lb measure zero) and $S \subseteq \mathcal{S}_{<\aleph_1}(H(\aleph_1))$ is stationary *then* for some $\mathbf{g} = \langle \mathbf{g}_a : a \in S \rangle$, we have: $(\bar{R}, S, \mathbf{g})$-covers and $a \in S \Rightarrow \mathbf{g}_a \in A$.

Proof. 1) We check that Possibility A holds, so we have to check $\oplus_k$. So in V^P let $Q, N, a, N_1, a_1, G^1, p, k, \underset{\sim}{f}_\ell, \beta_\ell, f^*_\ell$ $(\ell < k)$, x, y be given as there (so by 3.8A we have $d[a] \in a$). Let $\langle p_n : n < \omega \rangle$ be such that $p \le p_n \le p_{n+1} \in G^1$, (so p, $p_n \in Q \cap N_1$) and $\bigwedge_{q \in G^1} \bigvee_{n < \omega} q \le p_n$. Let $N_2 \prec (H(\chi_1), \in, <^*_{\chi_1})$ be countable such that

$$\{N_1, \langle p_n : n < \omega \rangle, \langle \underset{\sim}{f}_\ell, f^*_\ell : \ell < k \rangle, x, y\} \in N_2$$

and $a_2 \stackrel{\text{def}}{=} N_2 \cap \bigcup S \in S$ and $N_2 \in N$. Let $\langle p^n_m : m < \omega \rangle$, $f^*_{\ell,n}$ be such that: $p^n_0 = p_n$, $p^n_m \le p^n_{m+1}$, $\langle p^n_m : m < \omega \rangle$ is a generic sequence for (N_2, Q) and $p^n_m \Vdash$ "$\underset{\sim}{f}_\ell \restriction m = f^*_{\ell,n} \restriction m$"; without loss of generality $\langle f^*_{\ell,n}, p^n_m : \ell < k, n < \omega, m < \omega \rangle \in N$. Clearly for some $m^*_{\ell,n} < \omega$ we have $f^*_{\ell,n} R_{m^*_{\ell,n}} \mathbf{g}_a$. As we can thin the sequence $\langle \langle p_n, p^n_m : n < \omega \rangle : n < \omega \rangle$ as long as it belongs to N without loss of generality for some rational $u_n \in \mathbb{Q}$, $0 \le u < 1$, and $p_n \Vdash \sum\limits_i \mathrm{Lb}(\underset{\sim}{f}_\ell(i)) \in (u^\ell_n, u^\ell_n + 1/k2^{2^n}]$ and $\langle u^\ell_n : n < \omega \rangle \in N$ is strictly increasing and $\langle u^\ell_n + 1/k2^{2^n} : n < \omega \rangle$ is strictly decreasing, and p_n forces a value to $\underset{\sim}{f}_\ell \restriction m_{\ell,n}$ such that $\sum\limits_{i < m_{\ell,n}} \underset{\sim}{f}_\ell(i) > u^\ell_n$ and $\langle m_{\ell,n} : n < \omega \rangle \in N$. So it is forced by p_n that $\underset{\sim}{f}_\ell \restriction m_{\ell,m}$ has the value above, and $\sum\limits_{i \ge m_{\ell,n}} \underset{\sim}{f}(i) < 1/k2^{2^n}$, so $f^*_{\ell,n}$ satisfy this too. For every n we have $\sum_{\ell<k} \sum_{i \ge m_{\ell,n}} \mathrm{Lb}\,(f^*_\ell(i)) < 1/2^{2^n}$ and $p^n_0 \Vdash$ "$\underset{\sim}{f}_\ell \restriction m_{\ell,n} = f^*_\ell \restriction m_{\ell,n}$", hence $\sum\limits_i \sum \{\mathrm{Lb}\,(I_\rho)$: for some $\ell < k$, $n < \omega$ we have $I_\rho = f^*_{\ell,n}(i)\} \le \sum\limits_{\ell<k} \sum\limits_i \mathrm{Lb} f^*_\ell(i) + \sum\limits_\ell \sum\limits_n \sum\limits_{i \ge m_{\ell,n}} \mathrm{Lb} f^*_{\ell,n}(i) \le \sum\limits_\ell \sum\limits_i \mathrm{Lb} f^*_\ell(i) + \sum\limits_\ell \sum_n 1/2^{2^n}$ so this is a sum of two reals $< \infty$ (note that in first sum for each

i it is on the set double appearances not counted). Hence $\mathbf{g}_a$ belongs to only finitely many of the sets $\bigcup\{I_\rho :$ for some $\ell < k, n < \omega, I_\rho = \underset{\sim}{f}_{\ell,n}(i)\}$, so the rest is easy.

2), 3) are left to the reader.

4) Straightforward. $\square_{3.8B}$

3.8C Claim.

(1) Assume $S \subseteq \mathcal{S}_{<\aleph_1}(H(\aleph_1))$, and S is stationary as a subset of $\mathcal{S}_{<\aleph_1}(\bigcup S)$, and $\mathbf{g} : S \to {}^\omega 2$ is such that:

$(*)_{S,\mathbf{g}}$ if x, $S \in H(\chi)$ *then* for some countable $N \prec (H(\chi), \in <^*_\chi)$ we have $\{x, S, \mathbf{g}\} \in N$ and $N \cap (\bigcup S) = a \in S$ and $\mathbf{g}_a$ belongs to no measure zero set from N.

Then: if $\langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$ is a CS iteration of proper forcing proper notions, each Q_i preserving $(*)_{S^1,\mathbf{g}^1}$ whenever $\bigcup S \in \bigcup S^1$, $(\forall a \in S^1)(a \cap (\bigcup S) \in S)$, $\mathbf{g}^1_a = \mathbf{g}_{a\cap(\bigcup S)}$, this means $V^{P_i} \models$ " if $(*)_{S^1,\mathbf{g}^1}$ then $\Vdash_{Q_i}$ $(*)_{S^1,\mathbf{g}^1}$") then P_α preserves $(*)_{S,\mathbf{g}}$.

(2) Assume $X \subseteq {}^\omega 2$ has positive (outer) Lebesgue measure. If $\langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$, is CS iteration of proper forcing, each Q_i preserve the property $(*)_{S,\mathbf{g}}$ whenever $\mathbf{g} : S \to X$, *then* P_α preserves the property of "being of positive outer measure" for $X' \subseteq X$.

Proof. 1) As we can replace $\bar{Q}$ by $\langle P_{\beta+i}/P_\beta, \underset{\sim}{Q}_j : i \leq \alpha-\beta, j < \alpha-\beta\rangle$, and S by $S_1 \subseteq S$ as long as $(*)_{S_1,\mathbf{g}\restriction S_1}$ holds in V^{P_β}, it is enough to prove $\Vdash_{P_\alpha}$ "$(*)_{S,\mathbf{g}}$". Now letting $S^* = \{a \in S : \mathbf{g}_a$ is random over $a\}$, clearly $S^* \subseteq S$ is stationary and S^*, $\mathbf{g}\restriction S^*$ fit 3.8A.

We prove by induction on i that

(a) P_i is $(\bar{R}, S^*, \mathbf{g}\restriction S^*)$-preserving (for possibility A) and

(b) $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is $(\bar{R}, S^*, \mathbf{g}\restriction S^*)$-preserving".

Arriving to i, clause (a) holds by 3.8B(3). To prove clause (b) first we deal only with clause (b) of definition 3.4 and, for χ large enough in V^{P_i} we let

$$W = \{N : \text{(i) } N \text{ is a countable elementary submodel of } (H(\chi), \in, <^*_\chi), \text{ to which } S^*, \mathbf{g}, \underset{\sim}{Q}_i \text{ belong and } a = N \cap (\bigcup S) \in S, \text{ and } \mathbf{g}_a \text{ is random over } N$$
$$\text{(ii) for some } p \in \underset{\sim}{Q}_i \cap N \text{ there is no } q \text{ such that } p \leq q \in \underset{\sim}{Q}_i, q \text{ is } (N, \underset{\sim}{Q}_i)\text{-generic and } q \Vdash_{Q_i} \text{“}N[\underset{\sim}{G}_{Q_i}] \text{ is } (\bar{R}, S^*, \mathbf{g}{\restriction}S^*)\text{-good}\}.$$

If (b) fails then W is stationary (otherwise if $\chi' = (2^\chi)^+$, $\{W, \chi\} \in N \prec (H(\chi'), \in, <^*_\chi)$ then for N the required conclusion holds and we clearly finish). For $p \in \underset{\sim}{Q}_i$ let W_p be defined like W with p (in clause (ii)) fixed. So by normality for some p, W_p is stationary. But defining $\mathbf{g}^1 \stackrel{\text{def}}{=} \langle \mathbf{g}^1_N = \mathbf{g}_{N \cap \bigcup S} : N \in W_p \rangle$, clearly $V^{P_i} \Vdash$ “$(*)_{W_p, \mathbf{g}^1}$” but $V^{P_i} \models$ “ $\Vdash_{Q_i} \neg(*)_{W_p, \mathbf{g}^1}$” contradicting the assumption.

But we have to deal also with clause (b) in the conclusion of Definition 3.4, so define

$$W' = \{N : (i) \text{ as before}$$
$$(ii) \text{ for some } p \in \underset{\sim}{Q}_i \cap N \text{ and } k < \omega \text{ and } \underset{\sim}{f}_\ell \in N (\ell < k)$$
$$N_1, a_1, \langle p_n : n < \omega \rangle, f^*_\ell$$
$$\text{as in } (*) \text{ of Definition 3.4}$$
$$\text{there is no } q \text{ satisfying } (a) + (b) \text{ of Definition 3.4 }\}.$$

Assume toward contradiction that clause (b) here fails, hence $W \subset \mathcal{S}_{\leq \aleph_0}(H(\chi))$ is stationary and w.l.o.g. let $\langle m_{\ell,n} : \ell < k, n < \omega \rangle \in N$ be as in the proof of 3.8B. So for some $x = \langle p, h, \langle p_n : n < \omega \rangle, N_1, a_1, \langle f^*_\ell : \ell < k \rangle, \langle m_{\ell,n} : \ell, n \rangle \rangle$ we have

$$W'_x \stackrel{\text{def}}{=} \{N \in W' : x \in N \text{ gives a counterexample in (ii) of the Definition of } W'\}$$

is stationary. Let $\chi_1 >> \chi$. In $(V^{P_i})^{\underset{\sim}{Q_i}}$, clearly $\{\mathbf{g}_a : a = N \cap \bigcup S, N \in W'_x\}$ is not null. Also for every club E of $\mathcal{S}_{\leq\aleph_0}(H(\aleph_1)^{V^{P_i}})$ we have: the set $\{\mathbf{g}_a : a \in U_e\}$ is not null where $U_E = \{N \cap H(\chi_1) : N \in W''_x\} \cap E$.

