XIV. Iterated Forcing with Uncountable Support

§0. Introduction

This chapter is [Sh:250], revised. Here we consider revised support for the not necessarily countable case. In §1 we define and present the basic properties of κ-RS iterated. This includes the case $\kappa = \aleph_1$ and so it can serve as a substitute to X §1. The main difference is that here we have to use names which sometimes have no value as we cannot use rank as there.

Unlike Chapter X, we do not have a useful properness to generalize, so naturally the generalizations of completeness are in the center. In 2.1 we introduce, and in 2.4 we show it does not matter much if we use the version with games of length $\kappa = \mathrm{cf}\kappa$ or the version with a side order $\leq_0$, the "pure" extension which is κ-complete. Then we define iterations of such forcing notions and prove the basic properties (2.5–2.8). This repeats §1, so against dullness this time we waive the associativity law and simplify somewhat the definition of the iteration. In the definition of the order except finitely many places (which are names) the extensions are pure (i.e. $\leq_0$) in the old places. The first use of "pure" extensions is Prikry [Pr], and the first use of iterations with the distinction between old and new places (in normal support of course) is Gitik [Gi] which uses Easton support iteration $\bar{Q}$'s for high inaccessibles, each Q_i is $(\{2\}, \kappa_i)$-complete where for the important i's $\kappa_i = i$; a subsequent proof more similar to our case is [Sh:276, §1]. The application we have in mind is

$\kappa = \mu^+$, $\{\theta : \aleph_0 < \theta = \mathrm{cf}(\theta) \leq \kappa\} \subseteq S \subseteq \{2, \aleph_0\} \cup \{\theta : \theta = \mathrm{cf}(\theta) \leq \kappa\}$, and we shall iterate the forcing notion Q_i which has $(S, < \kappa)$-Pr, (so necessarily the cardinals $\theta \leq \kappa$ remain cardinals), the iteration being κ-Sp_2-iteration, and characteristically the length of iteration is some quite large cardinal λ, $i < \lambda \Rightarrow |P_i| < \lambda$, and we collapse all cardinals $> \kappa$, $< \lambda$ (so μ, κ play the role of $\aleph_0$, $\aleph_1$ in Chapter X). So we need to know of such iterations of forcing notions having $(S, < \kappa)$-Pr_1, which is done in 2.9. We could also deal similarly with iterations $\bar{Q}$ of length λ, λ strongly inaccessible $[i < \lambda \Rightarrow |P_i| < \lambda]$ and $S \subseteq \kappa$ unbounded in λ. In 2.18, 2.19 we look at the case essentially cofinalities are preserved (i.e. no $\theta = \mathrm{cf}(\theta) \geq \kappa$ becomes of cardinality $< \kappa$).

In the third section we indicate what forcing axioms we can get (3.4), and show how e.g. Mathias forcing fits in assuming $\mathrm{MA}_{<\kappa}$ (in 3.3). We then give a solution of the first Abraham problem (3.5).

In the fourth section we show how to fit Sacks forcing. The last section is a real application- to the second Abraham problem. In it we consider a forcing e.g. preserving $\theta \leq \kappa$, making the cofinality of κ^+ to $\aleph_0$, assuming only a weak form of "on κ^+ there is a large ideal" in which there the ideal disappears.

§1. κ-Revised Support Iteration

$\mathcal{D}_\kappa$ is the closed unbounded filter on κ.

A work of Groszek and Jech (see [J86] deal with making the continuum large (in a different way and effect, done about the same time independently).

1.1 Definition. Here κ is an infinite cardinal, but when it is an infinite ordinal which is not a cardinal we mean $|\kappa|^+$ (this is intended just for the case κ is collapsed during the iteration). We define the following notions and those in 1.2 and prove 1.4 by simultaneous induction on α:

(A) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a κ-RS interaction (RS stands for revised support)

(B) a $\bar{Q}$-named ordinal

(C) a $\bar{Q}$-named atomic condition $\underset{\sim}{q}$, and we define $\underset{\sim}{q}\restriction\xi, \underset{\sim}{q}\restriction\{\xi\}$ for a $\bar{Q}$-named atomic condition $\underset{\sim}{q}$ and ordinal ξ when $\xi \le \alpha$, $\xi < \alpha$ respectively and $\underset{\sim}{q}\restriction[\zeta, \xi)$ when $\zeta < \xi \le \alpha$.

(D) the κ-RS limit of $\bar{Q}$, $\mathrm{Rlim}_\kappa \bar{Q}$ which satisfies $P_i \lessdot \mathrm{Rlim}_\kappa \bar{Q}$ for every $i < \alpha$ and we define $p\restriction\beta \in P_\beta$ for $p \in \mathrm{Rlim}_\kappa \bar{Q}$ and $\beta \le \alpha$ (We may omit κ if clear from the context).

Let us define and prove

(A) We define "$\bar{Q}$ is a κ-RS iteration"

$\alpha = 0$: no condition.

α is limit: $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a κ-RS iteration iff for every $\beta < \alpha, \bar{Q}\restriction\beta$ is one.

$\alpha + 1 : \bar{Q}$ is an κ-RS iteration iff $\bar{Q}\restriction\beta$ is one, $P_\beta = \mathrm{Rlim}_\kappa(\bar{Q}\restriction\beta)$, and $\underset{\sim}{Q}_\beta$ is a P_β-name of a forcing notion.

(B) We define "$\underset{\sim}{\xi}$ is a $\bar{Q}$-named ordinal". It means:

(1) $\underset{\sim}{\xi}$ is a function, $\mathrm{Rang}(\underset{\sim}{\xi}) \subseteq \mathrm{Ord}$.

(2) for $r \in \mathrm{Dom}(\underset{\sim}{\xi})$, letting $\beta = \underset{\sim}{\xi}(r)$, we have $\beta < \alpha$, and $r \in P_\beta * \underset{\sim}{Q}_\beta$. (see an identification later).

(3) for every $r_1, r_2 \in \mathrm{Dom}(\underset{\sim}{\xi})$, if r_1, r_2 are compatible, then $\underset{\sim}{\xi}(r_1) = \underset{\sim}{\xi}(r_2)$.

[What do we mean by "r_1, r_2 are compatible"? Let $r_1 \in P_{\beta_1} * \underset{\sim}{Q}_{\beta_1}$ and $r_2 \in P_{\beta_2} * \underset{\sim}{Q}_{\beta_2}$. If $\beta_1 = \beta_2$, there is no problem in defining compatibility. Otherwise, without loss of generality $\beta_1 < \beta_2$. Then, as noted in 1.4, $P_{\beta_1} * \underset{\sim}{Q}_{\beta_1}$ is essentially the same as P_{β_1+1} and $P_{\beta_1+1} \lessdot P_{\beta_2} \lessdot P_{\beta_2} * \underset{\sim}{Q}_{\beta_2}$, so we can test compatibility in $P_{\beta_2} * \underset{\sim}{Q}_{\beta_2}$].

1.1A Remark. For α a limit ordinal, $\mathrm{Dom}(\underset{\sim}{\xi})$ is essentially a subset of $\bigcup_{\beta<\alpha} P_\beta$, so $\underset{\sim}{\xi}$ is a "partial name" for an ordinal. Note that $\mathrm{Dom}(\underset{\sim}{\xi})$ is not necessarily pre-dense (there is no point in requiring it to be pre-dense in $\bigcup_{\beta<\alpha} P_\beta$, since this will not imply pre-density in $P_\alpha = \mathrm{Rlim}_\kappa \bar{Q}$, which is the forcing we are interested in).

Continuation of 1.1.

(C) We say "$\underset{\sim}{q}$ is a $\bar{Q}$-named atomic condition" if:

(1) $\underset{\sim}{q}$ is a pair of functions $(\underset{\sim}{\zeta}_q, \mathrm{cnd}_{\underset{\sim}{q}})$ with common domain $D = D_{\underset{\sim}{q}}$.

(2) $\underset{\sim}{\zeta}_q$ is a $\bar{Q}$-named ordinal.

(3) if $r_1, r_2 \in D_{\underset{\sim}{q}}$ and r_1, r_2 are compatible (see above) then $\mathrm{cnd}_{\underset{\sim}{q}}(r_1) = \mathrm{cnd}_{\underset{\sim}{q}}(r_2)$.

(4) if $r \in D_{\underset{\sim}{q}}$, letting $\beta = \underset{\sim}{\zeta}_q(r)$, we have:

$$r \restriction \beta \Vdash \text{"}\mathrm{cnd}_{\underset{\sim}{q}}(r) \in \underset{\sim}{Q}_\beta\text{"}$$

(note: we can add: it is forced ($\Vdash_{P_\beta}$) that $Q_\beta \models$ "$r(\beta) \leq \mathrm{cnd}_{\underset{\sim}{q}}(r)$" with little subsequent change). We define $\underset{\sim}{q} \restriction \xi$ as $(\underset{\sim}{\zeta}_q \restriction D_1, \mathrm{cnd}_{\underset{\sim}{q}} \restriction D_1)$ where $D_1 = \{p \in D_{\underset{\sim}{q}} : \underset{\sim}{\zeta}_q(p) < \xi\}$. We define $\underset{\sim}{q} \restriction \{\xi\}$ as $(\underset{\sim}{\zeta}_q \restriction D_2, \mathrm{cnd}_{\underset{\sim}{q}} \restriction D_2)$ where $D_2 = \{p \in D_{\underset{\sim}{q}} : \underset{\sim}{\zeta}_q(p) = \xi\}$, and $\underset{\sim}{q} \restriction [\zeta, \xi)$ similarly.

1.1B Remark. The definition would become simpler if we demand $r \in P_\beta$ instead of $r \in P_\beta * Q_\beta$ in (B2). (e.g. we could then drop the clause "$r(\beta) \leq \mathrm{cnd}_{\underset{\sim}{q}}(r)$" in (4)). However, we need this more complicated definition if we want associativity i.e. 1.5(3):

Consider a $\kappa - RS$ iteration $\bar{Q} = \langle P_\alpha, Q_\alpha : \alpha < \alpha^* \rangle$. Then a condition in P_{α^*} could be of the form $p = (\{(r, \beta)\}, \{(r, \underset{\sim}{q})\})$ with $r \in P_\beta$, $\underset{\sim}{q} \in \underset{\sim}{Q}_\beta$. Now assume that $\langle \alpha(\xi) : \xi \leq \xi^* \rangle$ is an increasing continuous sequence with $\alpha(0) = 0, \alpha(\xi^*) = \alpha^*$, and $\alpha(\xi) < \beta < \beta + 1 < \alpha(\xi + 1)$. Then in the natural "isomorphic copy" of p in $P'_{\xi^*} = \mathrm{Rlim}_\kappa \bar{Q}'$, where

$$\bar{Q}' = \langle P_{\alpha(\xi)}, P_{\alpha(\xi+1)}/P_{\alpha(\xi)} : \xi < \xi^* \rangle = \langle P'_\xi, \underset{\sim}{Q}'_\xi : \xi < \xi^* \rangle$$

β would become ξ (as $\underset{\sim}{q}$ correspond to an element of $\underset{\sim}{Q}'_\xi$). However, r may not be in P'_ξ but can only be found in $P'_\xi * \underset{\sim}{Q}'_\xi$. However this is mainly an aesthetic problem– saving here costs us some cumbersomeness in application, but no real damage: when we prove statements on iteration $\bar{Q}$ we cannot restrict ourselves to length $\alpha = 0, 1, 2$, or $\alpha = \mathrm{cf}(\alpha)$ etc. For diversity, we do use this way in §2.

(D) We define $\text{Rlim}_\kappa \bar{Q}$ as follows:

if $\alpha = 0$: $\text{Rlim}_\kappa \bar{Q}$ is a trivial forcing with just two compatible conditions i.e. $\text{Rlim}_\kappa \bar{Q} = \{\emptyset, \{(\emptyset, \emptyset)\}\}$ say with $\emptyset \leq \{(\emptyset, \emptyset)\}$

if $\alpha > 0$: we call q an atomic condition of $\text{Rlim}_\kappa \bar{Q}$, if it is a $\bar{Q}$-named atomic condition.

The set of conditions in $\text{Rlim}_\kappa \bar{Q}$ is

$$\begin{aligned}\{p : \ & p \text{ a set of } \lambda \text{ atomic conditions for some } \lambda < \kappa; \\ & \text{and for every } \beta < \alpha, p{\restriction}\beta \stackrel{\text{def}}{=} \{q{\restriction}\beta : q \in p\} \in P_\beta, \\ & \text{and } p{\restriction}\beta \Vdash_{P_\beta} \text{“the set } \{q{\restriction}\{\beta\} : q \in p\} \text{ has an upper bound in } \underset{\sim}{Q}_\beta\text{”}\}\end{aligned}$$

More precisely, the last condition in the previous paragraph means

$$\begin{aligned} p{\restriction}\beta \Vdash_{P_\beta} \text{“}\exists q_0 \in \underset{\sim}{Q}_\beta \forall \underset{\sim}{q} \in p \forall r \in D_q : \\ & \text{if } \underset{\sim}{\zeta}_q(r) = \beta \text{ and } r{\restriction}\beta \in \underset{\sim}{G}_{P_\beta} \text{ then} \\ & q_0 \Vdash_{\underset{\sim}{Q}_\beta} [\text{if } r(\beta) \in \underset{\sim}{G}_{Q_\beta} \text{then } \text{cnd}_{\underset{\sim}{q}}(r) \in \underset{\sim}{G}_{Q_\beta}]\text{”}\end{aligned}$$

(where $\underset{\sim}{G}_P$ is the canonical name for the generic set for P).

Remember that we have defined $p{\restriction}\beta = \{q{\restriction}\beta : q \in p\}$ and $p{\restriction}[\beta, \gamma)$ for $\beta < \gamma \leq \alpha$, similarly.

The order: $p_0 \leq p_1$ iff $p_0 \subseteq p_1$ or just $p_0 \subseteq \{\underset{\sim}{q}{\restriction}\beta : \underset{\sim}{q} \in p_1 \text{ and } \beta \leq \alpha\}$.

The identification. Clearly for $\beta < \alpha$, we have $P_\beta \subseteq P_\alpha$. We can identify $P_\beta * \underset{\sim}{Q}_\beta$ with a subset of P_α when $\beta + 1 = \alpha : (p, \underset{\sim}{q})$ is identified with $p \cup \{[\underset{\sim}{q}]\}$ where $[\underset{\sim}{q}] = (\underset{\sim}{\zeta}, \text{cnd})$, $\text{Dom}(\underset{\sim}{\zeta}) = \{\emptyset\}$($\emptyset$ the empty condition of P_β), $\underset{\sim}{\zeta}(\emptyset) = \beta$ and $\text{cnd}(\emptyset) = \underset{\sim}{q}$.

It is easy to check the demands, e.g. under this identification $P_\beta * \underset{\sim}{Q}_\beta$ is a dense subset of $P_{\beta+1}$.

Now we have to show $P_\beta \lessdot \text{Rlim}_\kappa \bar{Q}$ (for $\beta < \alpha$). Note that any $\bar{Q}{\restriction}\beta$-named ordinal (or condition) is a $\bar{Q}$-named ordinal (or condition), and see Claim 1.4(1) below.

1.1C Remark. Note that for the sake of 1.5(3) we allow κ to be not a cardinal and then we really use $|\kappa|^+$.

1.1D Remark. We can obviously define $\bar{Q}$-named sets; but for condition (and ordinals for them) we want to avoid the vicious circle of using names which are interpreted only after forcing with them below.

1.2 Definition.

(1) Suppose $\bar{Q}$ is a κ-RS iteration, $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named ordinal, $\alpha = \ell g(\bar{Q})$, $G \in \mathrm{Gen}(\bar{Q})$ (see part (2) of the Definition below). We define $\underset{\sim}{\zeta}[G]$ by:

(i) $\underset{\sim}{\zeta}[G] = \gamma$ if ($\gamma < \alpha \stackrel{\text{def}}{=} \ell g(\bar{Q})$ and) for some $p \in \mathrm{Dom}(\underset{\sim}{\zeta}) \cap G_{\gamma+1}$ which is in $P_\gamma * \underset{\sim}{Q}_\gamma$ we have $\underset{\sim}{\zeta}(p) = \gamma$.

(ii) otherwise (i.e., $G \cap \mathrm{Dom}(\underset{\sim}{\zeta}) = \emptyset$) $\underset{\sim}{\zeta}[G]$ is not defined.

(1A) For a $\bar{Q}$-named condition $\underset{\sim}{q}$, we defined $\underset{\sim}{q}[G]$ similarly.

(2) We denote the set of $G \subseteq \bigcup_{i<\alpha} P_{i+1}$ such that $G \cap P_{i+1}$ is generic over V for each $i < \alpha$ by $\mathrm{Gen}(\bar{Q})$. We let $G_i = G \cap P_i$.

(3) For $\underset{\sim}{\zeta}$ a $\bar{Q}$-named ordinal and $q \in \bigcup_{i<\alpha} P_i$ let $q \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} = \xi$" if for every $G \in \mathrm{Gen}(\bar{Q})$ we have: $q \in G \Rightarrow \underset{\sim}{\zeta}[G] = \xi$, i.e. if $q \Vdash$ "$\underset{\sim}{\xi}[G]$ is defined and equal to ξ". Similarly for $p \Vdash_{\bar{Q}}$ "$\underset{\sim}{q} = r$" and for $p \Vdash_{\bar{Q}}$ "$q \in \underset{\sim}{G}$".

1.3 Remark. 1) From where is G taken in (2), (3)? E.g., V is a countable model of set theory, G taken from the "true" universe.
2) If $p, p' \in P_\alpha, p \subseteq p'$ and for all $(\underset{\sim}{\xi}, \mathrm{cnd}) \in p' \setminus p$ there is a $\beta < \alpha$ such that $p\restriction\beta \Vdash$ "$\underset{\sim}{\xi}[G]$ undefined", then p and p' are essentially equivalent, i.e. for all $q \supseteq p$ we have: q and p' are compatible; or equivalently, $p \Vdash$ "$p' \in G$".

Now we point out some properties of κ-RS iteration.

1.4 Claim. Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ be a κ-RS iteration, $P_\alpha = \mathrm{Rlim}_\kappa \bar{Q}$.

(1) If $\beta < \alpha$ then: $P_\beta \subseteq P_\alpha$ and $p \in P_\beta \Rightarrow p\restriction\beta = p$ & $P_\alpha \models p\restriction\beta \le p$; for $p_1, p_2 \in P_\beta$ we have $[P_\beta \models p_1 \le p_2$ iff $P_\alpha \models p_1 \le p_2]$; and $P_\beta \lessdot P_\alpha$.

Moreover, if $q \in P_\beta, p \in P_\alpha$, then q, p are compatible in P_α iff $q, p\restriction\beta$ are compatible in P_β; if $p\restriction\beta \le q$, a least upper bound of $q \in P_\beta$, $p \in P_\alpha$ is $q \cup p\restriction[\beta, \alpha)$. Also for $\beta_1 \le \beta_2 < \alpha$ and $p \in P_\alpha$ then $p\restriction\beta_1 = (p\restriction\beta_2)\restriction\beta_1$.

(2) If $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named ordinal and $G, G' \in \mathrm{Gen}(\bar{Q})$ and $G \cap P_{\xi+1} = G' \cap P_{\xi+1}$ and $\underset{\sim}{\zeta}[G] = \xi$ then $\underset{\sim}{\zeta}[G'] = \xi$; hence we write $\underset{\sim}{\zeta}[G \cap P_{\xi+1}] = \xi$.

(3) If $\underset{\sim}{\beta}, \underset{\sim}{\gamma}$ are $\bar{Q}$-named ordinals, then Max $\{\underset{\sim}{\beta}, \underset{\sim}{\gamma}\}$ (for a generic $G \in \mathrm{Gen}(\bar{Q})$, this name is defined if both are defined and its value is the maximum) is a $\bar{Q}$-named ordinal. Also Min$\{\underset{\sim}{\beta}, \underset{\sim}{\gamma}\}$ (defined if at least one of them is defined, if only one is defined the value is its value, if both are defined the value is the minimum).