So for some club E^*, the outer measure of $\{\mathbf{g}_a : a \in U_E\}$ is minimal. So really in V^{P_i} we have a $\underset{\sim}{Q}_i$-name $\underset{\sim}{E}^* \in H(\chi_1)$. We can find $\chi_0 < \chi$ large enough such that letting

$$\underset{\sim}{E}' = \{a \cap H(\chi_0) : a \in \underset{\sim}{E}^*\} \text{ and } W''_x = \{N \cap H(\chi_0) : N \in W'_x\}$$

we have all those properties and there are $<^*$-first hence belong to N_1. Replacing N_1, $\langle p_n : n < \omega\rangle$ by N_2, $\langle p'_n : n < \omega\rangle$ by 3.8B, we have $\{W''_x, \chi_1, \underset{\sim}{E}', x\} \in N_1$. So choose $N \in W'_x$.

Now for some n, p_n force outer Lebesgue measure of $\{\mathbf{g}_a : a \in U''_{\underset{\sim}{E}'}\}$ is $> 1/n^*$, $n^* > 0$, and if n is large enough, it forces value to $\underset{\sim}{f}_\ell \restriction m$, and force $\sum_{i \geq m} \mathrm{Lb}(\underset{\sim}{f}(i)) < 1/n^*(k+1)$. Let $p_n \in G_{Q_i} \subseteq Q_i$, G_{Q_i} generic over V^{P_i}.

So $E^{\otimes} = \{N \prec H(\chi) : N[G_{Q_i}] \cap {}^i\mathrm{Ord} \subseteq N\}$ is a club, so restricting ourselves to it does not change the outer measure. Let $N \in U_{\underset{\sim}{E}'} \cap E^{\otimes}$, then $\bigvee_\ell \neg \underset{\sim}{f}_\ell R_{\beta_\ell} g_{N \cap \bigcup S}$. There are $2k$ possibilities: which i, and if bad i is $\geq k m_{\ell,n}$ or $< m_{\ell,n}$, later is impossible.

The outer measure of former is $< k1/n^*(k+1) < 1/n^*$, but by the choice of the club $\underset{\sim}{E}^*$ contradiction.

Remark. Really this is part of a quite general theorem. We shall return to it elsewhere.

2) Should be clear. $\square_{3.8C}$

3.9 Application. Souslinity of an ω_1-tree.

Here we return to the issue of IX §4.

3.9A Context and Definition. Let T be an ω_1-tree, say with $[\omega\alpha, \omega\alpha + \omega) \setminus \{0\}$ being the $(1+\alpha)$-th level. Let $W \subseteq \omega_1$ be the set of limit ordinals $\delta = \omega\delta$ (for clarity). Let for $t \in T_\gamma, \beta \leq \gamma, t \restriction \beta$ be the unique $s \in T_\beta$, such that $s \leq_T t$. Let for $\delta \in W$, $a_\delta = \delta \cup {}^{\omega>}\delta$, $S = \{a_\delta : \delta \in W\}$, $d[a] = a$, $c[a] = a$ (so $d' = \omega_1$,

$c' = {}^{\omega>}\omega_1$) and $\{t^\delta_n : n < \gamma_\delta \leq \omega\}$ be a subset of T_δ for some non zero $\gamma_\delta \leq \omega$. Let $\alpha^* = \omega_1$ and lastly we choose[†] $\mathbf{g}_a$ such that: for $\alpha \in a \cap \alpha^*$, we let $f R_\alpha \mathbf{g}_a$ *iff* one of the following holds:

(α) $\alpha = 0$ and $f^{-1}(\{1\}) \cap \{s \in T_{<\delta} : 0 < s <_T t^\delta_{f(0)}\} \neq \emptyset$ or

(β) $0 < \alpha < \delta$ and $f^{-1}(\{1\}) \cap \{s \in T_{<\delta} : t^\delta_{f(0)} \restriction \alpha \leq s$ and $s \neq 0\} = \emptyset$ or

(γ) $\neg(f(0) \in \gamma_\delta)$.

Let $Y = \{t^\delta_n : n < \gamma_\delta, \delta \in W\}$; we say the tree T is Y-Souslin if: for χ large enough, for every $x \in H(\chi)$ for some N we have: x, $T \in N \prec (H(\chi), \in, <^*_\chi)$, N countable, $\delta \overset{\text{def}}{=} N \cap \omega_1$, and for $n < \gamma_\delta$, $\{s : s <_T t^\delta_n\}$ is (N, T)-generic. For $W' \subseteq W$ let

$$Y \restriction W' = \{t^\delta_n : n < \gamma_\delta \text{ and } \delta \in W'\}.$$

3.9B Claim. 1) If $\bigwedge_\delta T_\delta = \{t^\delta_n : n < \gamma_\delta\}$ *then* Y-Souslin means Souslin. If T is Y-Souslin then T is not special, even not W-special.

2) If T is a Y-Souslin tree then $(\bar{R}, S, \mathbf{g})$ fully covers (so for any forcing notion Q, if in V^Q the tree T is still Y-Souslin, then $(\bar{R}, S, \mathbf{g})$ still fully covers),

3) If $(\bar{R}, S, \mathbf{g})$ covers then $(\bar{R}, S, \mathbf{g})$ strongly covers for possibility A.

Proof. 1), 2) Straightforward.

3) Clearly each R_α is closed and as $[a \in S \Rightarrow d[a] \notin a]$ we are done.

3.9C Claim. A forcing notion Q is $(\bar{R}, S, \mathbf{g})$-preserving *iff* Q is $(\bar{R}, S, \mathbf{g})$-preserving for possibility A.

Proof.

The "only if" direction.

Let $N \prec (H(\chi), \in, <^*_\chi)$ be countable $(\bar{R}, S, \mathbf{g}) \in N$, and p, $\langle p_n : n < \omega\rangle$, $\langle f^*_\ell : \ell < k\rangle$, $\langle \underset{\sim}{f}_\ell : \ell < k\rangle$, $\langle \beta_\ell : \ell < k\rangle$ be as in Definition 3.4.

We can assume $2^{\aleph_1} < \chi_1 = \mathrm{cf}(\chi_1)$, $2^{\chi_1} < \chi$.

[†] As commented earlier, actually the identity of $\mathbf{g}_a$ does not matter only the sets $R_{\alpha,a} = \{f : f R_\alpha \mathbf{g}_a\}$

Let $w = \{\ell < k : f_\ell(0) < \gamma_\delta \text{ and } \beta_\ell \neq 0\}$. For $\ell \in k \setminus w$ choose $x_\ell \in T \cap N$ such that $x_\ell <_T t^\delta_{f_\ell(0)}$ and $\bigvee_{n<\omega} p_n \Vdash$ "$\underset{\sim}{f}_\ell(x_\ell) = 1$ or $\underset{\sim}{f}_\ell(0) \geq \gamma_\delta$". So for some $n(*) < \omega$,

$$p_{n(*)} \Vdash \text{“} \bigwedge_{\ell \in k \setminus w} [\underset{\sim}{f}_\ell(x_\ell) = 1 \text{ or } \underset{\sim}{f}_\ell(0) \geq \gamma_\delta]\text{”}.$$

Let

$$\mathcal{I} = \{q \in Q : \text{for each } \ell \in w, q \text{ forces a value to } \underset{\sim}{f}_\ell(0), \text{ say } m_\ell, \text{ and it forces a truth value to } (\exists x)(t^\delta_{m_\ell} \restriction \beta_\ell <_T x \ \&\ \underset{\sim}{f}_\ell(x) = 1)\}.$$

So for some $n > n(*)$, we have $p_n \in \mathcal{I}$, so those truth values which it forces are all false (as if $p_n \Vdash (\exists x)(t^\delta_{m_\ell} \restriction \beta_\ell <_T x \ \&\ \underset{\sim}{f}_\ell(x) = 1)$ then for some $n' > n$, $p_{n'}$ forces a specific such x so $F^*_\ell(x) = 1$, contradiction). So any (N, Q)-generic $q \in Q$ which is $\geq p_n$ and satisfies $(*)_q$ below is as required, where

$(*)_q$ for $n < \gamma_{N \cap \omega_1}$, the branch $\{t : t <_T t^\delta_n\}$ of $T \cap N$ is (N, T)-generic.

Its existence follows from "Q is $(\bar{R}, S, \mathbf{g})$-preserving."

The "if" direction.

It is trivial (reread Definition 3.4). $\square_{3.9C}$

3.9 D Definition. We say Q is Y-preserving when: if $N \prec (H(\chi), \in, <^*_\chi)$ countable, $\delta = N \cap W_1$, $\{Y, T\} \in N$, and $p \in Q$ such that $n < \gamma \Rightarrow \{t : t <_T t^\delta_n\}$ is (N, T)- generic, *then* there is q, $p \leq q \in Q$, $q \Vdash$ "$n < \gamma_\delta \Rightarrow \{t : t < t^\delta_n\}$ is $(N[\underset{\sim}{G}_Q], T)$-generic."

3.9 E Fact. Q is Y-preserving iff Q is $(\bar{R}, S, \mathbf{g})$-preserving.

3.9 F Conclusion.

If T is an ω_1 - tree, $Y \subseteq T$ *then* the property "Q is Y-preserving and is proper" is preserved by CS iterations (and composition).

3.10 Application. Being a nonmeager set

3.10A Context and Definition. Let $S \subseteq \mathcal{S}_{<\aleph_1}(H(\chi))$, for $a \in S$, $d[a] = c[a] = {}^{\omega>}\omega$. Let fRg iff (f is a function from ${}^{\omega>}\omega$ to ${}^{\omega>}\omega$ and g a function from ω to ω) & $(\exists^\infty m)[(g{\restriction}m)\char`^ f(g{\restriction}m) \trianglelefteq g]$. Let $\mathbf{g} = \langle \mathbf{g}_a : a \in S\rangle$ where $\mathbf{g}_a \in {}^\omega\omega$, (so $\alpha^* = 1, R = R_0$).

Remark. Note that if N is a model of ZFC$^-$, then: "g is Cohen over N" *iff*

$$(\forall f \in N)(f : {}^{\omega>}\omega \to {}^{\omega>}\omega \Rightarrow (\exists^\infty m)[(g{\restriction}m)\char`^ f(g{\restriction}m) \trianglelefteq g])$$

iff

$$\text{“}(\forall f \in N)(f : {}^{\omega>}\omega \to {}^{\omega>}\omega \Rightarrow (\exists m)[(g{\restriction}m)\char`^ f(g{\restriction}m) \trianglelefteq g])$$

(as N is closed enough).

3.10B Claim. 1) If $(\bar{R}, S, \mathbf{g})$ covers in V *then* it strongly covers in V (by Possibility B, C).

2) In V, if $A \subseteq {}^\omega\omega$ is not meager and S stationary subset of e.g. $\mathcal{S}_{<\aleph_1}(H(\aleph_1))$ *then* for some $\mathbf{g}$ we have $(\bar{R}, S, \mathbf{g})$ covers in V and $\mathbf{g}(a) \in A$ for $a \in S$.

Proof. 1) We can show that in Definition 3.3 Possibility B holds. The winning strategy is in stage n, to choose b_n so large that for $\ell \leq n$, there are at least n members in solutions of $\{m : (\mathbf{g}_a{\restriction}m)\char`^ f^n_\ell(\mathbf{g}_a{\restriction}m) \triangleleft g\}$ are guaranteed (similar to VI §3, because the property has the form $(\exists^\infty m)$) (i.e. G_δ Borel set) (remember $\alpha^* = 1$ so $\oplus_k$ is not needed). The proof for possibility C is similar.