(4) If $\alpha = \beta_0 + 1$, in Definition 1.1(D), in defining the set of elements of P_α, in the demand "$\beta < \alpha \Rightarrow p\restriction\beta \in P_\beta$", we can restrict ourselves to $\beta = \beta_0$.

(5) The following set is dense in P_α : $\{p \in P_\alpha$; for every $\beta < \alpha$, if $r_1, r_2 \in p$, then $\Vdash_{P_\beta}$ "if $r_1\restriction\{\beta\} \ne \emptyset, r_2\restriction\{\beta\} \ne \emptyset$ then they are equal" $\}$ where $Y \subseteq P_\alpha$ is dense iff for every $p \in P$ there is q, $p \le q$ and q is equivalent to some $q' \in Y$ (i.e. $q \Vdash$ "$q' \in \underset{\sim}{G}$" and $q' \Vdash$ "$q \in \underset{\sim}{G}$") (can even we $\Vdash_{\bar{Q}\restriction\beta}$).

(6) $|P_\alpha| \le (\prod_{i<\alpha} 2^{|P_i|})^{<\kappa}$ for limit α (where $|P|$ is the number of elements of P up to equivalence). Also if $\beta < \alpha \Rightarrow \mathrm{density}(P_\alpha) \le \lambda = \mathrm{cf}(\lambda)$ and $\alpha \le \lambda$ (or just $\alpha < \lambda^+$) *then* $\mathrm{density}(P_\alpha) \le 2^\lambda$.

(7) If $\Vdash_{P_i}$ "$|\underset{\sim}{Q}_i| \le \lambda$ & $\underset{\sim}{Q}_i \subseteq V$" (and $\lambda \ge 2$), then (essentially) $|P_{i+1}| \le \lambda^{|P_i|}$. (Why "essentially"? We have to identify P_i-names of members of Q_i which $\Vdash_{P_i}$ "they are equal".) We can replace $|P_i|$ by $\mathrm{density}(P_i)$ and get $\mathrm{density}(P_{i+1}) \le \mathrm{density}(P_i) + \lambda + \aleph_0$. Instead of "$\underset{\sim}{Q}_i \subseteq V$" it suffices that: $\lambda^{<\mu} = \lambda$ and: Q (i.e. set of members) is included in the closure of V under taking subsets of power $< \mu$.

(8) Suppose $\bar{Q}$ is an κ-RS iteration, $\varphi(x, y)$ is a formula (possibly with parameters from V) such that:

(a) for every $G \in \mathrm{Gen}(\bar{Q})$ there is at most one x such that $(V[G], V, G) \models$ "$\varphi(x, G)$", this x is called $\underset{\sim}{x}[G]$ if there is such x, and $\underset{\sim}{x}[G]$ is not defined otherwise.

(b) if G $\in$Gen$(\bar{Q})$, $\underset{\sim}{x}[G]$ defined then it is an ordinal $< \ell g(\bar{Q})$, call it β, moreover for some $p \in P_{\beta+1}$, such that if $\beta + 1 = \ell g(\bar{Q})$ then $p \in P_\beta * \underset{\sim}{Q}_\beta$, we have:

$[p \in G' \in \text{Gen}(\bar{Q}) \Rightarrow \underset{\sim}{x}[G'] = \beta]$.

Then there is a $\bar{Q}$-named ordinal $\underset{\sim}{\zeta}$ such that: for every $G \in \text{Gen}(\bar{Q})$, $\underset{\sim}{x}[G] = \underset{\sim}{\zeta}[G]$ (i.e. they are both defined with the same value, or they are both undefined).

(9) Suppose $\bar{Q}, \varphi(x,y), \underset{\sim}{x}[G]($ for $G \in \text{Gen}(\bar{Q}))$ are as in (8) except that clause (b) is replaced by:

(b)$'$ if G $\in$Gen$(\bar{Q})$ and $\underset{\sim}{x}[G]$ is defined, then it has the form $(\zeta, p), \zeta < \ell g(\bar{Q}), p \in \underset{\sim}{Q}_\zeta[G_\zeta]$ and for some q we have: $q \in G \cap P_{\zeta+1}$, and $[\zeta + 1 = \ell g(\bar{Q}) \Rightarrow q \in G \cap P_\zeta]$ (and if we make the addition in 1.1(C) clause (4) then $\underset{\sim}{Q}_\zeta[G \cap P_\zeta] \vDash$ "$q \restriction \{\zeta\} \leq p$") and $[q \in G' \in \text{Gen}(\bar{Q}) \Rightarrow \underset{\sim}{x}[G] = \underset{\sim}{x}[G']]$.

Then there is a $\bar{Q}$-named condition $\underset{\sim}{q}$ such that:

for every $G \in \text{Gen}(\bar{Q})$, $\underset{\sim}{x}[G] = \langle \underset{\sim}{\zeta}_q[G], \underset{\sim}{q}[G] \rangle$ (so both are defined and equal or both are not defined).

Proof: By induction on α.

1.4A Remark. The inverse of 1.4(8) and of 1.4(9) hold, of course.

1.5 Lemma. *The Iteration Lemma*

(1) Suppose F is a function, *then* for every ordinal α there is one and only one κ-RS iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha^\dagger \rangle$ such that:

(a) for every $i, \underset{\sim}{Q}_i = F(\bar{Q} \restriction i)$,

(b) $\alpha^\dagger \leq \alpha$,

(c) either $\alpha^\dagger = \alpha$ or $F(\bar{Q})$ is not an $(\text{Rlim}_\kappa \bar{Q})$-name of a forcing notion.

(2) Suppose $\bar{Q}$ is a κ-RS-iteration, $\alpha = \ell g(\bar{Q}), \beta < \alpha, G_\beta \subseteq P_\beta$ is generic over V. *Then* in $V[G_\beta], \bar{Q}/G_\beta = \langle P_i/G_\beta, \bar{Q}_i : \beta \leq i < \kappa \rangle$ is a κ-RS-iteration and $\text{Rlim}_\kappa \bar{Q} = P_\beta * (\text{Rlim}_\kappa \bar{Q}/G_\beta)$ (essentially).

(3) The Association Law: If $\alpha_\xi = \alpha(\xi)$ $(\xi \le \xi^*)$ is increasing and continuous, $\alpha_0 = 0; \bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha^* \rangle$ is a κ-RS-iteration, $P_{\alpha^*} = \mathrm{Rlim}_\kappa \bar{Q}$ and $\alpha_{\xi^*} = \alpha^*$ and κ is a successor cardinal (for κ inaccessible we need to assume more); *then* so are $\langle P_{\alpha(\xi)}, P_{\alpha(\xi+1)}/P_{\alpha(\xi)} : \xi < \xi^* \rangle$ and $\langle P_i/P_{\alpha(\xi)}, \underset{\sim}{Q}_i : \alpha(\xi) \le i < \alpha(\xi+1) \rangle$ (with κ-RS-Limits $P_{\alpha_{\xi^*}}$ and $P_{\alpha(\xi+1)}/P_{\alpha(\xi)}$ respectively) and vice versa.

1.5A Remark. In (3) we can use α_ξ's which are names.

Proof: (1) Easy.

(2) Pedantically, we should formalize the assertion as follows:

(*) There are function F_0, F_1(= definable classes) such that for every κ-RS-iteration $\bar{Q}$ with $\ell g(\bar{Q}) = \alpha$, and $\beta < \alpha, F_0(\bar{Q}, \beta)$ is a P_β-name $\underset{\sim}{\bar{Q}}^\dagger$ such that:

(a) $\Vdash_{P_\beta}$ "$\bar{Q}^\dagger$ is a κ-RS-iteration of length $\alpha - \beta$".

(b) $P_\beta * (\mathrm{Rlim}_\kappa \underset{\sim}{\bar{Q}}^\dagger)$ is equivalent to $P_\alpha = \mathrm{Rlim}_\kappa \bar{Q}$, by $F_1(\bar{Q}, \beta)$ (i.e., $F_1(\bar{q}, \beta)$ is an isomorphism between the corresponding completions to Boolean algebras)

(c) if $\beta \le \gamma \le \alpha$ then $\Vdash_{P_\beta}$ "$F_0(\bar{Q}\restriction\gamma, \beta) = F_0(\bar{Q}, \beta)\restriction(\gamma - \beta)$" and $F_1(\bar{Q}, \beta)$ extends $F_1(\bar{Q}\restriction\gamma, \beta)$ and $F_1(\bar{Q}\restriction\gamma, \beta)$ transfers the P_γ-name $\underset{\sim}{Q}_\gamma$ to a P_β-name of $(\mathrm{Rlim}_\kappa(\underset{\sim}{\bar{Q}}^\dagger\restriction(\gamma-\beta))$-name of $\underset{\sim}{Q}^\dagger_{\gamma-\beta}$ (where $\underset{\sim}{Q}^\dagger_{\gamma-\beta} = \mathrm{Rlim}_\kappa\langle \underset{\sim}{Q}^\dagger_{\beta+i} : i < \gamma - \beta \rangle$).

The proof is induction on α, and there are no special problems.

(3) Again, pedantically the formulation is: There are functions F_3, F_4 such that

(*) For $\bar{Q}$-iteration, $\ell g(\bar{Q}) = \alpha_{\xi^*}, \bar{\alpha} = \langle \alpha_\xi : \xi \le \xi^* \rangle$ increasing continuous, $F_3(\bar{Q}, \bar{\alpha})$ is a κ-RS-iteration $\bar{Q}^\dagger$ of length α_{ξ^*} such that

(a) $F_4(\bar{Q}, \bar{\alpha})$ is an equivalence of the forcing notions $\mathrm{Rlim}_\kappa \bar{Q}, \mathrm{Rlim}_\kappa \bar{Q}^\dagger$.

(b) $F_3(\bar{Q}\restriction\alpha_\xi, \alpha\restriction(\xi+1)) = F_3(\bar{Q}, \bar{\alpha})\restriction\xi$

(c) $\underset{\sim}{Q}^\dagger_\xi$ is the image by $F_4(\bar{Q}\restriction\alpha_\xi, \bar{\alpha}\restriction(\xi+1))$ of the $P_{\alpha_\xi} = \mathrm{Rlim}_\kappa(\bar{Q}\restriction\alpha_\xi)$-name $F_0(\bar{Q}\restriction\alpha_{\xi+1}, \alpha_\xi)$.

The proof is tedious but straightforward. $\square_{1.5}$

1.6 Claim. Suppose we add in Definition 1.1(B) also:

1.1(B)(4) if α is inaccessible $\geq \kappa$, and for some $\beta < \alpha$ for every γ satisfying $\beta \leq \gamma < \alpha$ we have $\Vdash_{P_\beta}$ "$|P_\gamma / P_\beta| < \alpha$" *then* $(\exists \beta < \alpha)[\mathrm{Dom}(\underset{\sim}{\zeta}) \subseteq P_\beta]$.

Then nothing changes in the above, and if λ is an inaccessible cardinal $> \kappa$ and $|P_i| < \lambda$ for every $i < \lambda$ and $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \lambda \rangle$ is an RS_κ-iteration, then

(1) every $\bar{Q}$-named ordinal is in fact a $(\bar{Q}{\restriction}i)$-named ordinal for some $i < \lambda$,

(2) like (1) for $\bar{Q}$-named conditions.

(3) $P_\lambda = \bigcup_{i<\lambda} P_i$.

(4) if λ is a Mahlo cardinal *then* P_λ satisfies the λ-c.c. (in a strong way).

1.6A Remark. As in XI §1, actually if "$\theta = \mathrm{cf}(\theta) \geq \kappa$" is preserved by every P_α for $\alpha < \alpha^*$, then: $\alpha < \alpha^*$ & $\mathrm{cf}(\alpha) = \theta$ implies $\bigcup_{\beta<\alpha} P_\beta$ is dense in P_α. In this case, if α^* is strongly inaccessible $> \theta$ and $[\alpha < \alpha^* \Rightarrow \mathrm{density}(P_\alpha) < \alpha^*]$ *then* P_{α^*} satisfies the α^*-c.c.

§2. Pseudo-Completeness

We think here of replacing $\aleph_1$ by, say, κ^+. So we want to deal with forcing notions not collapsing any cardinal $\leq \kappa^+$, but possibly collapsing κ^{++}, and possibly adding reals and changing the cofinality of κ^{++} to say $\aleph_0$. So on the one hand we want to have support $\leq \kappa$, and even a κ^+-RS; and on the other hand some amount of pseudo completeness (expressed in Definition 2.1 below). Further consideration lead to finite pure.

We deal with forcing notions Q satisfying:

2.1 Definition. Let γ be an ordinal, $S \subseteq \{2\} \cup \{\lambda : \lambda \text{ a regular cardinal }\}$.

1) Now Q satisfies (S, γ)-Pr_1 if:

(i) $Q = (|Q|, \leq, \leq_0)$ (here $|Q|$ is the set of elements of Q)

(ii) as a forcing notion, Q is $(|Q|, \leq)$, with a least element $\emptyset_Q$

(iii) $\leq_0$ is a partial order (of $|Q|$).

(iv) $[p \leq_0 q \Rightarrow p \leq q]$

(v) *pure decidability: for every* cardinal $\theta \in S$ and Q-name $\underset{\sim}{\tau}$, such that $\Vdash_Q$ "$\underset{\sim}{\tau} \in \theta$" and $p \in Q$ *for some* $q \in Q$ and $\beta \in \theta$ we have: $p \leq_0 q$ and $q \Vdash_Q$ "if $\theta = 2$ then $\underset{\sim}{\tau} = \beta$ and if $\theta \geq \aleph_0$ then $\underset{\sim}{\tau} < \beta$"

(vi) for each $p \in Q$ in the following game player I has a winning strategy: for $i < \gamma$ player I chooses $p_{2i} \in Q$ such that $p \leq_0 p_{2i} \wedge \bigwedge_{j<2i} p_j \leq_0 p_{2i}$ and then player II chooses $p_{2i+1} \in Q$ such that $p_{2i} \leq_0 p_{2i+1}$.

Player I loses if he has some time no legal move, which can occur in limit stages only.

2) $Q = (|Q|, \leq, \leq_0)$ satisfies (S,γ)-Pr_1^+, if (i)-(v) hold and $(Q, \leq_0)$ is γ-complete (i.e. if $p_i \in Q$ for $i < \beta < \gamma$, and $i < j < \beta \Rightarrow p_i \leq_0 p_j$ *then* for some $p \in Q$ we have: $i < \beta \Rightarrow p_i \leq_0 p$).

3) A forcing notion $(Q, \leq)$ satisfies (S,γ)-Pr_1 (or (S,γ)-Pr_1^+), if there is a relation $\leq_0$ such that $(Q, \leq, \leq_0)$ satisfies (S,γ)-Pr_1 (or (S,γ)-Pr_1^+)).

4) Q satisfies $(S,\gamma) - \mathrm{Pr}_1^-$ or $S - \mathrm{Pr}_1^-$ if it satisfies (i) - (v) of part (1) (note: the ordinal γ does not appear in conditions (i)-(v) of 2.1(1)).

5) If a member of S is an infinite ordinal δ which is not a regular cardinal, we mean $\mathrm{cf}(\delta)$ (occurs e.g. when $Q \in V^P$ and $S \in V$).

6) If $Q = (|Q|, \leq, \leq_0)$ then $\hat{Q}$ is defined as follows:

the set of elements is $\{u : u \subseteq Q$, and if $u \neq \emptyset$ then for some $q \in u$, $(\forall p \in u)(\exists r \in u)(p \leq r \;\&\; q \leq_0 r)$ and there is $r^* \in Q$ such that for every such q, r^* is a $\leq_0$-upper bound of $\{r \in u : q \leq_0 r\}\}$,

the order $u_1 \leq u_2$ iff $u_1 = u_2$ or for some $q_2 \in u_2$ for every $q_1 \in u_1$, $q_1 \leq q_2$,

the pure order $u_1 \leq_0 u_2$ iff $u_1 = u_2$ or for some $q_2 \in u_2$ witnessing $u_2 \in \hat{Q}$ for every $q_1 \in u_1$ witnessing $u_1 \in \hat{Q}$ we have $(\forall p \in u_1)(\exists r \in u_1)[p \leq r \,\&\, q_1 \leq_0 r \leq_0 q_2]$ (this is naturally used in 2.7; we usually identify $p \in Q$ with $\{p\} \in \hat{Q}$).

2.2 Fact.

(1) If $\kappa < \gamma_1, \gamma_2 < \kappa^+$ *then* (S,γ_1)-Pr_1 is equivalent to (S,γ_2)-Pr_1.

(2) If $\kappa + 1 \leq \gamma < \kappa^+$ and $\square_\kappa$ (which can be stated as, i.e. an equivalent formulation is: there is a sequence $\langle C_\alpha : \alpha < \kappa^+\rangle, C_\alpha \subseteq \alpha$ closed, for

limit α the set C_α is unbounded in α and $[\alpha_1 \in C_\alpha \Rightarrow C_{\alpha_1} = C_\alpha \cap \alpha_1]$, and $\mathrm{cf}(\alpha) < \kappa \Rightarrow |C_\alpha| < \kappa$) and Q satisfies (S, γ)-Pr_1 *then* Q satisfies (S, κ^+)-Pr_1.

(3) Assume Q satisfies (S, γ)-Pr_1. If $\lambda \leq \gamma$, and $\lambda \in S$ *then* in V^Q still λ is a regular cardinal (or at least $\Vdash_Q$ "$\mathrm{cf}(\lambda) = \mathrm{cf}^V(\lambda)$"). If $2 \in S$, *then* Q does not add bounded subsets to γ.

(4) If Q satisfies (S, γ)-Pr_1, $\lambda \in S$, λ regular, and for every regular μ, $\gamma \leq \mu < \lambda \Rightarrow \Vdash_Q$ "μ is not regular" (e.g., $[\gamma, \lambda)$ contains no regular cardinal) *then* λ is regular in V^Q.

(5) If Q satisfies (S, γ)-Pr_1, $\gamma \geq \omega + 1$, *then* Q is S-semiproper.

(6) Similar assertions (to 1-5) holds for (S, γ)-Pr_1^+ (but in (2) we do not need $\Box_\kappa$) and (S, γ)-Pr_1^+ implies (S, γ)-Pr_1.

(7) In 2.1(6), $\leq^{\hat{Q}}$, $\leq_0^{\hat{Q}}$ are quasi orders of the set of elements of $\hat{Q}$ and for p, $q \in Q$ we have

 (i) $Q \models p \leq q \Leftrightarrow \hat{Q} \models \{p\} \leq \{q\}$,

 (ii) $Q \models p \leq_0 q \Leftrightarrow \hat{Q} \models \{p\} \leq_0 \{q\}$

 (on incompleteness see inside 2.7(D)).

(8) Assume $Q = (|Q|, \leq, \leq_0)$ satisfies:

$$(*) \qquad p \leq q \leq r \ \&\ p \leq_0 r \Rightarrow p \leq_0 q$$

then in Definition 2.1(6):

if q', $q'' \in u_1$ witness $u_1 \in \hat{Q}$ *then* there is $q \in u_1$ such that $q' \leq_0 q$ & $q'' \leq_0 q$.

Also: if $q' \in u_1$ witness $u_1 \in \hat{Q}$ and $q' \leq q \in u_1$ then q witness $u_1 \in \hat{Q}$, provided that

$$(*)' \qquad p \leq q \leq r \ \&\ p \leq_0 r \Rightarrow q \leq_0 r.$$

(9) In definition 2.1(6), if $u_1 \neq u_2$ and q_2 witness $u_1 \leq_0 u_2$ then $u_1 \leq_0 q_2 \leq_0 u_2$ (or more formally $u_1 \leq_0 \{q_2\} \leq_0 u_2$) so $\{\{p\} : p \in Q\} \subseteq Q$ is dense.

Proof: Straightforward. E.g. for (2), note that $\mathrm{otp}(C_\delta) \leq \kappa$ for all $\delta < \kappa^+$. Without loss of generality $\alpha \in C_\delta \Rightarrow \alpha$ is even (so $C_{2\alpha+1} = \emptyset$). So in stages α, player I can apply his strategy to the play $\langle p_\gamma, p_{\gamma+1} : \gamma \in C_\alpha \rangle$). $\square_{2.2}$

2.2A Remark. Concerning 2.2(2) note that $\square_{\aleph_0}$ always holds trivially (see [Sh:351, §4]). The equivalence of this formulation of square to the standard one is similar to the proof in [Sh:351, §4].

2.3 Definition. $(S, < \kappa)$-Pr_1 will mean $(S, \gamma) - \mathrm{Pr}_1$ holds for every $\gamma < \kappa$.

2.4 Fact. The following three conditions on a forcing notion Q, a set $S \subseteq \{2\} \cup \{\lambda : \lambda$ a regular cardinal $\}$ and regular κ are equivalent:

(a) there is $Q' = (Q', \leq, \leq_0)$ such that $(Q', \leq)$, $(Q, \leq)$ are equivalent forcing notions and Q' satisfies (S, κ)-Pr_1.

(b) for each $p \in Q$, in the following game (which lasts κ moves) player II has a winning strategy:

in the ith move player I chooses $\lambda_i \in S$ and a Q-name $\underset{\sim}{\tau}_i$ of an ordinal $< \lambda_i$, then player II chooses an ordinal $\alpha_i < \lambda_i$. In the end player II wins if for every $\alpha < \kappa$ there is $p_\alpha \in Q$, $p \leq p_\alpha$ such that for every $i < \alpha$ we have $p_\alpha \Vdash$ "either $\lambda_i = 2$ & $\underset{\sim}{\tau}_i = \alpha_i$ or $\lambda_i \geq \alpha_0$ & $\underset{\sim}{\tau}_i < \alpha_i$".

(c) like (a) but moreover $(Q', \leq_0)$ is κ-complete (i.e. Q' satisfies $(S, \kappa) - Pr_1^+$).

(d) like (a) but moreover $(Q' \leq_0)$ is κ-directed complete, i.e.

if $B \subseteq Q$, $|B| < \kappa$ and for each finite $B' \subseteq B$ there is a $\leq_0$-upper bound to B', *then* B has a $\leq_0$-least upper bound.

Proof. *(d)* $\Rightarrow$ *(c)* $\Rightarrow$ *(a)*: trivial.

(a) $\Rightarrow$ *(b)*: As Q, Q' are equivalent, there is a forcing notion P and $f : Q \rightarrow P$, $f' : Q' \rightarrow P$ both preserving $\leq$ and incompatibility and with dense ranges. Choose $q \in Q'$ which essentially is above p i.e. $f'(q) \Vdash_P$ "$f(p) \in \underset{\sim}{G}_P$" . We describe a winning strategy (in the game from (b) of 2.4) for player II: he plays on the side a play (for q) of the game from 2.1(vi) for Q' where he uses a winning strategy (whose existence in guaranteed by (a)). In step i of the play

(for 2.4(b)) he already has the initial segment $\langle p_j : j < 2i \rangle$ of the simulated play for 2.1(vi). If player I plays λ_i, $\underset{\sim}{\tau}_i$ in the actual game, player II defines $p_{2i} \in Q'$ (for player I) in the simulated play by the winning strategy of player I there and then he chooses $p_{2i+1}, p_{2i} \le_0 p_{2i+1} \in Q'$, which force for some α_i: $\underset{\sim}{\tau}_i = \alpha_i$ if $\lambda_i = 2$, $\underset{\sim}{\tau}_i < \alpha_i$ if $\lambda_i \ge \aleph_0$ (exists by 2.1(v)) (more formally, for some $r_i \in Q$, we have $f'(p_{2i+1}) \Vdash_P$ "$f(r_i) \in \underset{\sim}{G}_P$" and r_i forces $\underset{\sim}{\tau}_i = \alpha_i$ & $\lambda_i = 2$ or $\underset{\sim}{\tau}_i < \alpha_i$ & $\lambda_i \ge \aleph_0$, alternatively we can interpret $\underset{\sim}{\tau}_i$ as a Q'-name using f, f') and then plays α_i in the actual play. In the end for $\alpha < \kappa$, there is $p^*_\alpha \in Q$ such that $f(p^*_{2\alpha}) \Vdash_P$ "$f'(p_{2\alpha}) \in \underset{\sim}{G}_P$", now p^*_α exists and is as required.

(b) $\Rightarrow$ *(d)*: Fix a winning strategy St_p for player II in the game from 2.4(b) for each $p \in Q$. We define Q' as follows

$$Q' = \{(p, \langle \lambda_i, \underset{\sim}{\tau}_i, \alpha_i : i < \xi \rangle) : p \in Q, \text{ and } \langle \lambda_i, \underset{\sim}{\tau}_i, \alpha_i : i < \xi \rangle \text{ is an initial segment of a play of the game from 2.4(b) for } p \text{ in which player II uses his winning strategy } \mathrm{St}_p\}.$$

The order $\le_0$ is:

$$(p, \langle \lambda_i, \underset{\sim}{\tau}_i, \alpha_i : i < \xi \rangle) \le_0 (p', \langle \lambda'_i, \underset{\sim}{\tau}'_i, \alpha'_i : i < \xi' \rangle)$$

iff (both are in Q') and

$$p = p', \ \xi \le \xi', \text{ and for } i < \xi$$
$$\lambda_i = \lambda'_i, \underset{\sim}{\tau}_i = \underset{\sim}{\tau}'_i, \ \alpha_i = \alpha'_i$$

and the order $\le$ on Q' is

$$\mathbf{p} = (p, \langle \lambda_i, \underset{\sim}{\tau}_i, \alpha_i : i < \xi \rangle) \le \mathbf{p}' = (p', \langle \lambda'_i, \underset{\sim}{\tau}'_i, \alpha'_i : i < \xi' \rangle)$$

iff (both are in Q' and) $\mathbf{p} \le_0 \mathbf{p}'$ or $Q \models p \le p'$, and $p' \Vdash_Q$ "$\lambda_i = 2$ & $\underset{\sim}{\tau}_i = \alpha_i$ or $\lambda_i \ge \aleph_0$ & $\underset{\sim}{\tau}_i < \alpha_i$" for every $i < \xi$.

The checking is easy. Note that

(α) the map $p \mapsto (p, <>)$ is a dense embedding of $(Q, \le)$ into $(Q', \le)$.

(β) hence Q'-names are essentially Q-names,

(γ) $(p, \langle \lambda_i, \underset{\sim}{\tau}_i, \alpha_i : i < \xi \rangle) \Vdash_{Q'}$ "$(\forall i < \xi)[(\lambda_i = 2 \rightarrow \underset{\sim}{\tau}_i = \alpha_i) \,\&\, (\lambda_i \geq \aleph_0 \rightarrow \underset{\sim}{\tau}_i < \alpha_i)]$".

(δ) for (d) note that every $\leq_0$-directed set is linearly ordered by $\leq_0$ and if its cardinality is $< \kappa$ then it has a $\leq_0$-lub. $\square_{2.4}$

2.4A Remark . So $(S, \kappa) - \mathrm{Pr}_1$ and $(S, \kappa) - \mathrm{Pr}_1^+$ are "essentially" the same (for κ regular).

2.5 Definition.

(1) Assume $\bar{P}$ is a $\lessdot$-increasing sequence of forcing notions.

(a) Let

$$\begin{aligned} \mathrm{Gen}^r(\bar{P}) \stackrel{\text{def}}{=} \{G : & \text{ for some (set) forcing notion } P^* : \bigwedge_{i<\alpha} P_i \lessdot P^* \\ & \text{and } G^* \subseteq P^* \text{ generic over } V \\ & \text{and } G = G^* \cap \bigcup_{i<\alpha} P_i\}. \end{aligned}$$

(b) For a set E of regular cardinals we say that $\bar{P}$ obeys E if for $\gamma \in E$ we have: $P_\alpha = \bigcup\limits_{\beta<\gamma} P_\beta$ and P_β satisfies the γ-c.c. for $\beta < \gamma$. We say that $\bar{P}$ strongly obey E if in addition $\beta < \gamma \in E \Rightarrow |P_\beta| < \gamma$.

(c) Let $E(\bar{P}) = \{\gamma \leq \ell g(\bar{P}) : \gamma$ is strongly inaccessible , uncountable and $\beta < \gamma \Rightarrow P_\gamma$ satisfies the γ-c.c.$\}$, $E_s(\bar{P}) = \{\gamma \in E(\bar{P}) : \beta < \gamma \Rightarrow |P_\beta| < \gamma\}$

(2) If $\bar{Q} = \langle P_i : i < \alpha \rangle$ or $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ where P_i is $\lessdot$-increasing, obeying E (so here we ignore the $\underset{\sim}{Q}_i$'s) we define a $\bar{Q}$-E-name $\underset{\sim}{\tau}$ almost as we define $\left(\bigcup_{i<\alpha} P_i\right)$-names, but we do not use *maximal* antichains of $\bigcup_{i<\alpha} P_i$:

(*) $\underset{\sim}{\tau}$ is a function, $\mathrm{Dom}(\underset{\sim}{\tau}) \subseteq \bigcup_{i<\alpha} P_i$ and for every directed $G \in \mathrm{Gen}^r(\bar{Q})$, $\underset{\sim}{\tau}[G]$ is defined iff $\mathrm{Dom}(\underset{\sim}{\tau}) \cap G \neq \emptyset$ and then $\underset{\sim}{\tau}[G] \in V[G]$ [from where "every $G\ldots$" is taken? e.g., V is countable, G any set from the true universe] and $\underset{\sim}{\tau}$ is definable with parameters from $V \cup \{G\}$ (so

$\underset{\sim}{\tau}$ is really a first-order formula with the variable G and parameters from V) and $(\forall \beta \in E \cap E(\bar{Q}))(\exists \gamma < \beta)[\mathrm{Dom}(\underset{\sim}{\tau}) \cap P_\beta \subseteq P_\gamma]$.

Now $\Vdash_{\bar{Q}}$ has a natural meaning. If E is not mentioned we mean: any fixed $E \cap (\ell g(\bar{Q}) + 1) = E(\bar{Q})$ understood from the context, normally just $E(\langle P_i : i < \alpha \rangle)$, below we fix $E = \mathbf{E}^*$.

(3) For $\bar{Q}$-names $\underset{\sim}{\tau}_0, \dots, \underset{\sim}{\tau}_{n-1}$ we let $\{\underset{\sim}{\tau}_0, \dots, \underset{\sim}{\tau}_{n-1}\}$ be the name for the set that contains exactly those $\underset{\sim}{\tau}_i[\bar{Q}]$ that are defined. For $p \in \bar{Q}$ (i.e., $p \in \bigcup_{i<\alpha} P_i$) we let $p \Vdash$ "$\underset{\sim}{\tau} = x$" if for every G such that $p \in G \in \mathrm{Gen}^r(\bar{Q})$ we have $\underset{\sim}{\tau}[G] = x$. (But see 2.6(2).)

(4) A $\bar{Q}$-E-named $[j, \beta)$-ordinal $\underset{\sim}{\zeta}$ is a $\bar{Q}$-E-name $\underset{\sim}{\zeta}$ such that if $\underset{\sim}{\zeta}[G] = \xi$ then $j \leq \xi < \beta$ and $(\exists p \in G \cap P_{\xi \cap \alpha}) p \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} = \xi$" (where $\alpha = \ell g(\bar{Q})$). If we omit "$[j, \beta)$" we mean $[0, \ell g((\bar{Q})) = [0, \alpha)$.

2.5A Remark. 1) We can restrict in the definition of $\mathrm{Gen}^r(Q)$ to P^* in some class K, and get a K-variant of our notions.

2) Note: even if in 2.5(1) we ask $\mathrm{Dom}(\underset{\sim}{\tau})$ to be a maximal antichain it will not be meaningful as in the appropriate P_δ, we have $\bigwedge_{i<\delta} P_i \lessdot P_\delta$ but it will not in general be a maximal antichain.

2.5B Remark. Note that we wrote $P_{\xi \cap \alpha}$ not $P_{(\xi+1) \cap \alpha}$. Compare this to the remark 1.1B. We will not have a general associativity law, but the definition of $Sp_2 - \mathrm{Lim}_\kappa \bar{Q}$ will be slightly simplified. As said earlier we can interchange decisions on this matter (this does not mean this is the same iteration, just that it has the same relevant properties). Of course also Ch.X can be represented with this iteration.

2.5C Remark. Note that a $\bar{Q} - \emptyset$-named ordinal $\underset{\sim}{\zeta}$ is $\bar{Q} - E^*$-named ordinal *iff* for every $\beta \in E^* \cap E(\bar{Q})$ for some $\gamma < \beta$ we have $\Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} \notin [\gamma, \beta)$".

2.6 Fact.

(1) For $\bar{P} = \langle P_i : i < \ell g \bar{P} \rangle$, a $\lessdot$ increasing sequence of forcing notions and $\bar{P}$-named $[j, \beta)$-ordinal $\underset{\sim}{\zeta}$ and $p \in \bigcup_{i<\alpha} P_i$ there are ξ, q and q_1 such that $p \leq q \in \bigcup_{i<\ell g \bar{P}} P_i$ and: either $q \Vdash_{\bar{P}}$ "$q_1 \in \underset{\sim}{G}$", $q_1 \in P_\xi$, $\xi < \alpha$, $[p \in P_\xi \Rightarrow q = q_1]$ and $q_1 \Vdash_{\bar{P}}$ "$\underset{\sim}{\zeta} = \xi$" or $q \Vdash_{\bar{P}}$ "$\underset{\sim}{\zeta}$ is not defined" (and even $p \Vdash_{\bar{P}}$ "$\underset{\sim}{\zeta}$ is not defined").

(2) For $\bar{P}$ as above, and $\bar{P}$-named $[j, \beta)$-ordinals $\underset{\sim}{\zeta}, \underset{\sim}{\xi}$, also $\text{Min}\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}$, $\text{Max}\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}$ (naturally defined, so $\text{Max}\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}[G]$ is defined iff a $\underset{\sim}{\zeta}[G]$, $\underset{\sim}{\xi}[G]$ are defined, and $\text{Min}\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}[G]$ is defined iff $\underset{\sim}{\zeta}[G]$ is defined or $\underset{\sim}{\xi}[G]$ is defined); both are $\bar{P}$-named $[j, \beta)$-ordinals.
Similarly for $\text{Min}\{\underset{\sim}{\xi}_0, \ldots, \underset{\sim}{\xi}_{n-1}\}$, $\text{Max}\{\underset{\sim}{\xi}_0, \ldots, \underset{\sim}{\xi}_{n-1}\}$ for $\bar{P} - E$-named ordinals.

(3) For $\bar{P}$ as above, $n < \omega$ and $\bar{P}$-named ordinals $\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n$ and $p \in \bigcup_{i<\ell g(\bar{P})} P_i$ there are $\zeta < \alpha$ and $q \in P_\zeta$ such that, first: $p \leq q$ or at least $q \Vdash_{P_\zeta}$ "$p \in P_i / \underset{\sim}{G}_{P_\zeta}$ for some $i < \ell g(\bar{P})$" (actually $i = \min\{i : p \in P_i\}$) and second: for some $\ell \in \{1, \ldots, n\}$ we have $q \restriction \zeta \Vdash_{\bar{P}}$ "$\zeta = \underset{\sim}{\xi}_\ell = \text{Max}\{\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n\}$" or $p \Vdash_{\bar{P}}$ "$\text{Max}\{\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n\}$ not defined", in the second case we can add $q \Vdash_{\bar{P}}$ "$\underset{\sim}{\xi}_\ell$ not defined". Similarly for Min.

(4) *Convention:* If $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$, P_i is $\lessdot$-increasing, we may write $\bar{Q}$ instead of $\langle P_i : i < \alpha \rangle$.

2.6A Convention. $\mathbf{E}^*$ is a class of strongly inaccessible cardinals $> \kappa$ fixed for this section, not mentioned usually. So a $\bar{P}$-named (e.g. ordinal) mean a $\bar{P}$-$(\mathbf{E} \cap E(\bar{P}))$-named (e.g. ordinal). Outside this section the default value is the class of strongly inaccessible $> \kappa$.
[The reader can simplify life using $\mathbf{E}^* = \emptyset$, he will lose only 2.7(4), hence case II of 3.4, so this is a reasonable choice.]

2.7 Definition and Claim. Let $e \in \{1, 2\}$. We define and prove by induction on α the following simultaneously (all forcing notions satisfying 2.1 (i)- (iv)):

(A) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a $\kappa - \mathrm{Sp}_e$-iteration or really $\kappa - \mathrm{Sp}_e - \mathbf{E}^*$-iteration (the $\bar{Q}$'s below will have this form).

(B) A $\bar{Q}$-named (that is $\bar{Q}$-$\mathbf{E}^*$-named) atomic condition $\underset{\sim}{q}$ (or atomic $[j, \beta)$-condition, $\beta \leq \alpha$) and we define $\underset{\sim}{q} \restriction \xi$, $\underset{\sim}{q} \restriction \{\xi\}$, $\underset{\sim}{q} \restriction [\xi, \zeta)$ for a $\bar{Q}$-named atomic condition $\underset{\sim}{q}$ and ordinal $\xi \leq \zeta \leq \alpha$ (or $\bar{Q}$-named ordinals $\underset{\sim}{\xi}, \underset{\sim}{\zeta}$ instead ξ, ζ).

(C) If $\underset{\sim}{q}$ is a $\bar{Q}$-named (or really $\bar{Q}$-$\mathbf{E}^*$-named) atomic $[j, \beta)$-condition, $\xi < \alpha$, *then* $\underset{\sim}{q} \restriction \xi$ is a $(\bar{Q} \restriction \xi)$-named atomic $[j, \text{Min}\ \{\beta, \xi\})$-condition and $\underset{\sim}{q} \restriction \{\xi\}$ is a P_ξ-name of a member of $\underset{\sim}{Q}_\xi$ or undefined (and then it may be assigned the value $\emptyset_{\underset{\sim}{Q}_\xi}$, the minimal member of $\underset{\sim}{Q}_\xi$).

(D) The $\kappa - \mathrm{Sp}_e$-limit of $\bar{Q}$, $\mathrm{Sp}_e - \mathrm{Lim}_\kappa \bar{Q}$, (really $\mathrm{Sp}_e - \mathbf{E}^* - \mathrm{Lim}_\kappa \bar{Q}$) denoted by P_α for $\bar{Q}$ as in clause (A), and $p \restriction \xi$ and $\mathrm{Dom}(p)$ for $p \in \mathrm{Sp}_e - \mathrm{Lim}_\kappa \bar{Q}$, ξ an ordinal $\leq \alpha$ (or $\bar{Q}$-named ordinal $\underset{\sim}{\xi}$ etc.).

(E) $\mathrm{Sp}_e - \mathrm{Lim}_\kappa \bar{Q}$ satisfies (i)-(iv) of Definition 1.2 and it obeys $\mathbf{E}^* \cap E(\bar{Q})$ (so if $\ell g(\bar{Q}) \in \mathbf{E}^* \cap E(\bar{Q})$ then $\mathrm{Sp}_e - \mathrm{Lim}_\kappa(\bar{Q}) = \bigcup\limits_{\beta < lg(\bar{Q})} P_\beta$). Also if $\beta \leq \alpha$, $\beta \in \mathbf{E}^* \cap E(Q)$ and $\underset{\sim}{\zeta}$ is a $(\bar{Q} \restriction \beta)$-named ordinal then it is a $(\bar{Q} \restriction \gamma)$-named ordinal for some $\gamma < \beta$; similarly for atomic condition.