2) Straightforward. $\square_{3.10B}$

3.10C Claim. If $(\bar{R}, S, \mathbf{g})$ covers, then "proper $+(\bar{R}, S, \mathbf{g})$-preserving" is preserved by composition and more generally by CS iterations.

Proof. Remember that $(\bar{R}, S, \mathbf{g})$-preserving means "for possibility C" (the case where Definition 3.4 is more transparent). Now use 3.6. $\square_{3.10C}$

3.10D Claim.

(1) Assume $S \subseteq \mathcal{S}_{<\aleph_1}(H(\aleph_1))$, and S is stationary as a subset of $\mathcal{S}_{<\aleph_1}(\bigcup S)$, and $\mathbf{g} : S \to {}^\omega 2$ is such that:

$(*)_{S,\mathbf{g}}$ if x, $S \in H(\chi)$ *then* for some countable $N \prec (H(\chi), \in <^*_\chi)$ we have $\{x, S, \mathbf{g}\} \in N \cap (\bigcup S) \in S$ and $\mathbf{g}_a$ belongs to no meagre set from N.

Then: if $\langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$ is a CS iteration of proper forcing proper notions, each Q_i preserving $(*)_{S^1,\mathbf{g}^1}$ whenever $\bigcup S \in \bigcup S^1$, $(\forall a \in S^1)(a \cap (\bigcup S) \in S)$, $\mathbf{g}^1_a = \mathbf{g}_{a \cap (\bigcup S)}$ (and $(*)_{S^1,\mathbf{g}^1}$ is defined as in part (1); this means $V^{P_1} \models$ " if $(*)_{S^1,\mathbf{g}^1}$ then $\Vdash_{Q_i} (*)_{S^1,\mathbf{g}^1}$") then P_α preserve $(*)_{S,\mathbf{g}}$.

(2) Assume $X \subseteq {}^\omega 2$ is not meagre. If $\langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$, is CS iteration of proper forcing, each Q_i preserves the property $(*)_{s,\mathbf{g}}$ whenever $\mathbf{g} : S \to X$, *then* P_α preserves the property of "being of not meagre" for $X' \subseteq X$.

Proof. Like 3.8C.

3.10E Claim. If $(\bar{R}, S, \mathbf{g})$ covers in V and Q is a forcing notion which is Souslin-proper in any extension (i.e., we have a Souslin definition which in any generic extension is Souslin-proper) and $\Vdash_Q$ "$V \cap {}^\omega 2$ is not meager" (in every extension) *then* in V^Q we have: $(\bar{R}, S, \mathbf{g})$ still covers and Q is $(\bar{R}, S, \mathbf{g})$-preserving.

Proof. It follows from Lemma 3.11 below.

3.11 Lemma. [Goldstern and Shelah] Assume that Q is a Souslin proper forcing, say definable with a real parameter r^*, with the property

$$\Vdash_Q \text{“}V \cap {}^\omega 2 \text{ is not meager”}$$

and continues to have these properties in any extension of V (by set forcing). If $N \prec (H(\chi), \in, <^*_\chi)$ countable, x_0 a Cohen real over N and $p \in N \cap Q$, then there exists a condition $q \geq p$, q is (N, Q)-generic (i.e. (N, Q^V)-generic), and $q \Vdash$ "x_0 is Cohen over $N[\underset{\sim}{G}_Q]$."

We will prove this through a sequence of lemmatas. We always assume that Q is a forcing notion satisfying the assumptions of our lemma, N is a countable elementary submodel of some $(H(\chi), \in)$ (χ big enough, regular), M is a countable transitive model satisfying a large enough fragment of ZFC. We let $\lambda \in N$ be a regular cardinal that is reasonably big (say $\lambda > \beth_2$) but still small compared to χ, say $2^{2^\lambda} < \chi$.)

3.11A. Fact. Assume B is a complete Boolean algebra, $B_0 \subseteq B$ a complete subalgebra and $\{B, B_0\} \in N$. If $G_0 \subseteq B_0$ is N-generic, *then* there exists an N-generic filter $G \subseteq B$ extending G_0.

Proof. Easy.

3.11B. Fact. Assume $B \in N$ is a forcing notion, $x_0 \in {}^\omega 2$ a Cohen real over N. Assume $\underset{\sim}{c}$ is a B-name such that

$$\Vdash_B \text{“}\underset{\sim}{c} \text{ is a Cohen real over } V\text{”}$$

Then there is a N-generic filter $G_B \subseteq B$ such that $\underset{\sim}{c}[G_B]$ is almost equal to x_0.

Proof. Without loss of generality we assume that B is a complete boolean algebra. For any formula φ in the forcing language of B we write $[\![\varphi]\!]$ for the Boolean value of φ. We write $[\![\varphi]\!] = 0$ if $\Vdash_B \neg\varphi$. Assume that $x_0 \in {}^\omega 2$ is Cohen-generic over N, and $\underset{\sim}{c} \in {}^\omega 2$ is forced to be Cohen-generic over V. Let

$$T := \{\eta \in {}^{\omega>}2 : [\![\eta \subseteq \underset{\sim}{c}]\!] \neq 0\}$$

Then T is a tree, and $\Vdash_B$ “$\underset{\sim}{c} \in \lim T$”. So Lim$T$ cannot be nowhere dense, so for some $\eta_0 \in T$ we must have $(\forall \eta)(\eta_0 \lhd \eta \in {}^{\omega>}2 \Rightarrow [\eta \in T])$. For notational simplicity only we assume $\eta_0 = \emptyset$ (otherwise we have to consider $\underset{\sim}{c}\restriction[\ell g(\eta_0), \omega)$ and $x_0\restriction[\ell g(\eta_0), \omega)$ instead of $\underset{\sim}{c}$ and x_0).
Let $B_0 \subseteq B$ be the complete Boolean algebra generated by the elements $[\![\eta \subseteq \underset{\sim}{c}]\!]$, where η ranges over ${}^{\omega>}2$. Then B_0 is a complete subalgebra of B, and the map that sends $\eta \in {}^{\omega>}2$ to $[\![\eta \subseteq \underset{\sim}{c}]\!]$ is a dense embedding of ${}^{\omega>}2$ into B_0. Thus x_0 induces an (N, Q^N)-generic filter $G_0 \subseteq B_0$. By 3.11A, G_0 can be extended to an N-generic filter $G \subseteq B$. Clearly $\underset{\sim}{c}[G] = x_0$, as for every $n \in \omega$, letting $\eta := x_0\restriction n$, we have $[\![\eta \subseteq \underset{\sim}{c}]\!] \in G_0 \subseteq G$. $\square_{3.11B}$

3.11C. Lemma. The formula "$M \subseteq \omega \times \omega$ codes a well founded model of ZFC$^-$ with universe ω, q is (M, Q)-generic, and $q \Vdash_Q \; x$ is Cohen over $M[G_Q]$" is equivalent to a Π^1_1-formula (about x, q, and M as parameters). (M-generic means M'-generic, where M' is the transitive collapse of M. We will not notationally distinguish between M and M'.)

Proof. First we note that "q is (M, Q)-generic" is a Π^1_1-statement, as it is equivalent to

for every $A \in M$ such that $M \models$ "A is pre-dense in Q", and

for every $r \geq q$ there is $a \in M$, $M \models$ "$a \in A$", and a, r are compatible.

(Recall that in a Souslin forcing notion the compatibility relation is Σ^1_1 and Π^1_1.)

If q is (M, Q)-generic, then we have: $q \Vdash$ "x is Cohen over $M[G]$" iff for all $\underset{\sim}{T} \in M$ such that $M \models$ "$\underset{\sim}{T}$ is a Q-name of a nowhere dense tree $\subseteq {}^{\omega>}2$", and *for all* $r \geq q$ there exists a condition $p' \in M \cap Q$ and a natural number n such that p', r are compatible, and $M \models$ "$p' \Vdash x{\restriction}n \notin \underset{\sim}{T}$". Again it is easy to see that this can be written as a Π^1_1-statement. $\square_{3.11C}$

3.11D. Lemma. Assume M is as above, $p \in Q \cap M$, A a comeager Borel set. Then there exist a real $x \in A$ and a condition $q \geq p$ such that q is (M, Q)-generic, and $q \Vdash$ "x is Cohen over $M[\underset{\sim}{G}_Q]$."

Proof. Let $q_0 \geq p$ be (M, Q)-generic. Work in $V[G]$, where $q_0 \in G \subseteq Q$, G is generic over V. Since $V \cap 2^\omega$ is not meager (in $V[G]$), $A^{V[G]}$ is comeager, and the union of all meager sets coded in $M[G]$ is meager, we can find $x \in V \cap A^{V[G]}$ which is Cohen over $M[G]$, by absoluteness $x \in A$. Now let $q \geq q_0$ be a condition which forces this. $\square_{3.11D}$

3.11E. *Proof of the Lemma 3.11:* Recall that $\lambda \in N$ is much bigger than ω, but much smaller than χ. Let $M \stackrel{\text{def}}{=} (H(\lambda), \in)$. So $M \in N$.

Let B be the algebra that collapses $H(\lambda)$ to a countable set (using finite conditions) i.e. Levy$(\aleph_0, |H(\lambda)|)$. Clearly $\Vdash_B$"M is a countable model of ZFC$^-$."

We assert that

(*) $\Vdash_B$ "There exists x (Cohen over V) and $q \in Q$, q is (M, Q)-generic, and $\Vdash_Q$ x is Cohen over $M[G_Q]$".

To prove this assertion, work in V^B. The set of all closed nowhere dense sets coded in V is now countable, so the set of Cohen reals over V is comeager, and hence contains some comeager Borel set A. Now apply the previous lemma 3.11D. This finishes the proof of the assertion (*).

From the assertion we can get names $\underset{\sim}{x}$ and $\underset{\sim}{q}$ such that all the above is forced by the trivial condition of B. Clearly we can assume that $\underset{\sim}{x}$ and $\underset{\sim}{q}$ are in N.

Now apply Fact 3.11B to get an N-generic filter $G \subseteq B$ (in V!) such that $x \stackrel{\text{def}}{=} \underset{\sim}{x}[G]$ is almost equal to x_0. Let $q \stackrel{\text{def}}{=} \underset{\sim}{q}[G]$. Then

$$N[G] \models \text{"}q \text{ is } (M, Q)\text{-generic, and } q \Vdash_Q c \text{ is Cohen over } M[\underset{\sim}{G}_Q]\text{"}$$

and $N[G] \cap \mathrm{Ord} = N \cap \mathrm{Ord}$.

Since Π^1_1-formulas are absolute, we can replace $N[G]$ by V (remember $N[G] \subseteq V$). We can also replace x by x_0, since modifying a Cohen real in finitely many places still leaves it a Cohen real. Thus,

$$V \models \text{"}q \text{ is } (M \cap N, Q)\text{-generic, and } q \Vdash_Q x_0 \text{ is Cohen over } M[\underset{\sim}{G}_Q].\text{"}$$

(Why $M \cap N$ and not M? As we should look at M as interpreted in $N[G]$, note $N[G] \models$ "M is countable"). As $M \cap N$ and N have the same dense sets of Q, q is (M, Q)-generic iff it is N-generic. Similarly, x_0 is Cohen over $M[G]$ iff it is Cohen over $N[G]$, so we are done. $\square_{3.11}$

Remark. We shall deal with more general theorems in [Sh:630].

3.12 Concluding Remarks. 1) We may consider the following variant of this section's framework concentrating on $d[a] \in a$ (of course if $R_{\alpha,s} = R_\alpha$ we get back the previous version).

(A) We replace R_α by $R_{\alpha,t}$ for $t \in \mathbb{Q}$ such that $s < t \Rightarrow R_{\alpha,s} \subseteq R_{\alpha,t}$; we may use $R_\alpha = \bigvee_{s \in \mathbb{Q}} R_{\alpha,s}$.

(B) N is $(\bar{R}, S, \mathbf{g})$-good *iff* $a \stackrel{\text{def}}{=} N \cap (\bigcup S) \in S$ and for every $f \in N$ satisfying $f \in {}^{d[a]}c[a]$ for some $\alpha < \alpha^*$ and $t \in \mathbb{Q}$ we have $f R_{\alpha,t} g$.

(C) "Strongly covers" is defined as before except that $\oplus_k$ is changed parallely to the change in (D) below, i.e. I.e.

$\oplus_k$ if (a) – (d) of $(*)$ of $\oplus_k$ from Definition 3.3 and

(e) $\beta_\ell \in a \cap \alpha^*$, $t_\ell < s_\ell$ are rationals and $\underset{\sim}{f}_\ell[G^1] R_{\alpha,t} \mathbf{g}_a$

then for any $y \in N \cap H(\chi)$ there are N_2, G_2 satisfying (the parallel of) clause (d) such that $y \in N_2$ and: for some $\gamma_\ell \in a$, $\gamma_\ell \leq \beta_\ell$, $s'_\ell \in Q$, $s'_\ell \leq s_\ell$ (for $\ell < k$) we have $\underset{\sim}{f}_\ell[G_2] R_{\gamma_\ell, s'_\ell} \mathbf{g}_a$ (for $\ell < k$).

(D) Q is $(\bar{R}, S, \mathbf{g})$-preserving means: if $(*)$ of Definition 3.4 holds (having now $f^*_\ell R_{\beta_\ell, t_\ell} \mathbf{g}_a$ and $s_\ell > t_\ell, s_\ell \in \mathbb{Q}$) *then* there is an (N, Q)-generic q, $p \leq q \in Q$ such that $q \Vdash_Q$ "for $\ell < k$ there are $\gamma_\ell \leq \beta_\ell$ and $s'_\ell \in \mathbb{Q}$ such that $[\gamma_\ell = \beta_\ell \Rightarrow s'_\ell \leq s_\ell]$ and $\underset{\sim}{f}_\ell R_{\gamma_\ell, s'_\ell} \mathbf{g}_a$.

2) If we have $\oplus^*_k$ of 3.3B(3) *then* (b) of the conclusion in Definition 3.4 can be omitted (as it follows from "q is (N, Q)-generic" under the circumstances).

3) Another variant of our framework is as follows.

(α) Let R be a definition of a forcing notion, i.e. partial order, $\{\mathcal{I}_y : y \in Y\}$ be a definition of a family of dense subsets of it (e.g. all), all absolute enough, K be a definition of a family of forcing notions closed under CS iterations (so e.g. if $Q_\ell \in K^{V_\ell}$, $V_{\ell+1} = V_\ell^{Q_\ell}$ then $Q_1 * \underset{\sim}{Q}_2 \in K^{V_1}$, similarly for limit. We have: if in $V_0^{Q_0}$, $p \leq q \in R$, $y \in Y^{(V^{Q_0})}$, $p \in \mathcal{I}_y$ *then* this holds in $V_0^{Q_0 * \underset{\sim}{Q}_1}$).

(β) $S \in V_0$, S a stationary subset of $\mathcal{S}_{<\aleph_1} H(\chi^*)^V$

(γ) for $a \in S$, $\mathbf{g}_a$ is a directed subset of $R \cap a$ not disjoint to $a \cap \mathcal{I}_y$ for $y \in Y \cap a$ (absolute as in (α)).

(δ) in $V_0^{Q_0}$, N is $(R, S, \mathbf{g})$-good if: $N \prec (H(\chi), \in, <^*_\chi)$ is countable, $(\bar{R}, S, \mathbf{g}) \in N$, $a \stackrel{\text{def}}{=} N \cap H(\chi^*)^{V_0} \in S$ and $[y \in N \cap Y^{V^{Q_0}} \Rightarrow \exists p \in \mathcal{I}_y \exists q \in \mathbf{g}_a (p \leq q)]$ (of course $\mathbf{g}_a$ is still a directed subset of $R^{V^{Q_0}} \cap a$).

3.13 Preservations connected to Norms.

3.13A Context and Definitions.

1) Assume we have $(\bar{n}^*, \bar{\text{nor}}, \bar{w})$ where

 (α) $\bar{n}^* = \langle n_i^* : i < \omega \rangle$ is strictly increasing.

 (β) $\bar{\text{nor}} = \langle \text{nor}_i : i < \omega \rangle$, where $\text{nor}_i : \mathcal{P}([n_i^*, n_{i+1}^*)) \to \omega$ satisfies:

$$u_1 \subseteq u_2 \subseteq [n_i^*, n_{i+1}^*) \Rightarrow \text{nor}_i(u_1) \leq \text{nor}_i(u_2)$$

$$\text{nor}_i([n_i^*, n_{i+1}^*)) > 0$$

$$\langle \text{nor}_i([n_i^*, n_{i+1}^*)) : i < \omega \rangle \text{ converge to infinity}$$

 (γ) $\bar{w} = \langle w_i : i < \omega \rangle$ where $\text{Dom}(w_i) = \omega$, $\text{Rang}(w_i) \subseteq \{U : U \subseteq \mathcal{P}([n_i^*, n_{i+1}^*))$ is downward closed, $U \neq \emptyset\}$ or even† $\text{Dom}(w_i) = {}^{i+1}\omega$ and for every $u \subseteq [n_i^*, n_{i+1}^*)$ with $\text{nor}_i(u) > 0$ and $x \in \omega$ (or $\bar{x} \in {}^{i+1}\omega$ otherwise) for some $u' \subseteq u$ we have $u' \in w_n(x)$ and $\text{nor}_i(u') \geq \text{nor}_i(u) - 1$ (if $\bar{w}$ is omitted then $w_i(x)$ is: let $x(i)$ code $w_{x(i)} \subseteq \mathcal{P}([n_i^*, n_{i+1}^*))$, now let $w_i(x) = w_{x(i)}$ if $\langle w_i(x) : i < \omega \rangle$ is O.K. and let $w_i(x) = \mathcal{P}([n_i^*, n_{i+1}^*))$ otherwise.

2) Let

$$\prod_{i<\omega}^{*} \mathcal{P}([n_i^*, n_{i+1}^*)) = \{\bar{t} : t_i \subseteq [n_i^*, n_{i+1}^*) \text{ and } \langle \text{nor}_i(t_i) : i < \omega \rangle \text{ converge to infinity } \text{nor}_i(t_i) > 0\}$$

3) We define two partial orders on $\prod_{i<\omega}^* \mathcal{P}([n_i^*, n_{i+1}^*))$:

$$\bar{t} \leq \bar{s} \text{ iff } t_i \supseteq s_i \text{ for every } i < \omega$$

$$\bar{t} \leq^* \bar{s} \text{ iff } t_i \supseteq s_i \text{ for every } i < \omega \text{ large enough}$$

(note that $\leq$ is a partial order, $\leq^*$ is a partial order such that every increasing ω-chain has an upper bound)

† Instead of $\omega = \text{Dom}(w_i)$ or ${}^i\omega = \text{Dom}(w_i)$ we can use other finite or countable set.

4) We call $\Gamma \subseteq \prod^*_{i<\omega} \mathcal{P}([n^*_i, n^*_{i+1}))$ a nice set if:

(α) Γ is $\leq^*$-directed

(β) every $\leq^*$-increasing ω-chain in Γ has an upper bound in Γ

(γ) for every $\bar{x} \in {}^\omega\omega$, for some $\bar{t} \in \Gamma$ we have $t_i \in w_i(x_i)$ (or $t_i \in w_i(\bar{x}\restriction(i+1))$) for every $i < \omega$ large enough.

3.13B Fact.

1) $(\prod^*_{i<\omega} \mathcal{P}([n^*_i, n^*_{i+1})), \leq)$ is a partial order.

2) $(\prod^*_{i<\omega} \mathcal{P}([n^*_i, n^*_{n+1})), \leq^*)$ is a partial order with any $\leq^*$-increasing ω-chain having an upper bound.

3) $\bar{t} \leq \bar{s} \Rightarrow \bar{t} \leq^* \bar{s}$.

4) If CH (or just MA) *then* there exists a nice Γ and hence S, $\mathbf{g}$ as in 3.13C (2), (3) below exist.

Proof. Straightforward.

We want to show that niceness of Γ is preserved under limit of CS proper iteration

3.13C Context and Definition.

1) Let Γ be nice in a universe V_0,

2) $S \subseteq \mathcal{S}_{<\aleph_1}(H(\aleph_1))$ be stationary,

3) $\mathbf{g} : S \to \Gamma$ be such that: $\mathbf{g}_a = \langle \mathbf{g}_{a,i} : i < \omega \rangle \in \prod^*_{i<\omega} \mathcal{P}([n^*_i, n^*_{i+1}))$ and

(α) for every $x \in ({}^\omega\omega) \cap a$ we have $(\forall^* i < \omega)[\mathbf{g}_{a,i} \in w_i(x_i)]$

(β) for $a_1 \in a_2$ from S we have $\mathbf{g}_{a_1} <^* \mathbf{g}_{a_2}$ (can ask that moreover $\mathbf{g}_{a_1} \in a_2$)

4) $d[a] = c[a] = \omega$

5) $\bar{R} = \langle R_n : n < \omega \rangle$ and $x R_n \mathbf{g}_a$ mean $(\forall i < \omega)[i \geq n \to \mathbf{g}_{a,i} \in W_i(x_i)]$

3.13D Claim.

1) $(\bar{R}, S, \mathbf{g})$ is as in 3.1, it covers (in V, see 3.2, we are assuming 3.13C of course)

2) If in V^P, we have "$(\bar{R}, S, \mathbf{g})$ covers" then it also strongly covers by Possibility A*

3) If $(\bar{R}, S, \mathbf{g})$ cover in V^P, Q is a proper forcing notion preserving "Γ is nice" *then* Q is $(\bar{R}, S, \mathbf{g})$-preserving.

Proof: 1) Read definitions.

2) Check (for $\oplus_1$, can apply 3.4B and the proof of $\oplus^+$ from there in the proof of part (3) below).

3) We use 3.4A(3), so the least trivial condition is $\otimes^+$ from there. Let V^P, N, a_1, a_2, f, $\langle f_n : n < \omega \rangle$, α, $\langle \alpha_n : n < \omega \rangle$ be as there. We can find a finite $d \subseteq \omega$ such that:

$(*)$ if $\ell \in \omega \setminus d$ then $\mathbf{g}_{a_1,\ell} \supseteq \mathbf{g}_{a_2,\ell}$,

w.l.o.g. also $\{0, \dots, \alpha - 1\} \subseteq d$, $\{0, \dots, \alpha_0 - 1\} \subseteq d$ (remember $\alpha < \alpha^* = \omega$). Let $k_i \geq i$ be maximal such that $f_i {\restriction} k_i = f {\restriction} k_i$, so $\lim\limits_{i \to \infty} k_i = \infty$.