(F) If $\beta < \alpha = \ell g(\bar{Q})$ then $P_\beta \subseteq \mathrm{Sp}_e - \mathrm{Lim}_\kappa \bar{Q}$ (as models with two partial orders, even compatibility is preserved) and $[p \in P_\beta \Rightarrow p \restriction \beta = p]$ and $[P_\alpha \vDash \text{“}p \leq q\text{”} \Rightarrow P_\beta \vDash \text{“}p \restriction \beta \leq q \restriction \beta\text{”}]$ and $[P_\alpha \vDash \text{“}p \leq_0 q\text{”} \Rightarrow P_\beta \vDash \text{“}p \restriction \beta \leq_0 q \restriction \beta\text{”}]$ and $P_\alpha \vDash \text{“}p \restriction \beta \leq p\text{”}$. Also $q \in P_\beta, p \in \mathrm{Sp}_e - \mathrm{Lim}_\kappa \bar{Q}$ are compatible iff $q, p \restriction \beta$ are compatible in P_β. In fact if $q \in P_\beta$, $P_\beta \models \text{“}p \restriction \beta \leq q\text{”}$ then $q \cup (p \restriction [\beta, \alpha)$ is a least upper bound of p, q, and if $P_\beta \models \text{“}p \restriction \beta \leq_0 q\text{”}$ even a $\leq_0$-least upper bound of q. Hence $P_\beta \lessdot (\kappa - \mathrm{Sp}_e - \mathrm{Lim}_\kappa(\bar{Q}))$ and so $\beta < \gamma < \ell g(\bar{Q}) \Rightarrow P_\beta \lessdot P_\gamma$.

(G) The set of $p \in P_\alpha$ such that for every $\beta < \alpha$ we have $\Vdash_{P_\beta}$ “$p \restriction \{\beta\}$ is a singleton or empty”, is a dense subset of P_α. Also we can replace $\underset{\sim}{Q}_\beta$ by $\hat{Q}_\beta$ (see Definition 2.1(6)) and the set of “old” $p \in P_\alpha$ is a dense subset of the new (but actually do not use this).

Proof and Definition.

(A) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a $\kappa - \mathrm{Sp}_e$-iteration if $\bar{Q}{\restriction}\beta$ is a $\kappa - \mathrm{Sp}_e$-iteration for $\beta < \alpha$, and if $\alpha = \beta + 1$ then $P_\beta = \mathrm{Sp}_e\text{-}\mathrm{Lim}_\kappa(\bar{Q}{\restriction}\beta)$ and $\underset{\sim}{Q}_\beta$ is a P_β-name of a forcing notion as in Definition 2.1(1)(i)-(iv).

(B) We say $\underset{\sim}{q}$ is a $\bar{Q}$-named atomic $[j, \beta)$-condition when : $\underset{\sim}{q}$ is a $\bar{Q}$-name (i.e. a $\bar{Q} - \mathbf{E}^*$-name), and for some $\underset{\sim}{\zeta} = \underset{\sim}{\zeta}_q$, a $\bar{Q}$-named $[j, \beta)$-ordinal (i.e. a $\bar{Q} - \mathbf{E}^*$-named $[j, \beta)$-ordinal), we have $\Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta}$ has a value iff $\underset{\sim}{q}$ has, and if they have then $j \le \underset{\sim}{\zeta} < \mathrm{Min}\{\beta, \ell g(\bar{Q})\}$ and $\underset{\sim}{q} \in \underset{\sim}{Q}_{\underset{\sim}{\zeta}}$". Now $\underset{\sim}{q}{\restriction}\xi$ will have a value iff $\underset{\sim}{\zeta}_q$ has a value $< \xi$ and then its value is the value of $\underset{\sim}{q}$. Lastly, $\underset{\sim}{q}{\restriction}\{\xi\}$ will have a value iff $\underset{\sim}{\zeta}_q$ has the value ξ and then its value is the value of $\underset{\sim}{q}$ (similarly for $\underset{\sim}{\xi}$ and $\underset{\sim}{q}{\restriction}[\zeta, \xi)$ and $\underset{\sim}{q}{\restriction}[\underset{\sim}{\zeta}, \underset{\sim}{\xi})$).

(C) Left to the reader.

(D) We are defining $\mathrm{Sp}_e - \mathrm{Lim}_\kappa \bar{Q}$ (where $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ of course). It is a triple $P_\alpha = (|P_\alpha|, \le, \le_0)$ where

(a) $|P_\alpha|$ is the set of $p = \{\underset{\sim}{q}_i : i < i^*\}$ satisfying:

  (i) $i^* < \kappa$,

  (ii) if $e = 1$, $\emptyset \le p$ (see below)

  (iii) each $\underset{\sim}{q}_i$ is a $\bar{Q}$-named atomic condition, and for every $\xi < \alpha$, $\Vdash_{P_\xi}$ "$p{\restriction}\{\xi\} \overset{\mathrm{def}}{=} \{q_i{\restriction}\{\xi\} : i < i^*\}$ if not empty, has a $\le_0$-upper bound in $\underset{\sim}{Q}_\xi$ or at least a weak $\le_0$-upper bound i.e. for some nonempty $u \subseteq i^*$ and $r \in \underset{\sim}{Q}_\xi$ we have $\bigwedge_{i<i^*} \bigvee_{j\in u} q_i{\restriction}\{\xi\} \le q_j{\restriction}\{\xi\}$ and $\bigwedge_{j\in u} q_i{\restriction}\{\xi\} \le_0 r$ and $\bigvee_{i\in u} \bigwedge_{j\in u} q_i{\restriction}\{\xi\} \le_0 q_j{\restriction}\{\xi\}$ (i.e. $q{\restriction}\{\xi\} \in \underset{\sim}{\hat{Q}}_\xi$)".

(b) for $p \in \mathrm{Sp}_e - \mathrm{Lim}_\kappa(\bar{Q})$ and $\xi < \ell g(\bar{Q})$ we let:
$p{\restriction}\xi \overset{\mathrm{def}}{=} \{r{\restriction}\xi : r \in p\}$
$p{\restriction}\{\xi\} \overset{\mathrm{def}}{=} \{r{\restriction}\{\xi\} : r \in p\}$,
we define similarly $p{\restriction}[\zeta, \xi)$, $p{\restriction}\{\underset{\sim}{\zeta}\}$, $p{\restriction}[\underset{\sim}{\zeta}, \underset{\sim}{\xi})$.

(c) $P_\alpha \models$ "$p^1 \le_0 p^2$" iff for every $\xi < \alpha$ we have (letting $p^\ell = \{q_i^\ell : i < i^\ell(*)\}$ for $\ell = 1, 2$):
$\{q_i^2{\restriction}\xi : i < i^2(*)\} \Vdash_{P_\xi}$ "$p^2{\restriction}\{\xi\} = \emptyset \Rightarrow p^1{\restriction}\{\xi\} = \emptyset$ and one of the following holds:

(i) $\{q_i^\ell \restriction \{\xi\}: i < i^\ell(*)\}$ are equal for $\ell = 1, 2$,

(ii) letting u^1, u^2 be as in clause (a)(iii) above for $q^1 \restriction \{\xi\}$, $q^2 \restriction \{\xi\}$ respectively for some $j_2 \in u^2$ for all $j_1 < i^1(*)$ if $\zeta_{\underset{\sim}{q}^1_{j_1}} = \xi$ and $j_1 \in u^1$ then $\hat{Q}_\zeta \vDash [q^1_{j_1} \restriction \{\xi\} \leq_0 q^2_{j_2} \restriction \{\xi\}]$
(note (i)$\vee$(ii) means $\underset{\sim}{\hat{Q}}_\xi \vDash p^1 \restriction \{\xi\} \leq_0 p^2 \restriction \{\xi\}$)

(iii) $e = 2$ and $p^1 \restriction \{\xi\} = \emptyset$".

(d) $P_\alpha \models p^1 \leq p^2$ iff

(i) for every $\xi < \ell g(\bar{Q})$ we have (letting $p^\ell = \{q_i^\ell : i < i^\ell(*)\}$, $\ell = 1, 2$): $\{q_i^2 \restriction \xi : i < i^2(*)\} \Vdash_{P_\xi}$ "$p^2 \restriction \{\xi\} = \emptyset \Rightarrow p^1 \restriction \{\xi\} = \emptyset$ and one of the following occurs: $p^1 \restriction \{\xi\}, p^2 \restriction \{\xi\}$ are equal as subsets of $\underset{\sim}{Q}_\xi$, or for some $j_2 < i^2(*)$ for all $j_1 < i^1(*)$ we have $Q_\xi \vDash$ "$[q^1_{j_1} \leq q^2_{j_2}]$" (i.e. the order of $\underset{\sim}{\hat{Q}}_\xi$).

(ii) for some $n < \omega$ and $\bar{Q}$-named ordinals $\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n$ we have:
for each $\zeta < \ell g(\bar{Q})$, $p_2 \restriction \zeta \Vdash_{P_\zeta}$ "if $\zeta \notin \{\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n\}$ then: $p^1 \restriction \{\zeta\} = \emptyset$ and $e = 2$ or $\underset{\sim}{\hat{Q}}_\xi \vDash$ "$[\{r[\underset{\sim}{G}_{P_\zeta}] : r \in p_1, \underset{\sim}{\zeta}_r = \zeta\} \leq_0 \{r[\underset{\sim}{G}_{P_\zeta}] : r \in p_2, \underset{\sim}{\zeta}_r = \zeta\}]$", note that the truth value of $\zeta = \underset{\sim}{\xi}_\ell$ is a P_ζ-name so this is well defined. Note: $p^1 \restriction \{\zeta\} = \emptyset$ not just $= \emptyset_{Q_i}$ but $\underset{\sim}{q} \in p^1 \Rightarrow \neg[\xi = \underset{\sim}{\zeta}_{\underset{\sim}{q}}[G_\zeta]]$.

We then (i.e. if (i)+(ii)) say: $p_1 \leq p_2$ over $\{\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n\}$.

Lastly (as said above) if $p \in \kappa - \mathrm{Sp}_e - \mathrm{Lim}_\kappa \bar{Q}$ then we let $p \restriction \xi \overset{\text{def}}{=} \{r \restriction \xi : r \in P\}$ and $\mathrm{Dom}(p) = \{\underset{\sim}{\zeta}_{\underset{\sim}{q}} : \underset{\sim}{q} \in p\}$ and similarly $p \restriction \zeta$, $p \restriction [\zeta, \xi)$, $p \restriction [\underset{\sim}{\zeta}, \underset{\sim}{\xi})$.

(E) : Let us check Definition 2.1 (1)(i)-(iv) for $P_\alpha \overset{\text{def}}{=} \mathrm{Sp}_e - \mathrm{Lim}_\kappa \bar{Q}$:

$\leq^{P_\alpha}$ is a partial order: Suppose $p_0 \leq p_1 \leq p_2$. Let n^ℓ, $\underset{\sim}{\xi}_1^\ell, \ldots, \underset{\sim}{\xi}_{n^\ell}^\ell$ appear in the definition of $p_\ell \leq p_{\ell+1}$. Let $n = n^0 + n^1$, and

$$\underset{\sim}{\xi}_i = \begin{cases} \underset{\sim}{\xi}_i^0 & \text{if } 1 \leq i \leq n^1 \\ \underset{\sim}{\xi}_{i-n}^1 & \text{if } n^1 < i \leq n^1 + n^2. \end{cases}$$

Now for $\ell = 0, 1$ and $\xi < \alpha$ we have $\Vdash_{P_\xi}$ "if $p_{\ell+1} \restriction \xi$ is in the set $\underset{\sim}{G}_{\bar{Q}}$ then $p_\ell \restriction \{\xi\} \leq p_{\ell+1} \restriction \{\xi\}$ in $\underset{\sim}{\hat{Q}}_\xi$ ", hence $\Vdash_{P_\xi}$ "if $p_2 \restriction \xi$ is in the set $\underset{\sim}{G}_\xi$ then $p_0 \restriction \{\xi\} \leq p_2 \restriction \{\xi\}$ in $\underset{\sim}{\hat{Q}}_\xi$ ".

Also for $\zeta < \alpha$ we have $p_2{\restriction}\zeta \Vdash_{P_\zeta}$ "if $\zeta \notin \{\xi_1, \dots, \xi_n\}$ then (in $\underset{\sim}{\hat{Q}}_\zeta$) $p_0{\restriction}\{\zeta\} \leq_0 p_1{\restriction}\{\zeta\} \leq_0 p_2{\restriction}\{\zeta\}$" or $e = 2$, $p_0{\restriction}\{\zeta\} = \emptyset$". So we finish.

$\leq_0$ is a partial order: check.

$p \leq_0 q \Rightarrow p \leq q$: By the definition; easy.

So in Definition 2.1, (i), (ii), (iii), and (iv) hold.

We still have to check that $\bar{Q}$ obeys $\mathbf{E}^* \cap E(\bar{Q})$, now by the induction hypothesis the only thing to check is: if $\alpha = \ell g(\bar{Q}) = \beta + 1$, $\beta \in \mathbf{E}^* \cap E(Q)$ then $P_\beta = \bigcup_{\gamma<\beta} P_\gamma$. This follows as $\beta \geq \kappa$, and each $(\bar{Q}{\restriction}\beta)$-named ordinal is a $(\bar{Q}{\restriction}\gamma)$-named ordinal for some γ. This is true by the definition of a $\bar{Q}-\mathbf{E}$-named ordinal.

(F) , (G) We leave the checking to the reader (for the first sentence of (G) see 2.10(1) below). $\square_{2.7}$

2.8 Claim. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a $\kappa - \mathrm{Sp}_e$-iteration (so $P_\alpha = \mathrm{Sp}_e{-}\mathrm{Lim}_\kappa(\bar{Q})$).

1) If $p \leq q$ in P_α *then* there are $r, n, \xi_1 < \ldots < \xi_n < \alpha$ such that:

 (a) $r \in P_\alpha$

 (b) $q \leq r$

 (c) $p \leq r$ above $\{\xi_1, \dots, \xi_n\}$.

2) We can find such r simultaneously for finitely many $p_k \leq q$.

Remark. In fact we can have $r{\restriction}[\xi_n, \alpha) = q{\restriction}[\xi_n, \alpha)$.

Proof. 1) We prove this by induction on α

Case 1: $\alpha = 0$. Trivial.

Case 2: $\alpha = \beta + 1$.

Apply the induction hypothesis to $\bar{Q}{\restriction}\beta, p{\restriction}\beta, q{\restriction}\beta$ (clearly $\bar{Q}{\restriction}\beta$ is an $\kappa{-}Sp_e$-iteration, $p{\restriction}\beta = P_\beta$, $q{\restriction}\beta \in P_\beta$ and $P_\beta \models$ "$p \leq q$", by 2.7).

So we can find $r', m, \{\xi'_1, \dots, \xi'_m\}$ such that:

(a)′ $r' \in P_\beta$

(b)′ $P_\beta \models q{\restriction}\beta \leq r'$

(c)′ $p \leq r'$ (in P_β) above $\{\xi_1, \ldots, \xi_m\}$.

Let $n \stackrel{\text{def}}{=} m+1$, and

$$\xi_\ell = \begin{cases} \xi'_\ell & \textit{if } \ell \in \{1\ldots, m\} \\ \beta & \textit{if } \ell = n \end{cases}$$

and lastly $r = r \cup (q\restriction\{\beta\})$.

Case 3: α is a limit ordinal.

Let $p \leq q$ (in P_α) above $\{\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n\}$. We choose by induction on $\ell \leq n$, $r_\ell, \beta_\ell, \xi^*_\ell$ such that

(α) $r_\ell \in P_{\beta_\ell}$,

(β) $r_\ell \leq r_{\ell+1}$

(γ) $q\restriction\beta_\ell \leq r_\ell$

(δ) $\beta_\ell \leq \beta_{\ell+1} < \alpha$

(ε) $\beta_0 = 0$, $r_0 = \emptyset_{P_0}$

(ζ) for $\ell \in \{0, \ldots, n-1\}$ we have: either $r_{\ell+1} \Vdash$ "$\underset{\sim}{\xi}_{\ell+1} = \xi^*_{\ell+1}$" and $\xi^*_{\ell+1} \leq \beta_{\ell+1}$ *or* $\beta_{\ell+1} = \beta_\ell \,\&\, r_{\ell+1} = r_\ell$ and $r_\ell \cup (q\restriction[\beta, \alpha)) \Vdash_{P_\alpha}$ "$\underset{\sim}{\xi}_{\ell+1}$ is not defined".

Carrying the definition is straight: for $i = 0$ use clause (ε). For $\ell + 1 \leq n$ when the second possibility of clause (ζ) fails there is r', such that $r_\ell \cup (q\restriction[\beta, \alpha)) \leq r' \in P_\alpha$, and $r' \Vdash_{P_\alpha}$ "$\underset{\sim}{\xi}_{\ell+1}$ is defined", so there are r'', $\xi^*_{\ell+1}$ such that $r' \leq r'' \in P_\alpha$ and $r'' \Vdash$ "$\underset{\sim}{\xi}_{\ell+1} = \xi^*_{\ell+1}$" so as "$\xi^*_{\ell+1}$ is a $\bar{Q}$-named ordinal" we know that $\xi^*_{\ell+1} < \alpha$ and $r''\restriction\xi^*_{\ell+1} \Vdash_{P_{\xi^*_{\ell+1}}}$ "$\underset{\sim}{\xi}_{\ell+1} = \xi^*_{\ell+1}$". Let $\beta_{\ell+1} \stackrel{\text{def}}{=} \max\{\beta_\ell, \xi^*_{\ell+1}\}$, and $r_{\ell+1} \stackrel{\text{def}}{=} r''\restriction\beta_{\ell+1}$. So we have carried the induction.

Apply the induction hypothesis to $\bar{Q}\restriction\beta_n$ $p\restriction\beta_n$, r_n; it is applicable as $\beta_n < \alpha$, and $P_{\beta_n} \models$ "$p\restriction\beta_n \leq q\restriction\beta_n \leq r_n$". So there are $m < \omega$, $\xi_1 < \ldots < \xi_m < \beta_n$ and r^* such that $P_{\beta_n} \models r_n \leq r^*$" and $p \leq r^*$ (in P_{β_n}) above $\{\xi_1, \ldots, \xi_m\}$. Now let $r \stackrel{\text{def}}{=} r^* \cup (q\restriction[\beta_n, \alpha))$, clearly $q \leq r$ and $p \leq r$ above $\{\xi_1, \ldots, \xi_m, \beta_n\}$.

2) Should be clear. $\square_{2.8}$

2.8A Claim. Let $\bar{Q}$ be a $\kappa - \mathrm{Sp}_2$-iteration (of length α).

(1) If $\beta < \alpha$ and $\underset{\sim}{\zeta}$ is a P_β-name of a $\bar{Q}$-named $[\beta, \alpha)$-ordinal *then* for some $\bar{Q}$-named $[\beta, \alpha)$-ordinal $\underset{\sim}{\xi}$

$$\Vdash_{\bar{Q}} \text{“}\underset{\sim}{\zeta} = \underset{\sim}{\xi}\text{”}$$

(2) The same holds if we replace "ordinal" by "atomic condition".

(3) If $\underset{\sim}{\beta}$ is a $\bar{Q}$-named ordinal, and for each $\beta < \alpha$, $\underset{\sim}{\zeta}_\beta$ is a $\bar{Q}$-named $[\beta, \alpha)$-ordinal then for some $\bar{Q}$-named ordinal $\underset{\sim}{\xi}$

$$\Vdash_{\bar{Q}} \text{“if } \underset{\sim}{\beta}[\underset{\sim}{G}] = \beta \text{ then } \underset{\sim}{\xi}[\underset{\sim}{G}] = \zeta_\beta[\underset{\sim}{G}]\text{”}$$

(4) Similarly for atomic conditions.