Also w.l.o.g. $\alpha_i > k_i > \sup(d)$ and we can find an infinite $A \subseteq \omega$ such that $\langle [k_i, \alpha_i) : i \in A \rangle$ are pairwise disjoint, and w.l.o.g. $A \in N$ and $n \in [\min A, \omega) \Rightarrow f_n(0) = f(0)$. Now define $g \in {}^\omega\omega$ by:

$$g {\restriction} [k_i, \alpha_i) = f_i {\restriction} [k_i, \alpha_i) \text{ for } i \in A \text{ and } g {\restriction} (\omega \setminus \bigcup_{i \in A} [k_i, \alpha_i)) \subseteq f,$$

so clearly $g \in {}^\omega\omega \cap N$, hence $\bigvee\limits_k g R_k \mathbf{g}_{a_2}$ so w.l.o.g. $\ell \in \omega \setminus d \Rightarrow \mathbf{g}_{a_2,\ell} \in w_\ell(g(\ell))$. Omitting finitely many members of A we can assume $i \in A \Rightarrow d \subseteq k_i$ and hence $f_i {\restriction} d = f {\restriction} d$. We will show that any $i \in A$, $d_i = \{0, \dots, \alpha_i - 1\}$ are as required in $\oplus^+$, so assume $f' \in {}^{d[a_2]}c[a_2] = {}^\omega\omega$, $f' {\restriction} d_i = f_n {\restriction} d_i$ and $f' R_{\alpha_i} \mathbf{g}_{a_1}$. So let $\ell \in \omega \setminus \alpha$, and we should prove $\mathbf{g}_{a_2,\ell} \in w_\ell(f'(\ell))$, thus proving $f' R_\alpha \mathbf{g}_{a_2}$ and finishing the proof of $\oplus^+$; we divide this to cases.

case 1: $\ell \notin d_i$

So $\ell \geq \alpha_i$ and we know $f' R_{\alpha_i} \mathbf{g}_{a_1}$ hence $\mathbf{g}_{a_1,\ell} \in w_\ell(f'(\ell))$; but also $\ell \notin d$ hence (see $(*)$ above) $\mathbf{g}_{a_1,\ell} \supseteq \mathbf{g}_{a_2,\ell}$, and $w_\ell(f'(\ell))$ is downward closed so $\mathbf{g}_{a_2,\ell} \in w_\ell(f'(\ell))$ as required.

case 2: $\ell \in d_i \setminus \{0, \dots, k_i - 1\}$

So $k_i \leq \ell < \alpha_i$. As $\ell \in d_i$ we know $f'(\ell) = f_i(\ell)$ and $f_i(\ell) = g(\ell)$, but as $\ell \notin d$ we have $\mathbf{g}_{a_2,\ell} \in w_\ell(g(\ell))$, together we finish.

case 3: $\ell < k_i$

But $f{\restriction}k_i = f_i{\restriction}k_i = f'{\restriction}k_i$, hence $f'(\ell) = f(\ell)$, and as $fR_\alpha \mathbf{g}_{a_2}$ we are done. So we have finished checking the condition $\otimes^+$ of 3.4A(3), thus proving 3.13D(3).

$\square_{3.13D}$

3.13E Conclusion. If Γ is nice, $\bar{Q} = \langle P_j, \underset{\sim}{Q}_i : j \leq \delta, i < \delta\rangle$ is a CS iteration of proper forcing, each Q_i preserves the niceness of Γ *then* P_δ preserves the niceness of Γ.

3.13F Remark. Similarly for the other variants in VI 0.1, for pure preserving.

3.14 Example (of 3.13).

3.14A Context. We work inside the subcontext of 3.13.

Let $\bar{n}^* = \langle n_i^* : i < \omega\rangle$, $n_i^* << m_i^* << k_i^* << n_{i+1}^*$.

By renaming we replace $[n_i^*, n_{i+1}^*)$ by $c_i^* = c^{t_i} = \{(\ell_1, \ell_2) : \ell_1, \ell_2 \in [n_i^*, k_i^*)\}$, so we consider subsets of C_i^* only, but actually can consider instead $e \in E_i$ only where:

$$E_i \overset{\text{def}}{=} \{e : e \text{ an equivalence relation on } [n_i^*, k_i^*) \text{ and each equivalence}$$
$$\text{class has exactly } n_i^* + 1 \text{ elements, except possibly one.}\}$$

For $e \in E_i$ we let $\mathrm{Dom}(e) = \bigcup\{x/e : |x/e| = n_i^* + 1\}$. To make it fit we identify e with

$$s_e = \{(\ell_1, \ell_2) : \ell_1 \in [n_i^*, k_i^*) \text{ and } \ell_2 \in [n_i^*, k_i^*) \text{ and } \neg(\ell_1 e \ell_2)\},$$

we will not continue to mention the minor changes; now we let

$$\mathrm{nor}^{t_i}(e) = \log_2\log_2(k_i^* - n_i^* - |\mathrm{Dom}(e)|)/m_i^*$$

rounded (to maximal natural number $\leq$ than this or zero if it is negative). For $\bar{x} = \langle x_j : j \leq i\rangle$ we define $w_i(\bar{x})$: we consider x_i as (being or just coding) a pair (f_{x_i}, A_{x_i}), where $A_{x_i} \subseteq \omega$ finite non empty and $f_{x_i} : [n_i^*, n_{i+1}^*) \to$

$({}^{A_{x_i}}\{0,\ldots,n_i^*\})$ (so $a \in A_{x_i} \Rightarrow f_{x_i}[a] : [n_i^*, n_{i+1}^*) \to \{0,\ldots,n_i^*\}$)

$$w_i(\bar{x}) = \Big\{e \in E_i : \text{ for } \geq (n_i^*)^{n_i^*+2} \text{ equivalence classes } u \text{ of } e,$$
$$1 - \frac{1}{(\log_2(n_i^*))^{x_0}} \leq \frac{|\{a \in A_{x_i} : (f^i_{x_i}[a]){\restriction}u \text{ is not one to one }\}|}{|A_{x_i}|}\Big\}.$$

We should check

$(*)_1$ if (i is large enough and) $e_0 \in E_i$ and $\mathrm{nor}^{t_i}(e_1) \geq \ell + 1$ and $\bar{x} = \langle x_j : j \leq i+1\rangle$ as above *then* for some $e_2 \in E_i$, $s_{e_2} \subseteq s_{e_0}$, $\mathrm{nor}^{t_i}(e_2) \geq \mathrm{nor}^{t_i}(e_0) - 1$ and $e_2 \in w_i(\bar{x})$.

[Why $(*)_1$ holds? Choose by induction on $m < n^* \stackrel{\text{def}}{=} (n_i^*)^{(n_i^*+2)}$ a set $u_m \subseteq [n_i^*, n_{i+1}^*)$ satisfying $|u_m| = n_{i+1}^*$ and u_m disjoint to $\bigcup\limits_{m'<m} u_{m'} \cup \bigcup\{x/e_1 : |x/e_1| = n_i^* + 1\}$ and:

$$1 - \frac{2}{n_i^* + 2} \leq |\{a \in A_{x_i} : (f_{x_i}[a]){\restriction}u_n \text{ is not one to one}\}|/|A_{x_i}|.$$

(Why? If $v \subseteq [n_i^*, n_{i+1}^*)$, $|v| = n_i^* + 2$ is disjoint to the set above then for each $a \in A_{x_i}$,

$$|\{u \in [v]^{n_i^*+1} : (f_{x_i}[a]){\restriction}u \text{ not one to one}\}| \geq (1 - \frac{2}{n_i^*+2}) \times |[v]^{n_i^*+1}|,$$

so by the "finitary Fubini", some $u \in [v]^{n_i^*+1}$ is (much more than) as required, increasing v we get better estimates.)

Let $e_2 \in E_i$ be such that: the set of e_2-equivalence classes of cardinality $n_i^* + 1$ is

$$\{x/e_1 : |x/e_1| = n_i^* + 1\} \cup \{u_m : m < (n_i^*)^{n_i^*} + 2\}.$$

Now

$$\begin{aligned}
\mathrm{nor}^{t_i}(e_2) &= \log_2\log_2(k_i^* - n_i^* - |\mathrm{Dom}(e_2)|)/m_i^* \\
&= \log_2\log_2(k_i^* - n_i^* - |\mathrm{Dom}(e_1)| - n_i^*(n_i^*)^{n_i^*+2})/m_i^* \\
&= \log_2\log_2((k_i^* - n_i^* - |\mathrm{Dom}(e_1)|) \times (1 - \frac{n_i^*(n_i^*)^{n_i^*+2}}{k_i^* - n_i^* - |\mathrm{Dom}(e_1)|}))/m_i^* \\
&= \log_2[\log_2(k_i^* - n_i^* - |\mathrm{Dom}(e_1)|) + \log_2(1 - \frac{n_i^*(n_i^*)^{n_i^*+2}}{k_i^* - n_i^* - |\mathrm{Dom}(e_1)|})]/m_i^*
\end{aligned}$$

but as $\mathrm{nor}^{t_i}(e_0) > 0$, necessarily

$$k_i^* - n_i^* - |\mathrm{Dom}(e_1)| \geq 2^{2^{m_i^*}} >> n_i^*(n_i^*)^{n_i^*+1},$$

hence

$$\log_2(1 - \frac{n_i^*(n_i^*)^{n_i^*+2}}{k_i^* - n_i^* - |\mathrm{Dom}(e_1)|}) \geq -1/n_i^*.$$

Hence $\mathrm{nor}^{t_i}(e_2) \geq \mathrm{nor}^{t_i}(e_0) - 1$.]

Moreover, the proof gives

$(*)_2$ if $e_0 \in E_i$, $\mathrm{nor}^{t_i}(e_0) \geq \ell + 1$ and X is a set of n_i^* (or less) $(i+1)$-tuples $\bar{x} = \langle x_j : j \leq i+1 \rangle$ as above then for some $e_2 \in E_i$, $s_{e_2} \subseteq s_{e_1}$ and $\mathrm{nor}^{t_i}(e_2) \geq \mathrm{nor}^{t_i}(e_1) - 1$ and $\bigwedge_{\bar{x} \in X} e_2 \in w_i(\bar{x})$

[Why? We define above u_m for $m < ((n_i^*)^{n_i^*+2}) \times |X|$ dealing with each $x \in X$ by u_m, $(n_i^*)^{n_i^*+2}$ times. As $|X| \leq n_i^*$ there is no problem.] $\square_{3.14A}$

Remark. 1) Think first for the case A'_x a singleton.
2) $(\log(n_i^*))^{x_0+2}$ serves as the $f(-,-)$ in [RoSh:470]

3.14B Claim. If the forcing P preserve "Γ is nice" then there is no Cohen real over V in V^P.