Proof. Easy. $\square_{2.8A}$

2.8B Discussion. Why do we use iteration of kind Sp_2 when Sp_1 may seem simpler? Think that we want say κ, κ^+ to play the roles of $\aleph_0$, $\aleph_1$ in Ch.X. Suppose $\langle P_i, \underset{\sim}{Q}_i : i < \kappa^+\rangle$ is an $\kappa^+ - \mathrm{Sp}_1$-iteration which is nice enough such that $\bigcup_{i<\kappa^+} P_i$ is a dense subset of P_{κ^+}. Suppose further that for $i < \kappa^+$, we have $\{p_i\} \in P_\alpha$ such that: $\Vdash_{P_i}$ "$p_i \in \underset{\sim}{Q}_i$, and for every q such that $q \in \underset{\sim}{Q}_i$, $\emptyset_{Q_i} \leq_0 q$ there is r such that $q \leq_0 r \in \underset{\sim}{Q}_i$ and r is incompatible with p_i (in $\underset{\sim}{Q}_i$)". These are reasonable assumptions for the iterations we have in mind.

Let $\underset{\sim}{u} = \{i < \kappa^+ : \{p_i\} \in \underset{\sim}{G}_{P_\alpha}\}$, so this is a P_α-name of a subset of κ^+. As $\bigcup_{i<\kappa^+} P_i$ is a dense subset of κ^+ clearly $\Vdash_{P_{\kappa^+}}$ "$\underset{\sim}{u}$ is an unbounded subset of κ^+". But for each $p \in P_{\kappa^+}$ and $\alpha < \kappa^+$ w.l.o.g. we have some n and $\xi_1 < \ldots < \xi_n < \kappa^+$ such that $i \in \kappa^+ \setminus \{\xi_1, \ldots, \xi_n\} \Rightarrow p{\restriction}i \Vdash$ "$\emptyset_{\underset{\sim}{Q}_i} \leq_0 p{\restriction}\{i\}$".

Now there is $q \in P_{\kappa^+}$, such that $p \leq_0 q$ (i.e. for every i we have $q{\restriction}i \Vdash_{P_i}$ "$p{\restriction}\{i\} \leq_0 q{\restriction}\{i\}$") and for every $i \in \alpha \setminus \{\xi_1, \ldots, \xi_n\}$ we have $\Vdash_{P_i}$ "$q(i)$, p_i are incompatible in $\underset{\sim}{Q}_i$". So $q \Vdash$ "$\underset{\sim}{u} \cap \alpha \subseteq \{\xi_1, \ldots, \xi_n\}$". As α was an arbitrary ordinal $< \kappa^+$, necessarily $\Vdash_{P_{\kappa^+}}$ "$\underset{\sim}{u}$ has order type $\leq \omega$", but as indicated earlier $\Vdash_{P_\alpha}$ "$\sup(\underset{\sim}{u}) = \kappa^+$". Together $\Vdash_{P_{\kappa^+}}$ "$\mathrm{cf}(\kappa^+) = \aleph_0$", certainly contrary to our desires.

So the use of our choice is $e = 2$. Where is this used? In the proof see end of the proof of 2.13 (hence also in 2.14, 2.15).

2.9 Claim. Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ be an κ-Sp_2-iteration, $P_\alpha = \mathrm{Sp}_2 - \mathrm{Lim}_\kappa \bar{Q}$ (as usual).

(1) If $\underset{\sim}{\beta}, \underset{\sim}{\gamma}$ are $\bar{Q}$-named $[j, \ell g(\bar{Q}))$-ordinals, then $\mathrm{Max}\{\underset{\sim}{\beta}, \underset{\sim}{\gamma}\}$ (defined naturally) is a $\bar{Q}$-named $[j, \ell g(\bar{Q}))$-ordinal.

(2) If $\alpha = \beta_0 + 1$, in Definition 2.7, part (D), in defining the set of elements of P_α we can restrict ourselves to $\beta = \beta_0$. Also in such a case, $P_\alpha = P_{\beta_0} * \underset{\sim}{Q}_{\beta_0}$ (essentially). More exactly, $\{p \bigcup \{\underset{\sim}{q}\} : p \in P_{\beta_0}, \underset{\sim}{q}$ a P_{β_0}-name of a member of $\underset{\sim}{Q}_{\beta_0}\}$ is a dense subset of P_α, and the order $p_1 \bigcup \{\underset{\sim}{q}_1\} \le p_2 \bigcup \{\underset{\sim}{q}_2\}$ iff $[p_1 \le p_2$ (in P_{β_0}) and $p_2 \Vdash_{P_{p_0}}$ "$\underset{\sim}{q}_1 \le \underset{\sim}{q}_2$ in $\underset{\sim}{Q}_{\beta_0}$"$]$ is equivalent to that of P_α, in fact is the restriction of $<^{P_\alpha}$ to this set, so we get the same completion to a Boolean Algebra.

(3) $|P_\alpha| \le (\sum_{i<\alpha} 2^{|P_i|})^{|\alpha|}$, for limit α (where of course $|P| = |\{p/\approx : p \in P\}|$ $p_0 \approx p_1$ iff $p_\ell \Vdash$ "$p_{1-\ell} \in \underset{\sim}{G}_P$" for $\ell = 0, 1$).

(4) If $\Vdash_{P_i}$ "$|\underset{\sim}{Q}_i| \le \mu$", μ a cardinal, then $|P_{i+1}| \le 2^{|P_i|} + \mu$.

(5) If $\Vdash_{P_i}$ "$d(Q_i) \le \mu$" then $d(P_{i+1}) \le d(P_i) + \mu$, where $d(P)$ is the density of P.

(6) For α limit $d(P_\alpha) \le 2^{\Sigma_{i<\alpha} d(P_i)}$.

(7) If $P = \hat{Q}$ *then* P is essentially complete, i.e. for every maximal antichain $\mathcal{I}_0 \cup \mathcal{I}_1$ of P with $\mathcal{I}_0 \cap \mathcal{I}_1 = \emptyset$, for some $q \in P$, for every $p \in \mathcal{I}_0 \cup \mathcal{I}_1$, q is compatible with p iff $p \in \mathcal{I}_0$.

Proof. Check. $\square_{2.9}$

2.10 Claim. Suppose (κ is regular and):

(a) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a $\kappa - \mathrm{Sp}_2$-iteration (and of course $P_\alpha = \mathrm{Sp}_2 - \mathrm{Lim}_\kappa(\bar{Q})$)

(b) $\Vdash_{P_i}$ "$(Q_i, \le_0)$ is θ-complete" for $i < \alpha$

(c) $\theta \le \kappa$

Then

(1) $(P_\alpha, \leq_0)$ is θ-complete, in fact: if $\delta < \theta$, $\langle p_i : i < \delta \rangle$ is $\leq_0$-increasing then it has an $\leq_0$-upper bound (in fact, as in 2.7(G))

(2) for $\beta < \alpha$, P_α / P_β is θ-complete.

(3) In fact we can get $\leq_0$ $-$lub if this holds for each Q_i.

Remark. We deal with Pr^+ and not Pr_1 (here and later) just for simplicity presentation, as it does not matter much by 2.4.

Proof: Straightforward.

(1) So assume $\delta < \theta$ and $p_i \in P_\alpha$ for $i < \delta$ and $[i < j < \delta \Rightarrow p_i \leq_0 p_j]$. Now it is enough to find $p \in P_0$ such that

$$i < \delta \Rightarrow p_i \leq_0 p$$

$$\Vdash_{\bar{Q}} \text{“}\mathrm{Dom}(p) = \bigcup_{i<\delta} \mathrm{Dom}(p_i)\text{”}$$

$$\Vdash_{P_\zeta} \text{“}p{\restriction}\{\zeta\} \text{ is a singleton or } \emptyset\text{”}.$$

Let $p_i = \{\underset{\sim}{q}^i_\gamma : \gamma < \gamma_i\}$ where $\gamma_i < \kappa$ and for each $\zeta < \zeta_i, \underset{\sim}{q}^i_\zeta$ is $\bar{Q}$-named atomic condition, say $\Vdash_{\bar{Q}}$ “$\underset{\sim}{q}^i_\gamma \in Q_{\underset{\sim}{\zeta}^i_\gamma}$”, where $\underset{\sim}{\zeta}^i_\gamma$ is a $\bar{Q}$-named ordinal which is $\zeta_{\underset{\sim}{q}^i_\gamma}$. Now for each $\beta < \alpha$ let $\underset{\sim}{\leq}^*_\beta$ be a P_β-name of a well ordering of $\underset{\sim}{Q}_\beta$. For each $i(*) < \delta, \gamma(*) < \gamma_i$ let $r^{i(*)}_{\gamma(*)}$ be the following $\bar{Q}$-named atomic condition:

Let $\zeta < \alpha$, $G_\zeta \subseteq P_\zeta$ generic over V and $\underset{\sim}{\zeta}^{i(*)}_{\gamma(*)}[G_\zeta] = \zeta$, now work in $V[G_\zeta]$, let $w_\zeta = \{i < \delta : \text{for some } \gamma \text{ we have } \underset{\sim}{\zeta}^i_\gamma[G_\zeta] = \zeta\}$, and for each $i \in w_\zeta$ let $u^\zeta_i = \{\gamma < \gamma_i : \underset{\sim}{\zeta}^i_\gamma[G_\zeta] = \zeta\}$, clearly not empty; moreover for some $\beta_i \in u^\zeta_i$ we have

$$(\forall \xi \in u^\zeta_i)(\exists \gamma \in u^\zeta_i)(\underset{\sim}{q}^i_{\beta_i}[G_\zeta] \leq_0 \underset{\sim}{q}^i_\gamma[G_\zeta] \;\&\; \underset{\sim}{q}^i_\xi[G_\zeta] \leq \underset{\sim}{q}^i_\gamma[G_\zeta])$$

and let $v^\zeta_i = \{\gamma \in u^\zeta_i : \underset{\sim}{q}^i_{\beta_i}[G_\zeta] \leq_0 \underset{\sim}{q}^i_\gamma[G_\zeta]\}$, clearly also v^ζ_i is not empty. (As p_i is $\leq_0$-increasing, w_ζ is an end segment of δ and $i(*) \in w_\zeta$.) We define $\underset{\sim}{r}^i_\gamma[G_\zeta] \in \underset{\sim}{Q}_\zeta[G_\zeta]$.

Case 1: For some $i \in w_\zeta$ we have:

$$(\forall j)\,[i \leq j \in w_\zeta \rightarrow p_i{\restriction}\{\zeta\}[G_\zeta] = p_j{\restriction}\{\zeta\}[G_\zeta]]\,.$$

By the definition there is $r \in \underset{\sim}{Q}_i$ such that $\gamma \in v_i^\zeta \Rightarrow \underset{\sim}{q}_\gamma^i[G_\zeta] \leq_0 r$.

We let $\underset{\sim}{r}_{\gamma(*)}^{i(*)}$ be the $\underset{\sim}{<}_\zeta[G_\zeta]$ - first such r.

Case 2: Not case 1.

Let $w'_\zeta = \{i \in w_\zeta :$ for no $j \in w_\zeta \cap i$ do we have $(p_j{\restriction}\{\zeta\})[G_\zeta] = (p_i{\restriction}\{\zeta\})[G_\zeta]\}$.

Note. w'_ζ has no last member and moreover is unbounded in δ; let $j(i) = \min(w'_\zeta \setminus (i+1))$ for $i \in w'_\zeta$.

For each $i \in w'_\zeta$, by 2.2(9) we know there is $\beta_i \in v_{j(i)}^\zeta$ such that

$$[\gamma \in v_i^\zeta \Rightarrow \underset{\sim}{Q}_\zeta[G_\zeta] \models \text{“}q_\gamma^i[G_\zeta] \leq_0 q_{\beta_i}^{j(i)}[G_\zeta]\text{”}].$$

Hence by 2.2(9) we know $\langle q_{\beta_i}^{j(i)}[G_\zeta] : i \in w'_\zeta\rangle$ is a $\leq_0$-increasing sequence in $\underset{\sim}{Q}_\zeta[G_\zeta]$, hence it has a $\leq_0$-upper bound; so let $r_{\gamma(*)}^{i(*)}$ be the $\underset{\sim}{<}_\zeta^*[G]$-least $\leq_0$-upper bound of such a sequence in $\underset{\sim}{Q}_\zeta[G_\zeta]$. Because of the “such” the choice depend on ζ, G_ζ but not on $i(*)$, $\gamma(*)$. Now

$$p \overset{\text{def}}{=} \{\underset{\sim}{r}_{\gamma(*)}^{i(*)} : i(*) < \delta \text{ and } \gamma(*) < \gamma_{i(*)}\}$$

is as required.

2), 3) Similar proof and will not be used (or use the associativity law, see 2.21(3) (which could be proved before 2.10)). $\square_{2.10}$

2.11 Definition. Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ be an $\kappa - \mathrm{Sp}_2$-iteration

(1) We say $\underset{\sim}{y}$ is a $(\bar{Q}, \underset{\sim}{\zeta}) - \mathbf{E}^*$-name (again we usually omit $\mathbf{E}^*$) *if:* $\underset{\sim}{y}$ is a P_α-name, $\underset{\sim}{\zeta}$ is a $\bar{Q} - \mathbf{E}^*$-named $[0, \alpha)$-ordinal, and: if $\beta < \alpha$, $G_{P_\alpha} \subseteq P_\alpha$ is generic over V and for some $r \in G_{P_\alpha} \cap P_\beta$, $r \Vdash_{\bar{Q}}$ “$\underset{\sim}{\zeta} = \beta$”, *then* $\underset{\sim}{y}[G_{P_\alpha}] \in V[G_{P_\beta}]$ is well defined and depends only on $G_{P_\alpha} \cap P_\beta$ so we write $\underset{\sim}{y}[G_{P_\alpha} \cap P_\beta]$; and if $G_{P_\alpha} \subseteq P_\alpha$ is generic over V, $\underset{\sim}{\zeta}[G_{P_\alpha}]$ not well defined then $\underset{\sim}{y}[G_{P_\alpha}]$ is not well defined.

(2) If $p \in P_\alpha$, $G_{P_\alpha} \subseteq P_\alpha$ generic over V, (or just in $\mathrm{Gen}^r(\bar{Q})$), *then* $p[G_{P_\alpha}]$ is a function, $\mathrm{Dom}(p[G_{P_\alpha}]) = \{\underset{\sim}{\zeta}_{\underset{\sim}{q}}[G_{P_\alpha}] : \underset{\sim}{q} \in p\}$ and $(p[G_{P_\alpha}])(\varepsilon) = \{\underset{\sim}{q}[G_{P_\alpha}] : \underset{\sim}{q} \in p \text{ and } \underset{\sim}{\zeta}_{\underset{\sim}{q}}[G_{P_\alpha}] = \varepsilon\}$.

2.12 Claim. Suppose

(a) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ is a $\kappa - \mathrm{Sp}_2$-iteration

(1) Assume that $\underset{\sim}{q}$ is an atomic $\bar{Q}$-named condition and $\underset{\sim}{\xi}$ is a $\bar{Q}$-named ordinal and $\underset{\sim}{q}{\restriction}\underset{\sim}{\xi}$ is defined naturally (i.e. if $G \in \mathrm{Gen}(\bar{Q})$, and $\xi_0 = \underset{\sim}{\xi}[G]$ and $\zeta = \underset{\sim}{\zeta}_{\underset{\sim}{q}}[G]$ then $\zeta < \xi \Rightarrow (\underset{\sim}{q}{\restriction}\underset{\sim}{\zeta})[G] = \underset{\sim}{q}[G]$, and $\zeta \geq \xi \Rightarrow (\underset{\sim}{q}{\restriction}\underset{\sim}{\xi})[G]$ not defined). Then $\underset{\sim}{q}{\restriction}\underset{\sim}{\xi}$ is an atomic $\bar{Q}$-condition with $\underset{\sim}{\zeta}_{(\underset{\sim}{q}{\restriction}\underset{\sim}{\xi})} = \min\{\underset{\sim}{\zeta}_{\underset{\sim}{q}}, \underset{\sim}{\xi}\}$ (see 2.9(1)). Similarly for $\underset{\sim}{q}{\restriction}[\underset{\sim}{\xi}, \alpha)$. we can let $P_{\underset{\sim}{\xi}} \stackrel{\text{def}}{=} \{p \in P_\alpha : p{\restriction}\underset{\sim}{\xi} = p\}$ and it has the natural properties.

(2) Assume in addition

(b) $p \in P_\alpha$, $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[0, \alpha)$-ordinals

(c) $\underset{\sim}{r}$ is a $(\bar{Q}, \underset{\sim}{\zeta})$-named member of $P_\alpha/P_{\underset{\sim}{\zeta}}$

(d) κ a successor (or just not a limit cardinal)

Then: there is $q \in P_\alpha$ such that:

(*) *if* $\xi < \alpha$, $G_\xi \subseteq P_\xi$ generic over V, and $\underset{\sim}{\zeta}[G_\xi] = \xi$ *then* $(p{\restriction}\xi)[G_\xi] = (q{\restriction}\xi)[G_\xi]$ and $(q{\restriction}[\xi, \alpha))[G_\xi] = \underset{\sim}{r}{\restriction}[\xi, \alpha)[G_\xi]$

In fact $q = (p{\restriction}\underset{\sim}{\zeta}) \cup [\underset{\sim}{r}{\restriction}[\underset{\sim}{\zeta}, \alpha)]$ will do where $p{\restriction}\underset{\sim}{\zeta} = \{\underset{\sim}{p}'{\restriction}\underset{\sim}{\zeta} : \underset{\sim}{p}' \in p\}$, $\underset{\sim}{r}{\restriction}[\underset{\sim}{\zeta}, \alpha) = \{\underset{\sim}{r}'{\restriction}[\underset{\sim}{\zeta}, \alpha) : \underset{\sim}{r}' \in r\}$.

(3) If in (2) in addition

(c)′ $\underset{\sim}{r}$ is a $(\bar{Q}, \underset{\sim}{\zeta})$-name of a member of $P_\alpha/P_{\underset{\sim}{\zeta}}$ above $p{\restriction}[\underset{\sim}{\zeta}, \alpha)$

then we can add $p \leq q$ (but now $q = p \cup (r{\restriction}[\underset{\sim}{\zeta}, \alpha))$.

(4) If in (2) in addition

(c)$^+$ $\underset{\sim}{r}$ is a $(\bar{Q}, \underset{\sim}{\zeta})$-named of a member of $P_\alpha/P_{\underset{\sim}{\zeta}}$ purely above $p{\restriction}[\underset{\sim}{\zeta}, \alpha)$

then we can add $p \leq_0 q$ (and now $q = p \cup (r{\restriction}[\underset{\sim}{\zeta}, \alpha))$.

Proof. Straightforward (think particularly on the case $\underset{\sim}{\zeta}[G] \in \mathbf{E}^* \cap E(\bar{Q})$).

2.13 Claim. Suppose

(a) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a $\kappa - \mathrm{Sp}_2$-iteration

(b) each Q_i satisfies $(\{2, \theta_1\}, \aleph_0) - \mathrm{Pr}_1$

(c) each $\underset{\sim}{Q}_i$ satisfies 2.1(1)(v) for $\theta \in \{2, \theta_1\}$

(d) κ is successor

Then

1) P_α satisfies 2.1(1)(v) for $\theta \in \{2, \theta_1\}$

2) for $\beta < \alpha$, P_α / P_β satisfies 2.1(1)(v) for $\theta \in \{2, \theta_1\}$

Proof. 1) Let $p \in P_\alpha$ and $\underset{\sim}{\tau}$ be a P_α-name of an ordinal $< \theta$, $\theta \in \{2, \theta_1\}$. We define a $\bar{Q}$-named $[0, \alpha)$-ordinal $\underset{\sim}{\zeta}$: for $r \in P_\beta$, $r \Vdash$ "$\underset{\sim}{\zeta} = \beta$" *iff*

(a) there are q, γ such that $r \cup (p{\restriction}[\beta, \alpha)) \leq_0 q \in P_\alpha$ and $q{\restriction}\beta = r{\restriction}\beta$ $(= r)$ and $\gamma < \theta$ and $q \Vdash$ "if $\theta = 2$, then $\underset{\sim}{\tau} = \gamma$ and: if $\theta > 2$ then $\underset{\sim}{\tau} < \gamma$"

(b) for no $\beta' < \beta$ and r', $r{\restriction}\beta' \leq r' \in P_{\beta'}$ does (r', β') satisfies (a).