Proof. For this the case A_{x_i} is a singleton suffices. If $\eta \in {}^\omega\omega$ is Cohen over V then

$$(\forall \bar{s} \in \Gamma)(\exists^\infty i)(\eta \text{ is not 1-to-1 on any equivalence class from } s_i)$$

(better look at $\{\eta \in {}^\omega\omega : \ell < n_{i+1} \Rightarrow \eta(\ell) \leq n_i^*\}$) $\square_{3.14B}$

3.14C Claim. Random real forcing preserves a nice Γ.

Proof. Let $p \in$Random be such that $p \Vdash$ "$\underset{\sim}{\bar{x}} = \langle \underset{\sim}{x}_\ell : \ell < \omega \rangle \in {}^\omega\omega$". W.l.o.g. $p \Vdash$ "$\underset{\sim}{x}_0 = x_0$, and $\underset{\sim}{x}_i = (\underset{\sim}{f}_{x_i}, \underset{\sim}{A}_{x_i})$, $\emptyset \neq \underset{\sim}{A}_{x_i} \subseteq \omega$ finite, $\underset{\sim}{f}_{x_i} \in {}^{\underset{\sim}{A}_{x_i}}\{0, \ldots, n_i^*\}$".

As Random forcing is ${}^{\omega}\omega$-bounding w.l.o.g. $p \Vdash_Q$ "$|\underset{\sim}{A}^i_x| \le \ell_i$", where $\langle \ell_i : i < \omega\rangle \in V \cap {}^{\omega}(\omega \setminus \{0\})$ and as we can replace A by any $A \times B$ w.l.o.g. $\underset{\sim}{A}_{x_i} = A^*_i$ (not name). Now define g_i, $\mathrm{Dom}(g_i) = [n^*_i, n^*_{i+1}) \times A^*_i \times \{0, \ldots, n^*_i\}$, as follows: if $m \in [n^*_i, n^*_{i+1})$, $a \in A^*_i$ and $\ell \in \{0, \ldots, n^*_i\}$ then

$$g_i(m, a, \ell) \overset{\mathrm{def}}{=} \mathrm{Lb}\Big(\text{maximal } q \ge p \text{ forcing } (\underset{\sim}{f}^i_{\underset{\sim}{x}_i}[a](m)) = \ell)\Big)/\mathrm{Lb}(\lim p)$$

W.l.o.g. $p = \lim(T)$ where T is a closed subtree of ${}^{\omega>}2$ and we can choose for each $i < \omega$, a natural number t_i large enough so that from $\eta \in p \cap T \cap {}^{t_i}2$ we can read $\underset{\sim}{f}_{x_i}$ that is for any $\eta \in T \cap {}^{t_i}2$ we have $\lim(T^{[\eta]})$ force a value to $\underset{\sim}{f}_{x_i}$ (where $T^{[\eta]} = \{\nu \in T : \nu \trianglelefteq \eta \ \vee \ \eta \trianglelefteq \eta\}$ of course). For $i < \omega$ we let $A'_i = A^*_i \times ({}^{(t_i)}2 \cap p)$, and we let g'_i be the function from $[n^*_i, n^*_{i+1})$ to ${}^{(A'_i)}\{0, \ldots, n^*_i\}$ defined as follows: for $(a, \eta) \in A'_i$ and $m \in [n^*_i, n^*_{i+1})$ we let $(g'_i[(a, \eta)])(m) = \ell$ iff $\lim(T^{[\eta]}) \Vdash (\underset{\sim}{f}^i_{\underset{\sim}{x}_i}[a])(m) = \ell$. So apply "$\Gamma$ nice" to $\bar{x}'' = \langle x_0 + 1, g_1, g_2, \ldots\rangle$. $\square_{3.14C}$

3.14D Claim. If Q has the Laver property or just is (f, g)-bounding with $f(i) = 2^{2^{k_i k_i}}$, $g(i) = n^*_i$, *then* Q preserves any nice Γ.

Proof. Assume $p \in Q$, $p \Vdash_Q$ "$\underset{\sim}{\bar{x}} = \langle \underset{\sim}{x}_n : n < \omega\rangle$, $x_0 < \omega$, and x_n codes $\underset{\sim}{f}_{\underset{\sim}{x}_i} : [n^*_i, n^*_{i+1}) \to {}^{(\underset{\sim}{A}_{\underset{\sim}{x}_i})}\{0, \ldots, n^*_i\}$" and we shall find p', a such that $p \le p' \in Q$, $a \in S$ and $p' \Vdash_Q \le \bar{x} R \mathbf{g}_a$", this is enough. So w.l.o.g. $p \Vdash$ "$\underset{\sim}{x}_0 = x_0$". For each $u \subseteq [n^*_i, k^*_i)$, $|u| = n^*_i + 1$, we let $\underset{\sim}{\mathbf{t}}_{i,u}$ be the truth value of the statement

$$1 - 1/(\log(n^*_i))^{x_0+2} \le |\{a \in \underset{\sim}{A}_{\underset{\sim}{x}_i} : (\underset{\sim}{f}_{\underset{\sim}{x}_i}[a])\restriction u \text{ is not one to one}\}|/|\underset{\sim}{A}_{\underset{\sim}{x}_i}|.$$

Let $\underset{\sim}{\bar{\mathbf{t}}}_i = \langle \mathbf{t}_{i,u} : u \subseteq [n^*_i, k^*_i), |u| = n^*_i + 1\rangle$. The number of possible u is $\le 2^{k_i k_i}$, hence the number of possible interpretation of $\underset{\sim}{\bar{\mathbf{t}}}$ is $\le 2^{2^{k_i k_i}}$. By the assumption w.l.o.g. for each i we have $\langle \bar{\mathbf{t}}^{i,\ell} : \ell < n^*_i\rangle$ (all in V not names) such that $p \Vdash$ "$\bigvee\limits_{\ell < n^*_i} \bar{\mathbf{t}}_i = \bar{\mathbf{t}}^{i,\ell}$".

So we can find, in V, $\langle (A^i_\ell, \underset{\sim}{f}^i_\ell) : \ell < n^*_i, i < \omega\rangle$ such that $(A^i_\ell, \underset{\sim}{f}^i_\ell)$ is a possible case of $(\underset{\sim}{A}^i_{\underset{\sim}{x}_i}, \underset{\sim}{f}^i_{\underset{\sim}{x}_i})$. By the way the norm was defined (for i large enough) by

dropping the norm by 1 we can deal not just with one case (i.e. one possible $\bar{\mathbf{t}}^{i,\ell}$ i.e one (A^i_ℓ, f^i_ℓ)) but even with n^* of them. This is $(*)_2$ of 3.15A.
Note: if $p \in G \subseteq Q$, G generic over V then for some $\ell < n^*_\ell$, $(\underset{\sim}{A}'_{x_i}, \underset{\sim}{f}^i_{x_i}) = (A^i_\ell, f^i_\ell)$, and they have the same $w_i(-)$.

3.15E Claim. The forcing as in [FrSh:406] is like that.

Proof. W.l.o.g. the i-th splittings are included in $(2^{2^{k^*_i}}, \log_2\log_2(n^*_{i+1}))$, so follows by 3.15D the $\langle(2^{(k^*_i)^{n*_i+1}}, n^*_i) : i < \omega\rangle$-bounding version.

§4. There May Be a Unique P-Point

This section continues VI §5.

4.1 Theorem. Assume V satisfies $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, F_0 is a Ramsey ultrafilter on ω. Then for some $\aleph_2$-c.c. proper, ${}^\omega\omega$-bounding forcing notion P of cardinality $\aleph_2$, in V^P there is a unique P-point, and it is F_0 (i.e. the filter it generates in V^P).

4.1A Remark. In fact, in V^P, F_0 is a Ramsey ultrafilter (actually this follows).

Proof. By the proof of VI 5.13, it suffices to prove the following lemma:

4.2 Lemma. Suppose

$(*)_0$ F_0, F_1 are ultrafilters on ω, F_0 is a Ramsey ultrafilter, F_1 is a P-point, $F_0 \leq_{\mathrm{RK}} F_1$ but not $F_1 \leq_{\mathrm{RK}} F_0$.

Then there is a forcing notion Q such that:

(a) Q has the PP-property, (hence is ${}^\omega\omega$ bounding) and is of cardinality $2^{\aleph_0}$ and

(b) $\Vdash_Q$ "F_0 is an ultrafilter", but

(c) if $Q \lessdot Q'$, Q' has the PP-property *then* in $V^{Q'}$ we have: F_1 cannot be extended with to a P-point (ultrafilter).

4.2A Remark. During the proof of 4.1 we use the forcing notions SP*(F) from Definition VI 5.4 to kill the P-points F with $F_0 \not\leq_{\mathrm{RK}} F$.

The rest of this section is dedicated to the proof of this Lemma.

Proof. Since $F_0 \leq_{\mathrm{RK}} F_1$ and F_1 is a P-point, there is a function $h : \omega \to \omega$ such that

$(*)_1$ $h(F_1) = F_0$ and for each $\ell < \omega$ the set $I(\ell) = I_\ell \stackrel{\mathrm{def}}{=} h^{-1}(\{\ell\})$ is finite.

Note that then $[A \subseteq \omega \ \& \ \bigwedge_\ell 1 \geq |I_\ell \cap A| \Rightarrow A \notin F_1]$ because $F_1 \not\leq_{\mathrm{RK}} F_0$.

Now in Definition 4.4 below we define a forcing notion $Q = SP^*(F_0, F_1, h)$ and then prove in 4.3 — 4.9 that it has all the required properties thus finishing the proof of 4.2 and 4.1.

4.3 Claim. In the following game player I has no winning strategy: In the n'th move player I chooses $A_n \in F_0$ and $B_n \in F_1$; player II chooses $k_n \in A_n$ ($k_n > k_\ell$ for $\ell < n$) and $w_n \subseteq B_n \cap I_{k_n}$. In the end player II wins the play if $\{k_n : n < \omega\} \in F_0$ and $\bigcup\{w_n : n < \omega\} \in F_1$ (the first demand follows from the second).

4.3A Remark. Clearly player II has no better choice than $w_n = B_n \cap I_{k_n}$. Remember $I_{k_n} = h^{-1}(\{k_n\})$ is finite.

Proof. Suppose H is a wining strategy of player I. Let λ be big enough, $N \prec (H(\lambda), \in, <^*_\lambda)$ be such that $\{F_0, F_1, h, H\} \in N$ and N is countable. As F_ℓ is a P-point there are, for $\ell \in \{0,1\}$ sets $A^*_\ell \in F_\ell$ such that $A^*_\ell \subseteq_{ae} B$ (i.e. $A^*_\ell \setminus B$ finite) for every $B \in F_\ell \cap N$.

Now we can find an increasing sequence $\langle M_n : n < \omega\rangle$ of finite subsets of N, $N = \bigcup_{n<\omega} M_n$ such that it increases rapidly enough; more exactly:

α) $H, F_0, F_1, h \in M_0$ and $M_n \in M_{n+1}$,

β) if $\varphi(x, a_0, \ldots)$ is a formula of length $\leq 1000 + |M_n|$ with parameters from $M_n \cup \{M_n\}$ satisfied by some $x \in N$, *then* it is satisfied by some $x \in M_{n+1}$,

γ) if $\ell \in \{0,1\}$, $B \in F_\ell \cap N$, $B \in M_n$ then $B \cup M_{n+1} \supseteq A^*_\ell$,

δ) $M_0 \cap \omega = \emptyset$,

ε) if $\ell \in M_n$ then $I(\ell) \subseteq M_{n+1}$ and M_n is closed under h (we can demand $m \in M_n \Leftrightarrow h(m) \in M_n$ if we make the domains of F_0, F_1 disjoint).