Note that: if β is a limit cardinal we can get (by 2.8) a contradiction to clause (b).

However we would like to apply 2.12(2) and for this we need to prove that $\underset{\sim}{\zeta}$ is a $\underset{\sim}{Q}$-named ordinal, i.e. a $\underset{\sim}{Q} - \mathbf{E}^*$-named ordinal. So (by 2.5C) let $\beta \leq \lambda$, $\beta \in \mathbf{E}^* \cap E(\bar{Q})$, and it suffice to find $\gamma < \beta$ such that $\Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} \notin [\gamma, \beta)$". But $p{\restriction}\beta \in P_\beta = \bigcup_{\gamma < \beta} P_\gamma$, so for some $\gamma < \beta$, $p{\restriction}\beta \in P_\gamma$ and this γ is as required because we are using Sp_2 (and not Sp_1), that is; because if $p{\restriction}\beta \leq q \in P_\beta$ then $(q{\restriction}\beta) \cup p{\restriction}[\beta, \gamma) = q{\restriction}\beta \leq_0 q$.

Let $\underset{\sim}{q}^*$ be a $(\bar{Q}, \underset{\sim}{\zeta})$-named member of P_α as in clause (a). Let $p_0 \stackrel{\text{def}}{=} p$ and $p_1 = p_0 \cup (\underset{\sim}{q}^*{\restriction}[\underset{\sim}{\zeta}, \alpha))$, now $p_0 \leq_0 p_1 \in P_\alpha$ by 2.12(4). We now define $p_2 = p_1 \cup \{\underset{\sim}{r}_q : \underset{\sim}{q} \in p_1\}$ where $\underset{\sim}{r}_q$ is defined as follows

(*) if $\beta < \alpha$, $G_{P_\beta} \subseteq P_\beta$ generic over V and $\underset{\sim}{\zeta}_q[G_{P_\beta}] = \beta$ and in $V[G_{P_\beta}]$ there is r such that

(i) $\hat{Q}_\beta[G_{P_\beta}] \models$ "$p{\restriction}\{\beta\} \leq_0 \{r\} \in \hat{Q}_\beta[G_{P_\beta}]$"

(ii) for some $r_1 \in P_{\beta+1}$ we have: $r_1{\restriction}\beta \in G_{P_\beta}$ and $r_1{\restriction}\{\beta\} = r$ and: r_1 forces $(\Vdash_{P_{\beta+1}})$ $\underset{\sim}{\zeta} = \beta + 1$ or r_1 forces $\underset{\sim}{\zeta} \neq \beta + 1$ and in the former case if $\theta = 2$ the condition $r_1 \cup (p_1 \restriction [\beta+1, \alpha))$ forces a value to $\underset{\sim}{\tau}$, if $\theta \neq 2$ forces a bound to $\underset{\sim}{\tau}$

then $\underset{\sim}{r}_q[G_\beta]$ is the $\underset{\sim}{\leq}^*_\beta[G_\beta]$-first such a member otherwise it is $\emptyset_{\underset{\sim}{Q}_\beta}$.

Let us choose now $\beta \leq \alpha$ minimal such that

$\otimes$ there are $r_1 \in P_\beta$ and $\gamma < \theta$ such that $p_2 \restriction \beta \leq r_1$ and $r_1 \cup (p_2 \restriction [\beta, \alpha)) \Vdash$ "$\theta = 2, \underset{\sim}{\tau} < \gamma$ or $\theta \neq 2$, $\underset{\sim}{\tau} \neq \gamma$".

There is such β as for some $\beta < \alpha$ and r we have $p_2 \restriction \beta \leq r \in P_\beta$, and $r \Vdash$ "$\underset{\sim}{\zeta} = \beta$" and see the choice of $\underset{\sim}{\zeta}$ and p_1, p_2; actually we can use $\beta = \alpha$. If $\beta = 0$ we are done. If β is limit without loss of generality, by 2.8 for some $n < \omega$ and $\xi_1 < \ldots < \xi_n < \beta$ we have: $p_2 \restriction \beta \leq r_1$ above $\{\xi_1, \ldots, \xi_n\}$.

By the choice of r_1 there are $q \in P_\alpha$ and $\gamma < \theta$ such that: $q \restriction \beta = r_1$ and $r_1 \cup (p_2 \restriction [\beta, \alpha)) \leq_0 q$ and $q \Vdash_{P_\alpha}$ "$\theta = 2$, $\underset{\sim}{\tau} = \gamma$ or $\theta \neq 2$, $\underset{\sim}{\tau} < \gamma$", just use $q = r_1 \cup (p_2 \restriction [\beta, \alpha))$. Hence $\beta' = \xi_n + 1$, $r' = r_1 \restriction (\xi_n + 1)$ satisfy: $r' \in P_{\xi_n+1}$, $p_2 \restriction \beta' \leq r'$, $q \restriction \beta' = q \restriction (\xi_n + 1) = r_1 \restriction (\xi_n + 1) = r'$, and $q \Vdash_{P_\alpha}$ "$\theta = 2$, $\underset{\sim}{\tau} = \gamma$ or $\theta \neq 2$, $\underset{\sim}{\tau} < \gamma$" and $r' \cup (p \restriction [\beta', \alpha)) \leq_0 r' \cup (q \restriction [\beta', \alpha))$. So by the definition of $\underset{\sim}{\zeta}$ we have $r' \Vdash$ "$\underset{\sim}{\zeta} \leq \beta'$" and of course $\beta' < \beta$. So we get a contradiction to the choice of β. Lastly assume $\beta = j + 1$, and let $G_{P_j} \subseteq P_j$ be generic over V such that $r_1 \restriction j \in G_{P_j}$. If for some $\underset{\sim}{q} \in p_2$ we have $\underset{\sim}{\zeta}_{\underset{\sim}{q}}[G_{P_j}] = j$ (so we could use $\underset{\sim}{r}_q \in p_1$) the contradiction is gotten similarly using the definition of p_2 (note that for $\theta \neq 2$ we use the result for $\theta = 2$!). In the remaining case we can decrease β by the definition of $\leq_0^{P_\alpha}$ (as we use Sp_2 rather than Sp_1!).

2) Same proof (or 2.21(2)). $\square_{2.13}$

Now 2.10 + 2.13 suffice to show that no bounded subset of κ is added by the $\kappa - \mathrm{Sp}_2$- iteration (if say each $\underset{\sim}{Q}_i$ has $(\{\theta : \aleph_0 \leq \theta = \mathrm{cf}(\theta) < \kappa\} \cup \{2\}, \gamma) - \mathrm{Pr}_1^+$ for $\gamma < \kappa$. But we may like to deal with iterations which e.g. add reals. The next claim does better.

2.14 Claim. Assume

(a) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a $\kappa - \mathrm{Sp}_2$-iteration

(b) each Q_i satisfies the $(\{\theta\}, \aleph_1) - \mathrm{Pr}_1^+$, and $\theta = \mathrm{cf}(\theta) > \aleph_0$

(c) κ is a successor cardinal.

Then

(1) P_α satisfies $(\{\theta\}, \aleph_1) - \mathrm{Pr}_1^+$

(2) also for $\beta < \alpha$, P_α / P_β satisfies $(\{\theta\}, \aleph_1) - \mathrm{Pr}_1^+$.

Proof of 2.14. Before proving, (in 2.14E) we define and prove in 2.14A – 2.14D, retaining our θ, κ, $\kappa - \mathrm{Sp}_2$-iteration $\bar{Q}$, and $\alpha = \ell g(\bar{Q})$. We can assume that 2.14 holds for any case with $\alpha' < \alpha$ instead of α

2.14A Definition.

(1) $\Gamma_1 = \{(p, \underset{\sim}{\zeta}, \underset{\sim}{\tau}) : p \in P_\alpha$ and $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[0, \alpha)$-ordinal and $\underset{\sim}{\tau}$ is a P_α-name of an ordinal $< \theta$ such that: if $p \in G_\alpha \subseteq P_\alpha$, G_α generic over V and $\underset{\sim}{\zeta}[G_\alpha] = \zeta$ then for some $r \in G_\alpha \cap P_\zeta$ and $\varepsilon < \theta$ we have $r \Vdash_{P_\alpha}$ "$\underset{\sim}{\tau} < \varepsilon$"$\}$

(2) $\Gamma_2 = \{(p, \underset{\sim}{\zeta}, \underset{\sim}{\tau}) \in \Gamma_1$: $\Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta}$ is a non limit ordinal", and for every $\beta < \alpha$ we have: if there are r and $\underset{\sim}{q}$ such that $P_\beta \models$ "$p{\restriction}\beta \le r$", $r \in P_\beta$, $\Vdash_{P_\beta}$ "$\underset{\sim}{q} \in \underset{\sim}{Q}_\beta$ & $p{\restriction}\{\beta\} \le_0 \underset{\sim}{q}$" and $r \cup \{\underset{\sim}{q}\} \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} = \beta + 1$" or $r \cup \{\underset{\sim}{q}\} \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} \neq \beta + 1$" *then* we can use $q = p{\restriction}\{\beta\}\}$

(3) For $y = (p, \underset{\sim}{\zeta}, \underset{\sim}{\tau})$ let us define a P_α-name $\underset{\sim}{w}_y$: for $G_\alpha \subseteq P_\alpha$ generic over V

$$
\begin{aligned}
\underset{\sim}{w}_y[G_\alpha] = \{\beta < \alpha : {} & \text{for some } r \in G_\alpha \cap P_\beta \text{ and } \underset{\sim}{q} \in p \text{ we have} \\
& r{\restriction}\beta \Vdash_{\bar{Q}} \text{“}\underset{\sim}{\zeta}_{\underset{\sim}{q}}[\underset{\sim}{G}_{P_\beta}] = \beta\text{” and} \\
& p{\restriction}\beta \le r \text{ and } \neg(\exists r' \in P_{\beta+1})(r \le (r'{\restriction}\beta) \\
& \quad \& \ r'{\restriction}\beta \Vdash_{P_\beta} \text{“}p{\restriction}\{\beta\} \le_0 r'{\restriction}\{\beta\}\text{”} \\
& \quad \& \ r' \Vdash_{\bar{Q}} \text{“}\underset{\sim}{\zeta} \text{ is not } = \beta + 1\text{”})\}.
\end{aligned}
$$

2.14B Subclaim.

(1) If $\ell \in \{1, 2\}$ and $(p, \underset{\sim}{\zeta}, \underset{\sim}{\tau}) \in \Gamma_\ell$ and $p \le_0 p_1 \in P_\alpha$ *then* $(p_1, \underset{\sim}{\zeta}, \underset{\sim}{\tau}) \in \Gamma_\ell$

(2) If $p \in P_\alpha$ and $\underset{\sim}{\tau}$ is a P_α-name of an ordinal $< \theta$ *then* for some $(p', \underset{\sim}{\zeta}', \underset{\sim}{\tau}') \in \Gamma_1$ we have $p \le_0 p'$ and $p' \Vdash_{P_\alpha}$ "$\underset{\sim}{\tau}' = \underset{\sim}{\tau}$"

(3) If $(p,\underset{\sim}{\zeta},\underset{\sim}{\tau}) \in \Gamma_1$ *then* for some $(p',\underset{\sim}{\zeta}',\underset{\sim}{\tau}') \in \Gamma_2$ we have $p \le_0 p'$, $\Vdash_{P_\alpha}$ "$\underset{\sim}{\zeta}' \le \underset{\sim}{\zeta}$ and $\underset{\sim}{\tau}' \ge \underset{\sim}{\tau}$".

Proof. 1) Read the definitions.

2), 3) Like the proof of 2.13. $\square_{2.14B}$

2.14C Subclaim. If $y = (p,\underset{\sim}{\zeta},\underset{\sim}{\tau}) \in \Gamma_2$ then the (P_α-name of a) set $\underset{\sim}{w}_y$ satisfies:

(a) $\Vdash_{P_\alpha}$ "$\beta \in \underset{\sim}{w}_y \Rightarrow \beta < \underset{\sim}{\zeta}$"

(b) if $G_\alpha \subseteq P_\alpha$ is generic over V and $\beta \in \underset{\sim}{w}_y[G_\alpha]$ *then* some $r \in G_\alpha \cap P_\beta$ forces this; in fact if $r \Vdash_{P_\alpha}$ "$\beta \in \underset{\sim}{w}_y$" then $r{\restriction}\beta \Vdash_{P_\alpha}$ "$\beta \in \underset{\sim}{w}_y$".

(c) $\Vdash_{P_\alpha}$ "$\underset{\sim}{w}_y$ is a finite subset of α"

Proof. For clauses (a), (b) read 2.14A(3), so let us prove clause (c).

If not, for some $G_{P_\alpha} \subseteq P_\alpha$ generic over V, and $\underset{\sim}{w}_y[G_{P_\alpha}]$ is infinite, and let $\zeta_0 < \zeta_1 \dots$ be the first ω members. Let $\delta \stackrel{\text{def}}{=} \bigcup_{n<\omega} \zeta_n$, so $V[G_{P_\alpha}] \models$ "$\text{cf}(\delta) = \aleph_0$".

Let $\underset{\sim}{\delta}$ and $\langle \underset{\sim}{\zeta}_n : n < \omega \rangle$ be the corresponding P_α-names, so there are $r \in G_{P_\alpha}$ and β and $\delta \le \alpha$ such that $r \Vdash_{P_\alpha}$ "$\underset{\sim}{\zeta} = \beta$ and $\underset{\sim}{\delta} = \delta$ (and $\underset{\sim}{w}_y$ is infinite)". Now as $\underset{\sim}{\zeta}_n[G_\alpha] \in \underset{\sim}{w}_y$, by Definition 2.14A(3) (the clause $p{\restriction}\beta \le r$) we have $p{\restriction}\underset{\sim}{\zeta}_n[G_\alpha] \in G_\alpha$; and as this hold for every n we have $p{\restriction}\delta \in G_\alpha$, so as we can increase r w.l.o.g. $p{\restriction}\delta \le r$. Hence by 2.8 without loss of generality for some $n < \omega$, and $\xi_1 < \dots, < \xi_n < \alpha$, we have $p{\restriction}\delta \le r$ above $\{\xi_1,\dots,\xi_n\}$ so letting ξ_0 be: β if $\beta < \delta$, 0 if $\beta \ge \delta$ and letting $\xi = \sup[\delta \cap \{\xi_0,\xi_1,\dots,\xi_n\}]$ we know that $\xi < \delta$ and $r \Vdash_{P_\alpha}$ "$\underset{\sim}{w}_y \cap (\xi,\delta) \ne \emptyset$" hence for some ε and r_1, we have $r \le r_1 \in P_\alpha$ and $\varepsilon \in (\xi,\delta)$ and $r_1 \Vdash_{P_\alpha}$ "$\varepsilon \in \underset{\sim}{w}_y$". But by the definition of $\underset{\sim}{w}_y$, i.e. by 2.14C(b) we have: $r_1{\restriction}\varepsilon \Vdash_{P_\varepsilon}$ "$\varepsilon \in \underset{\sim}{w}_y$" and clearly $r_1{\restriction}\varepsilon \Vdash_{P_\varepsilon}$ "$p{\restriction}\{\varepsilon\} \le_0 r{\restriction}\{\varepsilon\}$" hence by the definition of $\underset{\sim}{w}_y$ we have $(r_1{\restriction}\varepsilon) \cup (r{\restriction}\{\varepsilon\}) \nVdash_{P_{\varepsilon+1}}$ "$\underset{\sim}{\zeta}$ is not $= \varepsilon + 1$" hence (as $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named ordinal) there is r_2 such that $(r_1{\restriction}\varepsilon) \cup (r{\restriction}\{\varepsilon\}) \le r_2 \in P_{\varepsilon+1}$ and $r_2 \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} = \varepsilon + 1$". But $\varepsilon + 1 \ne \beta$ by the choice of ξ_0 and ξ and ε, and $r \Vdash_{P_\alpha}$ "$\underset{\sim}{\zeta} = \beta$" so r_2, r should be incompatible in P. But

$$r{\restriction}(\varepsilon+1) \le (r_1{\restriction}\varepsilon) \cup (r{\restriction}\{\varepsilon\}) \le r_2 \in P_{\varepsilon+1}.$$

[Why? As $r \le r_1$, and choice of r_2 (twice).]
Hence by 2.7 clause (F) we know r, r_2 are compatible, contradiction. $\square_{2.14C}$

2.14D Subclaim. If $y = (p, \underset{\sim}{\zeta}, \underset{\sim}{\tau}) \in \Gamma_2$ *then* we can find p^+, $\underset{\sim}{\zeta}_n$, $\underset{\sim}{\tau}_n$ for $n < \omega$ such that:

(a) $p \le_0 p^+ \in P_\alpha$

(b) $(p^+, \underset{\sim}{\zeta}_n, \underset{\sim}{\tau}_n) \in \Gamma_1$

(c) if $G_\alpha \subseteq P_\alpha$ is generic over V, then $\underset{\sim}{\zeta}_n[G_\alpha]$ is the n-th member of $\underset{\sim}{w}_y[G_\alpha]$ if there is one, if so then for some $r \in G_{\underset{\sim}{\zeta}_n[G_\alpha]}$ we have $p^+ \restriction \underset{\sim}{\zeta}_n[G_\alpha] \le r$ and $r \cup (p^+ \restriction \{\underset{\sim}{\zeta}_n[G_\alpha]\}) \Vdash_{P_\alpha}$ " if $\underset{\sim}{\zeta}[\underset{\sim}{G}_{P_\alpha}] = \underset{\sim}{\zeta}_n[G_\alpha] + 1$ then $\underset{\sim}{\tau}[G_\alpha] \le \underset{\sim}{\tau}_n[\underset{\sim}{G}_\alpha]$".