Let $u_{n+1} = (M_{n+1} \setminus M_n) \cap \omega$. So $\langle u_n : n < \omega \rangle$ forms a partition of ω into finite sets. As F_0 is Ramsey, we can find $A \in F_0$ such that $\bigwedge_n |u_n \cap A| \leq 1$ and $A \subseteq A_0^*$ and

$$u_n \cap A \neq \emptyset \;\&\; u_m \cap A \neq \emptyset \;\&\; n < m \;\Rightarrow\; m - n \geq 10.$$

Let $A = \{i_\zeta : \zeta < \omega\}$ (increasing), $i_\zeta \in u_{n_\zeta}$. Now we define by induction on ζ, A_ζ, B_ζ, k_ζ, w_ζ such that

(a) $\langle A_\xi, B_\xi, k_\xi, w_\xi : \xi < \zeta \rangle$ is an initial segment of a play of the game in which Player I uses his winning strategy.

(b) $\langle A_\xi, B_\xi, k_\xi, w_\xi : \xi \leq \zeta \rangle$ belongs to $M_{n_\zeta + 3}$.

(c) $k_\zeta = i_\zeta$ and $w_\zeta = B_\zeta \cap I(k_\zeta) \cap A_1^*$.

There is no problem to carry out the definition, and clearly Player II wins because not only $\{k_\zeta : \zeta < \omega\} = \{i_\zeta : \zeta < \omega\} = A \subseteq A_0^*$ but also

$$\bigcup_{\zeta<\omega} w_\zeta = A_1^* \cap \bigcup_{\zeta<\omega} w_\zeta = A_1^* \cap \{j < \omega : h(j) = i_\zeta \text{ for some } \zeta < \omega\}$$

$$= A_1^* \cap \{j : h(j) \in A\} \in F_1.$$

[Why? As respectively: $w_\zeta \subseteq A_1^*$; as $A_1^* \setminus A_\xi \subseteq \bigcup\{w_\zeta : \zeta \leq i_\xi + 4\}$ by clause (γ) above; as $A = \{i_\zeta : \zeta < \omega\}$; as $A_1^* \in F_1$ and $A \in F_0$ hence $\{j : h(j) \in A\} \in F_1$.] Contradiction. $\square_{4.3}$

4.4 Definition. Let $T_n^h = \prod_{\ell<n} {}^{I(\ell)\times\ell}2$ and let $T^h = \bigcup_{n<\omega} T_n^h$. Note that T^h is a perfect tree with finite branching ordered by $\triangleleft$ (being initial segment). Let $Q = \mathrm{SP}^*(F_0, F_1, h) = \{T :\ T$ is a perfect subtree of T^h and for each $k < \omega$ for some $A_k \in F_0$ and $B_k \in F_1$ we have: if $\ell \in A_k$ and $\eta \in T^{[\ell]} \overset{\text{def}}{=} T \cap T_\ell^h$ and $\rho \in {}^{(B_k \cap I(\ell))\times k}2$ then for some $\nu \in {}^{I(\ell)\times\ell}2$ we have $\rho \subseteq \nu$ and $\eta\,\hat{}\,\langle\nu\rangle \in T\}$. The order: inverse inclusion.

4.5 Claim. 1) If $T \in Q$, $T^{[n]} = \{\eta_1, \dots, \eta_k\}$ (with no repetition) $T_\ell = T_{[\eta_\ell]} \stackrel{\text{def}}{=} \{\nu \in T : \eta_\ell \trianglelefteq \nu \text{ or } \nu \trianglelefteq \eta_\ell\}$, $T^\dagger_\ell \in Q$, $T_\ell \leq T^\dagger_\ell$ (i.e. $T^\dagger_\ell \subseteq T_\ell$) *then* $T \leq T^\dagger \stackrel{\text{def}}{=} \bigcup_{\ell=1}^{k} T_\ell \in Q$.
2) If $\underset{\sim}{\tau}$ is a Q-name of an ordinal and $n < \omega$ *then* there is $T^\dagger$, $T \leq T^\dagger \in Q$ such that $T^\dagger \Vdash_Q$ "$\underset{\sim}{\tau} \in A$" for some A satisfying $|A| \leq |T^{[n]}|$, and $T \cap \bigcup_{\ell \leq n} T^{[\ell]} = T^\dagger \cap \bigcup_{\ell \leq n} T^{[\ell]}$. Moreover for each $\eta \in T^{[n]}$, $T^\dagger_{[\eta]}$ determines $\underset{\sim}{\tau}$.

Proof. Same as in the proof of VI 5.5. $\square_{4.5}$

4.6 Claim.
Q is proper, in fact α-proper for every $\alpha < \omega_1$, and has the strong PP-property (see VI 2.12E(3)).

Proof. First we prove properness. Let λ be regular $> 2^{\aleph_1}$, $N \prec (H(\lambda), \in, <^*_\lambda)$ be countable, $\{Q, F_0, F_1, h\} \in N$ and $T \in N \cap Q$.
Let $\{\underset{\sim}{\tau}_\ell : \ell < \omega\}$ list the Q-names of ordinals from N. We now define a strategy for player I in the game from Claim 4.3. In the n'th move player I chooses $A_n \in F_0 \cap N$, $B_n \in F_1 \cap N$ and player II chooses $k_n \in A_n$ and $w_n \stackrel{\text{def}}{=} B_n \cap I_{k_n}$ (remember 4.3A); on the side player I chooses $T_n \in N \cap Q$ and m_n such that $T_0 = T$, $T_n \leq T_{n+1}$, $T_n^{[m_n+1]} = T_{n+1}^{[m_n+1]}$ and $m_n > \max\{m_{n-1}, k_{n'} : n' < n\}$ and $m_0 = 1$.

In the $(n+1)$'th move, player I first chooses m_{n+1} as above then he chooses $T_{n+1} \in Q$, $T_n \leq T_{n+1}$, $T_{n+1}^{[m_n+1]} = T_n^{[m_n+1]}$ such that for every $\eta \in T_n^{[m_n+1]}$, $(T_{n+1})_{[\eta]}$ forces a value to $\underset{\sim}{\tau}_\ell$ for $\ell \leq m_{n+1}$. This is possible by 4.5. Then as $T_{n+1} \in Q \cap N$ there are sets $A_{n+1} \in F_0 \cap N$, $B_{n+1} \in F_1 \cap N$ such that for every $k \in A_{n+1}$, $\eta \in (T_{n+1})^{[k]}$ and $\rho \in {}^{(B_{n+1} \cap I(k)) \times n}2$ for some $\nu \in {}^{I(k) \times k}2$, we have: $\rho \subseteq \nu$ and $\eta\char`^\langle \nu \rangle \in T_{n+1}$ and for simplicity $A_{n+1} \cap m_n = A_n \cap m_n$. Note that the amount of free choice player II retains is in N.

So by 4.3 for some such play, player II wins. Now $T^* \stackrel{\text{def}}{=} \bigcap_{n<\omega} T_n \in Q$ as $\{k_n : n < \omega\} \in F_0$ and $\bigcup\limits_{n<\omega} B_n \cap I(k_n) \in F_1$ witness; of course $T_n \leq T^*$ for each n hence $T = T_0 \leq T^*$ and $T^* \Vdash$ "$\underset{\sim}{\tau}_\ell[\underset{\sim}{G}_Q] \in N \cap Q_n$" (as $T_{\ell+1} \leq T^*$, see its choice).

So Q is proper. The proof also shows that Q has the strong PP-property (see VI 2.12: for more details see the proof of VI 4.4.). The proof of α-properness is as in VI 4.4 (and anyhow it is not used). $\square_{4.6}$

4.7 Lemma. Suppose $((*)_0$ of 4.2, Q =SP$^*(F_0, F_1, h)$ as defined in 4.4 of course and) $Q \lessdot P$ and P has the PP-property.
Then in V^P, F_1 cannot be extended to a P-point.

Proof. Suppose $p \in P$ forces that $\underset{\sim}{E}$ is an extension of F_1 to a P-point (in V^P). Let $\langle \underset{\sim}{r}_n : n < \omega \rangle$ be the sequence of reals which Q introduces, i.e. $r_n(i) = \ell \in \{0,1\}$ is defined as follows: clearly for a unique $k < \omega$, $i \in I_k$; now $\underset{\sim}{r}_n(i) = \ell$ iff: $n \geq k$, $\ell = 0$ *or* for some $T \in \underset{\sim}{G}_Q$, $T^{[k+1]} = \{\eta\}$ and $(\eta(k))(i,n) = \ell$ (remember that $\eta(k)$ is a function from $I(k) \times k$ to $\{0,1\}$). Define a P-name $\underset{\sim}{h}$:

$\underset{\sim}{h}(n)$ is 1 if $\{i < \omega : \underset{\sim}{r}_n(i) = 1\} \in \underset{\sim}{E}$ and

$\underset{\sim}{h}(n)$ is 0 if $\{i < \omega : \underset{\sim}{r}_n(i) = 0\} \in \underset{\sim}{E}$

So $p \Vdash$ "$\underset{\sim}{h} \in {}^{\omega}2$". Now as P has the PP-property, by VI 2.12D, there are $p_1 \geq p$, ($p_1 \in P$), and $\langle \langle \langle k(n), \langle i_n(\ell), j_n(\ell) \rangle : \ell \leq k(n) \rangle : n < \omega \rangle$ in V such that $k(n) < \omega$, $i_n(0) < j_n(0) < i_n(1) < j_n(1) < \ldots < i_n(k(n)) < j_n(k(n))$, and $j_n(k(n)) < i_{n+1}(0)$ such that:

$p_1 \Vdash_P$ " for every $n < \omega$ for some $\ell \leq k(n)$ we have $\underset{\sim}{h}(i_n(\ell)) = \underset{\sim}{h}(j_n(\ell))$"

Now define the following P-names:

$$\underset{\sim}{A}_n = \{m < \omega : \text{ for some } \ell \leq \underset{\sim}{k}(n), \underset{\sim}{r}_{i_n(\ell)}(m) = \underset{\sim}{r}_{j_n(\ell)}(m)\}.$$

We can conclude as in the proofs of VI 4.7, VI 5.8. $\square_{4.7}$

4.8 Claim. In V^Q, F_0 still generates an ultrafilter.

Proof. If not, then for some $T_0 \in Q$, and Q-name $\underset{\sim}{A}$ we have $T_0 \Vdash_Q$ "$\underset{\sim}{A} \subseteq \omega$ and $\underset{\sim}{A}, \omega \setminus \underset{\sim}{A}$ are $\neq \emptyset$ mod F_0".