Proof. Straight using 2.10 (for $\theta = \aleph_1$) to have a $\le_0$-upper bound, and taking care of n work as in 2.14B(2) i.e. as in 2.13. $\square_{2.14D}$

2.14E Completion of the proof of 2.14. We concentrate on part (1) (the proof of part (2) being similar or use 2.21(3), and it is not used). By 2.7, clause (E) we know that (i) - (iv) of Definition 2.1(1) holds, and by 2.10(1) not only clause (vi) of Definition 2.1(1) and the extra demand from Definition 2.1(2) hold, so the problem is to verify clause (v) in Definition 2.1(1), i.e. the pure decidability. So let $p \in P_\alpha$ and $\underset{\sim}{\tau}$ is a P_α-name and $p \Vdash$ "$\underset{\sim}{\tau} < \theta$", we have to find q, j such that $p \le_0 q \in P_\alpha$ and $j < \theta$ and $q \Vdash_{P_\alpha}$ "$\underset{\sim}{\tau} \le j$". So we can replace p, $\underset{\sim}{\tau}$ by p', $\underset{\sim}{\tau}'$ if $p \le_0 p'$ and $p' \Vdash$ "$\underset{\sim}{\tau} \le \underset{\sim}{\tau}' < \theta$". By subclaim 2.14B(2)+(3) w.l.o.g. for some $\underset{\sim}{\zeta}$ the triple $(p, \underset{\sim}{\zeta}, \underset{\sim}{\tau})$ belongs to Γ_2. We choose by induction on $n < \omega$, p_n, p_n^+ and $\langle (\underset{\sim}{\zeta}_\eta, \underset{\sim}{\tau}_\eta, \underset{\sim}{j}_\eta, \underset{\sim}{\tau}'_\eta) : \eta \in {}^n\omega \rangle$ such that:

(a) $p_0 = p$, $\underset{\sim}{\tau}_{\langle\rangle} = \underset{\sim}{\tau}$, $\underset{\sim}{\zeta}_{\langle\rangle} = \underset{\sim}{\zeta} = \underset{\sim}{j}_{\langle\rangle}$

(b) $(p_n, \underset{\sim}{\zeta}_\eta, \underset{\sim}{\tau}_\eta) \in \Gamma_2$ for each $n < \omega$, $\eta \in {}^n\omega$

(c) $p_n \le_0 p_n^+ \le_0 p_{n+1}$

(d) for each n and $\eta \in {}^n\omega$ we have:

  (i) $\{\underset{\sim}{j}_{\eta\frown\langle k\rangle} : k < \omega\}$ list $\underset{\sim}{w}_{(p_n, \underset{\sim}{\zeta}_\eta, \underset{\sim}{\tau}_\eta)} \cup \{0\}$

  (ii) $\underset{\sim}{j}_{\eta\frown\langle k\rangle}$ is a $\bar{Q}$-named $[0, \alpha)$-ordinal

  (iii) $\underset{\sim}{\tau}'_{\eta\frown\langle k\rangle}$ is a $P_{\underset{\sim}{j}_{\eta\frown\langle k\rangle}}$-name of an ordinal $< \theta$

(iv) if $G_\alpha \subseteq P_\alpha$ is generic over V, and $\beta = \underset{\sim}{j}_\eta[G_\alpha]$, and $p_n^+ \restriction (\beta+1) \in G_\alpha$ and $\beta = \underset{\sim}{\zeta}_\eta[G_\alpha]$ *then* $\underset{\sim}{\tau}'_{\eta\frown\langle k\rangle}[G_\alpha] \geq \underset{\sim}{\tau}_\eta[G_\alpha]$

(v) $(p_n^+, \underset{\sim}{j}_{\eta\frown\langle k\rangle}, \underset{\sim}{\tau}'_{\eta\frown\langle k\rangle}) \in \Gamma_1$

(e) $p_{n+1} \Vdash_{P_\alpha}$ "$\underset{\sim}{\zeta}_{\eta\frown\langle k\rangle} \leq \underset{\sim}{j}_{\eta\frown\langle k\rangle}$, and $\underset{\sim}{\zeta}_{\eta\frown\langle k\rangle}$ is non limit and $\underset{\sim}{\tau}_{\eta\frown\langle k\rangle} \geq \underset{\sim}{\tau}'_{\eta\frown\langle k\rangle}$" for $n < \omega$, $\eta \in {}^n\omega$, and $k < \omega$.

The case $n = 0$ is straight. Having arrived to stage n, i.e. p_n and $\underset{\sim}{\zeta}_\eta$, $\underset{\sim}{\tau}_\eta$ for $\eta \in {}^{n\geq}\omega$ are defined and as required, list $({}^n\omega)$ as $\langle \eta_\ell : \ell < \omega\rangle$ and choose by induction on $\ell < \omega$, $p_{n,\ell}^+$, $\underset{\sim}{j}_{\eta_\ell\frown\langle k\rangle}$, $\underset{\sim}{\tau}'_{\eta_\ell\frown\langle k\rangle}$ for $k < \omega$ such that $p_{n,0}^+ = p_n$, $p_{n,\ell}^+ \leq_0 p_{n,\ell+1}^+$ and $p_{n,\ell+1}^+$ satisfies the requirements of p_n^+ for $\eta = \eta_\ell$ which is possible by Subclaim 2.14D. Then let $p_n^+ = \bigcup_\ell p_{n,\ell}^+$ (it is a $\leq_0$-upper bound of $\{p_{n,\ell}^+ : \ell < \omega\}$; by 2.10 + assumption (b) of 2.14 it exists, but why it still satisfies the demands? By Subclaim 2.14B(1)). Now the choice of p_{n+1}, $\underset{\sim}{\zeta}_{\eta\frown\langle k\rangle}$, $\underset{\sim}{\tau}_{\eta\frown\langle k\rangle}$ for $\eta \in {}^n\omega$, $k < \omega$ is by Subclaim 2.14B(3) again using 2.10(1).

Again there is p^+, a $\leq_0$-upper bound of $\{p_n : n < \omega\}$, it satisfies $p \leq_0 p_n \leq_0 p^+ \in P_\alpha$ and letting $\gamma = \sup\{\underset{\sim}{\tau}_\eta : \underset{\sim}{j}_\eta = 0\} < \theta$ we can prove $p^+ \Vdash$ "$\underset{\sim}{\tau}_\eta \leq \gamma$ when $\underset{\sim}{\tau}_\eta$ is defined". For this we prove by induction on $j < \alpha$ that $p\restriction j \Vdash_{P_j}$ "if $\eta \in {}^{\omega>}\omega$ and $\underset{\sim}{j}_\eta \leq j$ then $\underset{\sim}{\tau}_\eta \leq \gamma$"(similarly to the proof of 2.13). As $\underset{\sim}{\tau} = \underset{\sim}{\tau}_{\langle\rangle}$ we are done. $\square_{2.14}$

Remark. Actually the tree we use is of finite splitting.

2.15 Conclusion. Assume

(a) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ is a $\kappa - \mathrm{Sp}_2 - \mathbf{E}^*$-iteration

(b) each $\underset{\sim}{Q}_i$ satisfies $(S, < \kappa) - \mathrm{Pr}_1^+$ and $(\aleph_0 \in S \Rightarrow 2 \in S)$

(c) κ successor

Then

(1) P_α satisfies $(S, < \kappa) - \mathrm{Pr}_1^+$

(2) for $\beta < \alpha$, P_α/P_β satisfies $(S, < \kappa) - \mathrm{Pr}_1^+$

Remark. 1) Note: if κ is not a cardinal we can replace it by $|\kappa|^+$; *but* during the iteration $|\kappa|^+$ may increase.

2) We can replace Pr_1^+ by Pr_1 with minor changes but because of 2.6 the gain is doubtful.

Proof. By 2.10, 2.14 (and 2.7 of course). $\square_{2.15}$

2.16 Discussion. Suppose α is an ordinal and $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ is a $\kappa - \mathrm{Sp}_2$-iteration (so $P_\alpha = \mathrm{Sp}_2 - \mathrm{Lim}_\kappa(\bar{Q})$). We may wonder whether:

(a) If α is strong inaccessible and density$(P_i) < \alpha$ for $i < \alpha$ *then* density$(P_\alpha) = \alpha$.

(b) *If* α is a Mahlo and $\forall i < \alpha[|P_i| < \alpha]$, *then* P_α satisfies the α-c.c.

As unlike X §1 we use antichains of $\bigcup_{i<\alpha} P_i$ (rather than antichains which are maximal in P whenever $\bigwedge_{i<\alpha} P_i \lessdot P$) this is not clear. Note that in 2.17 below, we can weaken the Pr_1 demand to $\Vdash_{P_\alpha}$ "θ remains regular".

2.17 Lemma. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ is a κ-$\mathrm{Sp}_2 - \mathbf{E}^*$-iteration, $\kappa > \aleph_0$ a successor cardinal, $S \subseteq \{2\} \cup \{\mu : \aleph_0 \le \mu, \mu \text{ regular }\}$, $\aleph_0 \in S = 2 \in S$, and each Q_i (in V^{P_i}), has $(S, < \kappa) - \mathrm{Pr}_1^+$ (see 2.1(4)). Then:

(1) If $\kappa \le \theta \in S$ and $\mathrm{cf}(\alpha) = \theta$ *then* $\bigcup_{i<\alpha} P_i$ is dense in P_α.

(2) If α is strongly inaccessible $> \min(S \setminus \kappa)$, $\alpha > |P_i| + \kappa$ for $i < \alpha$ (or just P_i satisfies the α-c.c.) and $\alpha \in \mathbf{E}^*$ *then* P_α satisfies the α-chain condition (in a strong sense).

(3) If each Q_i satisfies $(\mathrm{RCar} \setminus \kappa, \kappa) - \mathrm{Pr}_1^+$ and has power $\le \chi$, *then* P_α has a dense subset (even a $\le_0$-dense subset) of power $(2^{|\alpha|+\chi})^{<\kappa}$ and satisfies $(\chi^{<\kappa})^+$-c.c.

(3) If α is strongly inaccessible and $\mathbf{E}^* \cap \alpha$ is a stationary subset of α and $[i < \alpha \Rightarrow |P_i| < \alpha]$ or at least $[i < \alpha \Rightarrow P_i$ satisfies the λ_i-c.c. for some $\lambda_i < \alpha]$ *then* P_α satisfies the α-chain condition (a strong version indeed).

Proof. 1) Left to the reader.

2) Choose $\theta \in S \setminus \kappa$. Let $\langle p_j : j < \alpha\rangle$ be a sequence of elements of P_α now as $\alpha \in \mathbf{E}^*$ we have $p_j \in \bigcup_{i<\alpha} P_i$. Let $A = \{j < \alpha : \mathrm{cf}(j) = \theta\}$, this is a stationary set. For $j \in A$ choose $r_j \in \bigcup_{i<j} P_i$, $r_j \ge p_j \restriction j$ (why such r_j exist? by part (1)),

say $r_j \in P_{i(j)}$, $i(j) < j$. Let $C = \{i : j \text{ limit}, \forall i < j \exists \gamma < j [p_i \in P_\gamma]\}$. This is a club subset of α.

By Fodor's lemma we can find $B \subseteq C \cap A$ stationary such that for all $j_1, j_2 \in B$ we have $i(j_1) = i(j_2)$ and $r_{j_1} = r_{j_2} = r$ or at least r_{j_1}, r_{j_2} are compatible in $\bigcup_{i<i(j_1)} P_i$ and let $r_{j_1}, r_{j_2} \leq r \in \bigcup_{i<i(j_1)} P_i$.

(Remember that $|P_i| < \alpha$ or at least P_i satisfies the α-c.c. for $i < \alpha$)

But for any such $j_1 < j_2$ the condition $r \cup p_{j_1} \restriction [j_1, j_2) \cup p_{j_2} \restriction [j_2, \alpha)$ is a common upper bound for p_{j_1}, p_{j_2}.

(3) Like III 4.1 use only names which are hereditarily $< \kappa$ (see below).

(4) Like part (2) using 2.7(E) (so 2.5(1)(b)) instead using part (1). $\square_{2.17}$

We may wonder about $\kappa - \mathrm{Sp}_e$-iterations which essentially do not change cofinality.

2.18 Definition. We define for an Sp_e-iteration $\bar{Q}$, and cardinal μ (μ regular), what is a $\bar{Q}$-name hereditarily $< \mu$, and in particular a $\bar{Q}$-named $[j, \alpha)$-ordinal hereditarily $< \mu$ and a $\bar{Q}$-named $[j, \alpha)$-atomic condition hereditarily $< \mu$, and which conditions of $\mathrm{Sp}_e\text{-}\mathrm{Lim}_\kappa \bar{Q}$ are hereditarily $< \mu$ (formally they are not special cases of the corresponding notions without the "hereditarily $< \mu$"). For simplicity we are assuming that the set of members of Q_i is in V. This is done by induction on $\alpha = \ell g(\bar{Q})$.

First case. $\alpha = 0$

trivial

Second case. $\alpha > 0$

(A) A $\bar{Q}$-named $[j, \alpha)$-ordinal $\underset{\sim}{\xi}$ hereditarily $< \mu$ is a $\bar{Q}$-named $[j, \alpha)$-ordinal which can be represented as follows: there is $\langle (p_i, \xi_i) : i < i^* \rangle, i^* < \mu$, each ξ_i an ordinal in $[j, \alpha), p_i \in P_{\xi_i}$ is a member of P_{ξ_i} hereditarily $< \mu$ and for any $G \in \mathrm{Gen}^r(\bar{Q}), \underset{\sim}{\zeta}[G]$ is ζ iff for some i we have

(a) $p_i \in G, \zeta_i = \zeta$

(b) if $p_j \in G$ then $\zeta_i < \zeta_j \vee (\zeta_i = \zeta_j \& i < j)$

(B) A $\bar{Q}$-named $[j, \alpha)$-atomic condition $\underset{\sim}{q}$ hereditarily $< \mu$, is a $\bar{Q}$-named $[j, \alpha)$-atomic condition which can be represented as follows: there is $\langle (p_i, \zeta_i, q_i) : i < i^* \rangle, i^* < \mu, \zeta_i \in [j, \alpha), p_i \in P_{\zeta_i}, q_i \in V$, and for any $G \in \mathrm{Gen}^r(\bar{Q}), \underset{\sim}{q}[G]$ is q iff for some i:

(a) $p_i \in G, q = q_i$, and $p_i \Vdash_{P_{\zeta_i}}$ "$q \in \underset{\sim}{Q}_{\zeta_i}$"

(b) if $p_j \in G$ then $\zeta_i < \zeta_j \vee (\zeta_i = \zeta_j \;\&\; i < j)$

(C) A member p of $P_\alpha = \mathrm{Sp}_e - \mathrm{Lim}_\kappa(\bar{Q})$ is hereditarily $< \mu$ if each member of r is a $\bar{Q}$-named atomic condition hereditarily $< \mu$.

(D) A $\bar{Q}$-name of a member of V hereditarily $< \mu$ is defined as in clause (B), similarly for member $x \in V^{P_\alpha}$ such that $y \in$ transitive closure of $x \;\&\; y \notin V \Rightarrow |y| < \mu$.

2.19 Claim. Suppose $\langle P_i, Q_i : i < \alpha \rangle$ is an $\kappa - \mathrm{Sp}_e$-iteration, $P_\alpha = \mathrm{Sp}_e\text{-}\mathrm{Lim}_\kappa \bar{Q}$, κ a successor cardinal, each Q_i (in V^{P_i}) satisfies $(\mathrm{RCar}^{V^{P_i}} \setminus \kappa, \kappa) - \mathrm{Pr}_1^+$. *Then*

(1) $\{p \in P_\alpha : p$ hereditarily $< \kappa\}$ is a dense subset of P_α, even a $\leq_0$-dense subset

(2) P_α preserve "$\mathrm{cf}(\delta) \geq \kappa$"

(3) for every $p \in P_\alpha$ and $\underset{\sim}{\tau}$ a P_α-name of an ordinal there are $p^*, p \leq_0 p^* \in P_\alpha$, and $A \in V$, a set of $< \kappa$ ordinals such that $q \Vdash_{P_\alpha}$ "$\underset{\sim}{\tau} \in A$".

Proof. Should be clear. $\square_{2.19}$

2.20 Remark. We can also get a similar theorem for forcing notions $(Q, \leq, \leq_0)$ as in 2.1 where instead of $\leq_0$ is κ-directed complete (see 2.4(d)) we demand that (vi) ("strategical completeness" of $\leq_0$).

2.21 Claim. (1) Suppose F is a function and $e = \{1, 2\}$, *then* for every ordinal α there is Sp_e-iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha^\dagger \rangle$, such that:

(a) for every $i, \underset{\sim}{Q}_i = F(\underset{\sim}{Q} \restriction i)$,

(b) $\alpha^\dagger \leq \alpha$,

(c) either $\alpha^\dagger = \alpha$ or $F(\bar{Q})$ is not an $(\mathrm{Rlim}\bar{Q})$-name of a forcing notion.

(2) Suppose $\bar{Q}$ is an $\kappa-\mathrm{Sp}_e$-iteration of length α and $\beta < \alpha$, $G_\beta \subseteq P_\beta$ is generic over V, *then* in $V[G_\beta]$, $\bar{Q}/G_\beta = \langle P_i/G_\beta, \underset{\sim}{Q}_i : \beta \le i < \alpha\rangle$ is an Sp_e-iteration and $\mathrm{Sp}_e - \mathrm{Lim}(\bar{Q}) = P_\beta * (\mathrm{Sp}_e - \mathrm{Lim}\bar{Q}/\underset{\sim}{G}_\beta)$ (essentially).

(3) If $\bar{Q}$ is an $\kappa - \mathrm{Sp}_2$-iteration, $p \in \mathrm{Sp}_2 - \mathrm{Lim}(\bar{Q})$, $P'_i = \{q \in P_i : q \ge p{\restriction}i\}, \underset{\sim}{Q}'_i = \{p \in \underset{\sim}{Q}_i : p \ge p{\restriction}\{i\}\}$ *then* $\bar{Q} = \langle P'_i, Q'_i : i < \ell g\bar{Q}\rangle$ is (essentially) an Sp_e iteration (and $\mathrm{Sp}_2\mathrm{Lim}(\bar{Q}')$ is $P'_{\ell g\bar{Q}}$).

Proof. Should be clear. $\square_{2.12}$

§3. Axioms

We can get from the lemma of preservation of forcing with $(S,\gamma) - Pr_1^+$ by $\kappa - Sp_2$ iteration (and on the λ-c.c. for then) forcing axioms. We list below some variations.

3.1 Notation. 1) Reasonable choices for S are

(A) $S^0_\kappa = \mathrm{RUCar}_{\le\kappa} = \{\mu : \mu \text{ a regular cardinal, } \aleph_0 < \mu \le \kappa\}$

(B) $S^2_\kappa = \{2\} \cup \mathrm{RCar}_{\le\kappa} = \{2\} \cup \{\mu : \mu \text{ a regular cardinal, } \aleph_0 \le \mu \le \kappa\}$

(C) If we write "$< \kappa$" instead $\le \kappa$ (and $S^\ell_{<\kappa}$ instead S^ℓ_κ) the meaning should be clear.

2) [Convention] $\mathbf{E}^*$ is the class of strongly inaccessible cardinal $> \kappa$.

3.2 Fact. Suppose the forcing notion P satisfies $(S,\gamma) - \mathrm{Pr}_1$

(1) If $2 \in S$ *then* P does not add any bounded subset of γ.

(2) If μ is regular, and $\lambda_i (i < \mu)$ are regular, and $\{\mu\} \cup \{\lambda_i : i < \mu\} \subseteq S$, D here is a uniform ultrafilter on $\mu, \theta = \mathrm{cf}(\prod_{i<\mu} \lambda_i/D)$ (λ_i-as an ordered set) *then* P satisfies $(S \cup \{\theta\}, \gamma') - \mathrm{Pr}_1$ whenever $\mu\gamma' \le \gamma$, ($\mu\gamma'$ is ordinal multiplication). We can do this for all such θ simultaneously.

(3) If $\lambda \in S$ is regular, $\mu < \gamma$ *then* for every $f : \mu \to \lambda$ from V^P, for some $g : \mu \to \lambda$ from V for every $\alpha < \mu$, $f(\alpha) < g(\alpha)$.

Proof. (1) and (3) are clear.
For (2), fix an increasing cofinal (modulo D) sequence $\langle f^\theta_\alpha : \alpha < \theta\rangle$ in $\prod\limits_{i<\mu_\theta} \lambda^\theta_i$ if $\theta = \mathrm{cf}(\prod\limits_{i<\mu} \lambda_i/D_\theta)$ with $\lambda^\theta_i \in S$, $\mu = \mu_\theta \in S$.
To play the game $\eth_1$ for $(S\cup\{\theta\}, \gamma')-\mathrm{Pr}_1$ from 2.4(b), player II on the side plays the $\eth_2$ game for (S, γ) but for move β in $\eth_1$, he uses moves $\langle \zeta : \bigcup_{\gamma<\beta} \zeta_\gamma \le \zeta < \zeta_\beta\rangle$ in $\eth_2$, he also chooses ζ_α's during the play. If player I chooses a $\lambda_\beta \in S, \mathcal{I}_\beta$, in $\eth_1$, player II copies I's move to $\eth_2$ and plays his answer from there and let $\zeta_\beta = \bigcup_{\gamma<\beta} \zeta_\gamma + 1$. If player I plays in the β-th move $\lambda_\beta = \mathrm{cf}(\prod_{j<\mu} \lambda_{\beta,j}/D_\beta)$, $\mathcal{I}_\beta$, player II simulates μ moves of $\eth_2$:

$$\langle \lambda_{\beta,j},\, f^{\lambda_\beta}_{\mathcal{I}_\beta}(j),\, \alpha^\beta_j : j < \mu\rangle.$$

Then player II finds α_i such that $\{j : \alpha^\beta_j < f_{\alpha_i}(j)\} \in D$ and plays this α_i and let $\zeta_\beta = \bigcup_{\gamma<\beta} \zeta_\gamma + \mu$.
It is clear that α_i is as required, and as $\mu\gamma' \le \gamma$, $\eth_2$ does not end before $\eth_1$. $\square_{3.2}$

3.3 Claim. Suppose $\mathrm{MA}_{<\kappa}$ holds (i.e., for every P satisfying the $\aleph_1$-c.c. and dense $\mathcal{I}_i \subseteq P$ (for $i < \alpha < \kappa$) there is a directed $G \subseteq Q$ such that $\bigwedge_{i<\alpha} G\cap\mathcal{I}_i \neq \emptyset$). *Then* the following forcing notions are equivalent to forcing notions having the $(\mathrm{RUCar}, \kappa) - \mathrm{Pr}_1$.