By the proof of 4.6 without loss of generality, for some $A_0 \in F_0$ we have: for $k \in A_0$ and $\eta \in T_0^{[k+1]}$, $(T_0)_{[\eta]}$ forces a truth value to "$k \in \underset{\sim}{A}$" which we call $\mathbf{t}(T_0, \eta)$; without loss of generality for $\eta \in T_0^{[k]}, k \notin A_0 \Rightarrow |\mathrm{suc}_{T_0}(\eta)| = 1$.

Now for every $T \geq T_0$ and $\ell < \omega$ there are $A(T,\ell), B(T,\ell)$ as in Definition 4.4. For every $\ell < \omega$, $T \geq T_0$ and $k \in A(T,\ell)$ fix an arbitrary $\eta(T,\ell,k) \in T^{[k]}$.

Then, by observation 4.9 below, there are $m_{T,\ell,k} \in I(k) \cap B(T,\ell)$ and a partition $\langle u_i(T,\ell,k) : i < 3\rangle$ of $I(k) \cap B(T,\ell)$ and a triple $\langle \mathbf{t}_i(T,\ell,k) : i < 3\rangle$ of truth values and $j_k(T,\ell) \in \{0,1\}$ and truth value $\mathbf{s}_k(T,\ell)$ such that:

$(*)$ (a) if $j_k(T,\ell) = 0$ then for $i < 3$, for every $\rho \in {}^{u_i(T,\ell,k)\times\ell}2$ there is $\nu \in {}^{I(k)\times k}2$ such that $\rho \subseteq \nu$ and $\eta(T,\ell,k)\hat{}\langle\nu\rangle \in T$ and

$$T_{[\eta\hat{}\langle\nu\rangle]} \Vdash_Q \text{"}k \in \underset{\sim}{A} \text{ iff } \mathbf{t}_i(T,\ell,k)\text{"}.$$

(Clearly $\mathbf{t}_i(T,\ell,k) = \mathbf{t}(T_0, \eta\hat{}\langle\nu\rangle)$).

(b) if $j_k(T,\ell) = 1$ then for every $\rho \in {}^{(I(k)\cap B(T,\ell)\setminus\{m_{T,\ell,k}\})\times\ell}2$ there is $\nu \in {}^{(I(k)\times k)}2$ such that: $\rho \subseteq \nu$ and $(\eta(T,\ell,k))\hat{}\langle\nu\rangle \in T$ and $T_{[\eta\hat{}\langle\nu\rangle]} \Vdash_Q$ "$k \in \underset{\sim}{A}$ iff $\mathbf{s}_k(T,\ell)$".

So for some $j(T,\ell) < 2$ and $i(T,\ell) < 3$ and truth value $\mathbf{t}(T,\ell)$ we have

(α) if $j(T,\ell) = 0$, then

$$\bigcup\{u_{i(T,\ell)}(T,\ell,k) : \; j_k(T,\ell) = 0, \; k \in A(T,\ell), \mathbf{t}_{i(T,\ell)}(T,\ell,k) = \mathbf{t}(T,\ell) \in F_1$$

(β) if $j(T,\ell) = 1$ then $\{k \in A(T,\ell) : j_k(T,\ell) = 1, \mathbf{s}_k(T,\ell) = \mathbf{t}(T,\ell)\} \in F_0$.

Note:

$\otimes$ for (T,ℓ) as above there are $A = A^*(T,\ell) \in F_0$, $B = B^*(T,\ell) \in F_1$ satisfying: for every $k \in A$ there is $\eta \in T$, $\ell g(\eta) = k$ such that: every $\rho \in {}^{((I(k)\cap B)\times\ell)}2$ can be extended to $\nu \in {}^{I(k)\times k}2$ satisfying: $\eta\hat{}\langle\nu\rangle \in T$, $T_{[\eta\hat{}\langle\nu\rangle]} \Vdash_Q$ "$k \in \underset{\sim}{A}$ iff $\mathbf{t}(T,\ell)$"

[Why? If $j(T,\ell) = 0$ let

$$B = \bigcup\{u_{i(T,\ell)}(T,\ell,k) : j_k(T,\ell) = 0, \; k \in A(T,\ell), \mathbf{t}_{i(T,\ell)}(T,\ell,k) = \mathbf{t}(T,\ell)\}$$

and $A = \{k : I(k) \cap B \neq \emptyset\}$. Check the demand by clauses $(*)(a)$ and (α) above. So assume $j(T,\ell) = 1$ and let

$$B = \bigcup \{I(k) \cap B(T,\ell) \setminus \{m_{T,k,\ell}\} : k \in A(T,\ell) \text{ and } j_k(T,\ell) = 1, \\ \text{and } \mathbf{s}_k(T,\ell) = \mathbf{t}(T,\ell)\};$$

(why $B \in F_1$? because $F_1 \not\leq_{\mathrm{RK}} F_0$!). Put $A = \{k : I_k \cap B \neq \emptyset\}$ and check the demand by clauses $(*)$(b) and (β) above].
Note that we have been dealing with fixed T, ℓ.

As we can increase T_0 without loss of generality: for some truth value $\mathbf{t}^*$ for a dense set of $T' \geq T_0$ for the F_0-majority of $\ell < \omega$ we have and $\mathbf{t}(T',\ell) = \mathbf{t}^*$.

Now we can define a strategy for player I in the game from 4.3. So in the n'th move player I chooses A_n, B_n and player II chooses k_n, w_n; but we let player I play "on the side" also T_n, ℓ_n (chosen in the n'th move) such that:

(A) $T \leq T_n \leq T_{n+1}$, $T_n^{[k_n+1]} = T_{n+1}^{[k_n+1]}$, $\omega > \ell_{n+1} > \ell_n$, and $\mathbf{t}^* = \mathbf{t}((T_n)_{[\eta]}, \ell_n)$ for $n > 0$ and $\eta \in T_n^{[k_n+1]}$.

(B) For every $k \in A_{n+1}$ and $\eta \in T_n^{[k_n+1]}$ there is η_1, $\eta \trianglelefteq \eta_1 \in T_n^{[k]}$ such that for every $\rho \in {}^{(B_{n+1} \cap I(k)) \times \ell_{n+1}}2$ there is ν, $\rho \subseteq \nu$, $\eta_1{}^\frown\langle\nu\rangle \in T_n$, $\mathbf{t}(T_n, \ell_n, k_n) = \mathbf{t}(T_n, \ell_n) = \mathbf{t}^*$, (note T_{n+1} is chosen only after k_{n+1}, w_{n+1} were chosen).

We should prove that player I can carry out his strategy. For stage $n+1$ let $\{\eta_0^n, \dots, \eta_{m(n)}^n\}$ list $T_n^{[k_n+1]}$, so for some $\ell_{n+1} > \ell_n$, for each $\zeta \leq m(n)$ there is $T_{n,\zeta} \geq (T_n)_{[\eta_\zeta^n]}$ such that $\mathbf{t}(T_{n,\zeta}, \ell_{n+1}) = \mathbf{t}^*$. Let $B_{n+1} = \bigcap_{\zeta \leq m(n)} B^*(T_{n,\zeta}, \ell_{n+1})$ and $A_{n+1} = \{k \in A_n : k > k_n \text{ and } I(k) \cap B_{n+1} \neq \emptyset\}$.

By clause (B) above, after player II moves, we can choose T_{n+1} as required. As this is a strategy, by Claim 4.3 for some play in which player I uses it he looses. For this play $\{k_n : n < \omega\} \in F_0$, $\bigcup_{n<\omega} w_n \in F_1$, so $T \stackrel{\mathrm{def}}{=} \bigcap_{n<\omega} T_n \in Q$. By tracing the demands on the $\mathbf{t}$'s:

$\oplus$ for $n < \omega$, $\eta \in T$, $\ell g(\eta) = k_n + 1$ we have $T_{[\eta]} \Vdash$ "$k_n \in \underset{\sim}{A}$ iff $\mathbf{t}^*$".

We conclude: $T \Vdash$ "$\{k_n : n < \omega\} \cap \underset{\sim}{A}$ is $\emptyset$ or is $\underset{\sim}{A}$" as $\{k_n : n < \omega\} \in F_0$ we get the desired conclusion. $\square_{4.8}$

4.9 Observation. Suppose $\mathbf{t}$ is a function from $X^* = \prod_{t\in u} A_t$ to $\{0,1\}$, u finite.

Then at least one of the following holds:

(α) we can find u_i, X_i $(i<3)$ such that :

  (a) $\langle u_i : i<3\rangle$ is a partition of u,

  (b) $X_i \subseteq X^*$,

  (c) $\mathbf{t}{\restriction}X_i$ is constant,

  (d) for every $i<3$ and $\rho \in \prod_{t\in u_i} A_t$ there is $\nu \in X_i$, $\rho \subseteq \nu$,

(β) for some $x\in u$, there is $X\subseteq X^*$ such that $\mathbf{t}{\restriction}X$ is constant and for every $\rho \in \prod_{t\in u\setminus\{x\}} A_t$ there is $\nu\in X$, $\rho\subseteq\nu$.

Proof. Let for $j\in\{0,1\}, P_j = \{v : v\subseteq u$ and there is $X\subseteq X^*$ such that $\mathbf{t}{\restriction}X$ is constantly j and for every $\rho\in\prod_{t\in v} A_t$ there is $\nu\in X$, $\rho\subseteq\nu\}$. Clearly

(A) $u_1\in P_j, u_0\subseteq u_1$ implies $u_0\in P_j$. [Why? Same X witnesses this.]

(B) $u_1\subseteq u$ & $u_1\notin P_j$ implies $u\setminus u_1\in P_{1-j}$ [Why? As $u_1\notin P_j$, for some $\rho\in\prod_{t\in u_1} A_t$ for no $\nu\in\prod_{t\in u\setminus u_1} A_t$ does $\mathbf{t}(\rho\cup\nu)=j$; let $X \stackrel{\text{def}}{=} \{\nu\in\prod_{t\in u} A_t : \rho\subseteq\nu\}$, it is as required for $u\setminus u_1$.]

(C) $\emptyset\in P_0\cup P_1$. [Why? Trivially.]

Case (i): $P_0\cup P_1$ is not an ideal.

So there are $u_0,u_1\in P_0\cup P_1$ with $v\stackrel{\text{def}}{=} u_0\cup u_1\notin P_0\cup P_1$. By (A) without loss of generality $u_0\cap u_1=\emptyset$. Let $u_2=u\setminus v$, so $\langle u_0,u_1,u_2\rangle$ is a partition of u. Now by clause (B) we know that $u_2\in P_0$ (and to P_1) as $v=u\setminus u_2$ does not belong to P_1 (and to P_0). Now we know $u_0,u_1,u_2\in P_0\cup P_1$, so for some $\langle j_\ell : \ell<3\rangle$ we have $u_\ell\in P_{j_\ell}$ for $\ell<3$, and let X_ℓ be a witness. Now check that clause (α) in the conclusion holds.

Case (ii): $P_0\cup P_1$ is an ideal.

If $u\in P_0\cup P_1$, then $\mathbf{t}$ is constant so conclusion (α) is trivial, so assume not. By (B) above the ideal is a maximal ideal so it is principal (because u is finite), i.e. for some $x\in u$, $u\setminus\{x\}\in P_0\cup P_1$, $\{x\}\notin P_0\cup P_1$ so we have finished. (Reflection shows we get more than required in (β): reread the proof of (B)).

$\square_{4.2,4.1}$