(1) Mathias forcing; $\{(w, A) : w \subseteq \omega \text{ finite}, A \subseteq \omega \text{ infinite}\}$ with the order

$$(w_1, A_1) \le (w_2, A_2) \text{ iff } w_1 \subseteq w_2 \subseteq w_1 \cup A_1,\ A_2 \subseteq A_1.$$

(2) The forcing from VI §6(=[Sh:207], Sect. 2).

Proof. (1) Let P' be the set of (w, A, B) satisfying: $w \subseteq \omega$ finite, $B \subseteq \omega$ infinite, $B \subseteq A \subseteq \omega$, with the order

$$\begin{aligned}(w_1, A_1, B_1) \le (w_2, A_2, B_2) &\text{ iff } (w_1, A_1) \le (w_2, A_2)\\ &\text{and } B_2 \subseteq^* B_1 (\text{i.e. , } B_2 \setminus B_1 \text{ finite})\end{aligned}$$

$$(w_1, A_1, B_1) \leq_0 (w_2, A_2, B_2) \text{ iff: } w_1 = w_2$$
$$A_1 = A_2$$
$$B_2 \subseteq^* B_1.$$

Let us check Definition 2.1: (i) – (iv) easy.

Note that $\{(w, A, A) : (w, A, A) \in P'\}$ is dense in P', and isomorphic to P.

Proof of (v). Let $\mu > \aleph_0$ be a regular cardinal, $\underset{\sim}{\tau}$ a P'-name, $\Vdash_P$ "$\underset{\sim}{\tau} < \mu$. Let $p = (w, A, B)$ be given. Choose by induction on $i < \omega, n_i$, B_i such that

(a) $B_0 = B(\subseteq A)$

(b) $n_i = \mathrm{Min}(B_i)$

(c) $B_{i+1} \subseteq B_i \setminus \{n_i\}$

(d) for every $u \subseteq \{0, 1, 2, \dots, n_i\}$ (not just $\subseteq \{n_0, n_1, \dots, n_i\}$!) one of the following occurs:

for some $\alpha_{i,u} < \mu$, we have $(u, B_{i+1}, B_{i+1}) \Vdash_P$ "$\underset{\sim}{\tau} = \alpha_{i,u}$"

or for no infinite $C \subseteq B_{i+1}$ and $\alpha < \mu$ do we have $(u, C, C) \Vdash$ "$\underset{\sim}{\tau} = \alpha$"

There is no problem to do this, now $q \overset{\text{def}}{=} (w, A, \{n_i : i < \omega\})$ satisfies:

(e) $p \leq q \in P'$ and even $p \leq_0 q$.

(f) $q \Vdash_{P'}$ "$\underset{\sim}{\tau} \in \{\alpha_{i,u} : i < \omega \text{ and } u \subseteq \{0, 1, 2, \dots, n_i\}\}$".

[Why? If not, then for some $\alpha \in \mu \setminus \{\alpha_{i,u} : i < \omega \text{ and } u \subseteq \{0, 1, \dots, n_i\}\}$ and r we have $q \leq r \in P'$ and $r \Vdash_{P'}$ "$\underset{\sim}{\tau} = \alpha$". Let $r = (v, A', B')$ so $B' \subseteq A'$, B' is infinite, $B' \subseteq^* \{n_i : i < \omega\}$ and $A' \subseteq A$. As v is finite and by the definition of $\subseteq^*$ there is $i < \omega$ such that: $v \subseteq \{0, \dots, n_i\}$ and $B' \setminus \{0, \dots, n_i\} \subseteq \{n_j : j < \omega\}$. So without loss of generality $\mathrm{Min}(B') > n_i$, and $A' = B'$; so by the choice of B_{i+1}, $(v, B_{i+1}, B_{i+1}) \Vdash$ "$\underset{\sim}{\tau} = \alpha_{i,u}$", but $\alpha \neq \alpha_{i,v}$ so (v, B_{i+1}, B_{i+1}), $r = (v, A', B')$ are incompatible, contradiction]. So q is as required.

Proof of (vi). Suppose $p_i (i < \gamma)$ is $\leq_0$-increasing so $p_i = (w, A, B_i)$ and $B_i \subseteq A$, and B_i is $\subseteq^*$ decreasing. It is well known that for $\gamma < \kappa$, $MA_{<\kappa}$ implies the existence of an infinite $B \subseteq \omega$ such that $(\forall i < \gamma) B \subseteq^* B_i$. Now $(w, A, B) \in P'$ and $i < \gamma \Rightarrow p_i \leq_0 (w, A, B)$, as required.

(2) Left to the reader (similar to the proof of (1)). $\square_{3.3}$

3.4 Discussion - Proofs. Let $\kappa < \lambda$, λ regular. Each of the following gives rise naturally to a forcing axiom, stronger as λ is demanded to be a larger cardinal (so if λ is supercompact we get parallels to PFA).

If φ is a property of forcing notions, let $Ax_{<\alpha}(\varphi, \lambda, \mu)$ be the following statement: For every forcing notion P of size $< \mu$ if P satisfies φ and $\bar{\mathcal{I}} = \langle \mathcal{I}_i : i < i^* < \lambda \rangle$ is a sequence of dense subsets of P and $\langle (\kappa_j, \underset{\sim}{S}_j) : j < j^* < \alpha \rangle$ a sequence of pairs, with $\underset{\sim}{S}_j$ a P-name of stationary subset of κ_j, where κ_j is a regular uncountable cardinal $< \lambda$ *then* there is a $\bar{\mathcal{I}}$-generic subset G of P such that (as say $i < i^* \Rightarrow \underset{\sim}{\mathcal{I}}_i \cap G \neq \emptyset$ and) $j < j^* \Rightarrow \underset{\sim}{S}_j[G]$ a stationary subset of κ_j.

Case I. We assume κ is successor cardinal and use $\bar{Q}$ of length λ, a $\kappa - \mathrm{Sp}_2$-iteration, $\Vdash_{P_i}$ "$|Q_i| < \lambda$", each Q_i having (S^ℓ_κ, κ)- Pr^+_1 and $\ell \in \{0, 2\}$ and (usually) $\diamondsuit_{\{\mu : \mu < \lambda \text{ is strongly inaccessible}\}}$.

Now $P_\lambda = \kappa - \mathrm{Sp}_2\text{-}\mathrm{Lim}_\kappa \bar{Q}$ have the $(S^\ell_\kappa, \kappa) - \mathrm{Pr}^+_1$ by 2.15, so all regular $\mu \leq \kappa$ remain regular and every $\lambda' \in (\kappa, \lambda)$ is collapsed (in the general case i.e. if $\bar{Q}$ "generic" enough). But λ is not collapsed if it is strongly inaccessible (by 2.17(2)). If $2 \in S^\ell_\kappa$, no bounded subset of κ is added. We can get $Ax_{<\kappa}\Big((S^\ell_\kappa, \kappa) - \mathrm{Pr}_1, \lambda, \lambda^+\Big)$. Note: if λ is in V, supercompact with Laver diamond, we get $Ax\Big((S^\ell_\kappa, < \kappa) - \mathrm{Pr}_1, \lambda, \infty\Big)$ (see VII).

So (even if we assume $\mathbf{E}^* = \emptyset$) the theorems of §2 are strong enough to deal with such iterations get forcing axioms etc. Of course you may then look for forcing notions which can serve as iterant, of course κ-complete and θ-complete θ^+-c.c. forcing notions can serve. For some more see §4, §5 below.

Case II. Like Case I (but κ may be limit $> \aleph_0$) with $(\kappa + 1) - \mathrm{Sp}_2$-iteration each $\underset{\sim}{Q}_i$-having $(S^\ell_\kappa, \kappa + 1) - \mathrm{Pr}^+_1$ and every $\lambda' \in (\kappa, \lambda)$ is collapsed. Here we can get $Ax_{<\lambda}\Big((S^\ell_\kappa, \kappa + 1) - \mathrm{Pr}^+_1, \lambda, \lambda^+\Big)$. Here λ is not collapsed (even P_λ satisfies the λ-c.c.) if it is strongly inaccessible Mahlo (by 2.17(4)). If λ is supercompact with Laver diamond we get $Ax_{<\lambda}\Big((S^\ell_\kappa, \kappa + 1) - \mathrm{Pr}, \lambda, \infty\Big)$.

The situation is similar to that of case 1: this time better using a non empty $\mathbf{E}^*$ e.g. the one of 3.1(2).

Case III: Like case 1 but $\underset{\sim}{Q}_i$ satisfies $(S^\ell_{\kappa_i}, \kappa_i) - \mathrm{Pr}^+_1$, $\kappa_{i+1} >$density(P_i) and κ_i strictly increasing with i. So in V^{P_λ}, λ is still inaccessible (though not strongly inaccessible).

Here we better do a variant of §2 (i.e. 2.6A -2.21) without κ. Let $\mathbf{E}^*$ be the class of strongly inaccessible $> \aleph_0$. In Definition 2.7 there, the restriction of $|p|$ for $p \in \kappa - \mathrm{Sp}_2 - \mathrm{Lim}\bar{Q}$ is only: $\beta \in \mathbf{E}^* \cap E(\bar{Q}) \Rightarrow$ for some $\gamma < \beta$, $p{\restriction}\beta \in P_\gamma$ (this change 2.7(D), the above statement replaces (A)(i)). For any $\bar{Q}$ and $\beta < \alpha$ $(= \ell g(\bar{Q})$ we define a partial order $\leq_{0,\beta}$ on P_α: $p \leq_{0,\beta} q$ iff $p{\restriction}\beta = q{\restriction}\beta$ and $p \leq_0 q$. Now 2.10 is changed to

$(*)$ if $i \in (\beta, \alpha) \Rightarrow \Vdash_{P_i} (Q_i, \leq^{Q_i}_0)$ is θ-complete then $(P_\alpha, \leq_{0,\beta})$ is θ-complete.

In Claim 2.12 we can omit clause 2.12(2)(d).

In Claim 2.12 becomes

$(*)$ for (our kind of) $\bar{Q}$, and $\beta < \alpha = \ell g(\bar{Q})$, and regular θ assume $i \in \beta(\alpha) \Rightarrow \Vdash_{P_i}$ "Q_i satisfies $(\{\theta\}; \aleph_1) - \mathrm{Pr}^+_1$, and P_β satisfies the θ-c.c. and $p \Vdash_{P_\alpha}$ "$\underset{\sim}{\tau} < \theta$" then for some q and ζ, we have $p \leq_{0,\beta} q \in P_\alpha$, and $q \Vdash_{P_\alpha}$ "$\underset{\sim}{\tau} < \zeta < \theta$".

Case VI: Like case I but κ is an uncountable inaccessible (possibly weakly) cardinal.

The problem with applying §2 is rooted in assumption (d) in 2.12(2), which is needed for the iteration as presented. We should change 2.7 as follows in 2.7(D) allow $i^* \leq \kappa$, but demand $\Vdash_{\bar{Q}}$ "$\{\zeta_{\underset{\sim}{q}_i}[G] : i < i^*\}$ has order type $< \kappa$".

Of course we should assume each P_i has at least $(\{\kappa\}, \aleph_1) - \mathrm{Pr}^+_1$. However, does this really add compared to Case II?

3.5 Conclusion. Suppose λ is strongly inaccessible, limit of measurables, $\lambda > \kappa$, κ successor. Then for some λ-c.c. forcing P not adding bounded subsets of κ, $|P| = \lambda$, and $\Vdash_P$ "$2^\kappa = \lambda = \kappa^+$, and for every $A \subseteq \kappa$ there is a countable subset of λ not in $[L(A)]$".

Proof: Use case I of 3.4. We use $\kappa - \mathrm{Sp}_2$-iteration $\langle P_i, \underset{\sim}{Q}_i : i < \lambda\rangle$, $|P_i| < \lambda$. For i even: let κ_i be the first measurable $> |P_i|$, (but necessarily $< \lambda$). Then Q_i is

Prikry forcing on κ_i and Q_{i+1} is Levy collapse of κ_i^+ to κ. (Compare X 5.5.)

$\square_{3.5}$

§4. On Sacks Forcing

We continue 3.3, 3.4. Assume for simplicity λ is strongly inaccessible, $\lambda > \kappa$. We want to show that we can find an $\kappa-\mathrm{Sp}_2$-iteration which force some $Ax[\ldots]$. A natural way is to use a preliminary forcing notion R.

A natural candidate is: $R = \{\bar{Q} : \bar{Q} \in H(\lambda), \bar{Q}$ an $\kappa - \mathrm{Sp}_2$-iteration of forcing notions satisfying $(S,\kappa) - \mathrm{Pr}_1^+\}$. As an example, we will prove this for Sacks forcing.

4.1 Lemma. Suppose

(i) R is an $\aleph_1$-complete forcing notion.

(ii) For $r \in R$, $\bar{Q}^r = \langle P_i^r : i \le \alpha^r \rangle$, P_i^r is $\lessdot$-increasing in i and if $j \le \alpha^r$ has cofinality ω_1, *then* every countable subset of ω, which belongs to $V^{P_j^r}$ belongs to $V^{P_i^r}$ for some $i < j$. We write P^r for $P_{\alpha^r}^r$

(iii) If $r^1 \le r^2$ then $\bar{Q}^{r^1} \le \bar{Q}^{r^2}$ (end extension), so $P^{r_1} \lessdot P^{r_2}$

(iv) If $r \in R$ and $\underset{\sim}{Q}$ is a $P_{\alpha^r}^r$-name of a forcing notion, *then* for some $r^1 \ge r$, $\alpha^{r_1} > \alpha^r$ and $P_{\alpha^r+1}^{r_1} = P_{\alpha^r} * \underset{\sim}{Q}'$ and $\Vdash_{P_{\alpha^{r_1}}^{r_1}}$ " if $\underset{\sim}{Q}$ does satisfy the c.c.c. then $\underset{\sim}{Q}' = \underset{\sim}{Q}$"

(v) If r^ζ for $(\zeta < \delta)$ is increasing, $\delta \le \omega_1$, *then* for some r

$$\bigwedge_{\zeta<\delta} r^\zeta \le r \text{ and } \alpha_r = \bigcup_{\zeta<\delta} \alpha_{r^\zeta}.$$

Let $P[\underset{\sim}{G}_R]$ be $\bigcup\{P_i^r : r \in \underset{\sim}{G}_R, i \le \alpha_r\}$, so it is an R-name of a forcing notion. *Then* $\Vdash_R [\Vdash_{P[\underset{\sim}{G}_R]}$ "for any $\aleph_1$ dense subsets of Sacks forcing, there is a directed subset of Sacks forcing not disjoint to any of them"].

Remark. Remember $Q_{Sacks} = \{\tau : \tau \subseteq {}^{\omega>}2$ is closed under initial segments nonempty and $(\forall \eta \in \tau)(\exists \nu)(\eta \lhd \nu \;\&\; \nu\char`^\langle 0\rangle \in \tau \;\&\; \nu\char`^\langle 1\rangle \in \tau)\}$ and $\tau_1 \le \tau_2$ if $\tau_2 \subseteq \tau_1$.

Proof: Let for $\xi < \omega_1$, $\underset{\sim}{\mathcal{I}}_\xi$ be $R * P[\underset{\sim}{G}_R]$-name of dense subset of $Q_{Sacks}^{R*P[\underset{\sim}{G}_R]}$ for $\xi < \omega_1$ (Q^V_{Sacks} is Sacks forcing in the universe V). W.l.o.g. the $\underset{\sim}{\mathcal{I}}_\xi$ are open. We will find a c.c.c. subset $\underset{\sim}{Q}'$ of $Q^{R*P[G_R]}_{Sacks}$ such that $\underset{\sim}{\mathcal{I}}_\xi \cap \underset{\sim}{Q}'$ is dense in $\underset{\sim}{Q}'$ for each $\xi < \omega_1$. Then any generic subset of $\underset{\sim}{Q}'$ intersects all $\underset{\sim}{\mathcal{I}}_\xi$'s.

For a subset E of Sacks forcing let var(E) be $\{(n, \tau) : \tau \in E, n < \omega\}$ ordered by $(n_1, \tau_1) \le (n_2, \tau_2)$ iff $n_1 \le n_2$, $\tau_2 \subseteq \tau_1$ and $\tau_1 \cap {}^{n_1\ge} 2 = \tau_2 \cap {}^{n_1\ge} 2$. (If $D \subseteq \text{var}(Q_{Sacks})$ is sufficiently generic, then $r_D = \cup\{\tau{\restriction}n : (n, \tau) \in d\}$ is a condition in Q_{Sacks}). We now define by induction on $\zeta \le \omega_1$, $r(\zeta)$, and D_ζ such that (the order as the one on Q_{Sacks}):

(a) $r(\zeta) \in R$ is increasing, $\alpha_{r(\zeta)}$-increasing continuous.

(b) D_ζ is a $P^{r(\zeta+1)}$-name of a countable subset of Q_{Sacks}.

(c) If $\tau \in D_\zeta$, $\eta \in \tau$ then $\tau_{(\eta)} \overset{\text{def}}{=} \{\nu : \eta \,\hat{}\, \nu \in \tau\}$ belongs to D_ζ. (We use round parentheses to distinguish it from $\tau_{[\eta]}$, see clause (f)).

(d) If $\tau_1, \tau_2 \in D_\zeta$ then $\{\langle\,\rangle, \langle 0\rangle\hat{}\,\eta : \eta \in \tau_1\}$, $\{\langle\rangle, \langle 1\rangle\hat{}\,\eta : \eta \in \tau_2\}$ and their union belongs to D_ζ.

(e) Let $\xi < \zeta$, then for every $\tau_1 \in D_\xi$ there is $\tau_2 \in D_\zeta$ such that $\tau_1 \le \tau_2$ and for every $\tau_2 \in D_\zeta$ there is $\tau_1 \in D_\xi$ such that $\tau_1 \le \tau_2$.

(f) If $\tau \in D_\zeta$ then for some n for every $\eta \in {}^n2 \cap \tau$ we have $\tau_{[\eta]} \overset{\text{def}}{=} \{\nu \in \tau : \nu \trianglelefteq \eta$ or $\eta \trianglelefteq \nu\}$ belongs to $\underset{\sim}{\mathcal{I}}_\zeta$ (i.e. is forced ($\Vdash_R$) to belong to it).

(g) Suppose ζ is limit, then $P^{r(\zeta+1)}_{\alpha_r(\zeta)+1} = P^{r(\zeta)}_{\alpha_r(\zeta)} * Q_\zeta$, Q_ζ is $\left[\text{var} \bigcup_{\xi<\zeta} D_\xi\right]^\omega$ (the ω-th power, with finite support).

(h) the generic subset of $\underset{\sim}{Q}_\zeta$ gives a sequence of length ω of Sacks conditions; closing the set of those conditions by (c) + (d) + (f) we get D_ζ.

We have to prove that Q_ζ satisfies the $\aleph_1$-c.c. in $V^{R*P_{\alpha_r(\zeta)}}$: (to get a generic subset by (iv)). If $\zeta < \omega_1$ this follows by countability. Let $\zeta = \omega_1$. It suffices to prove that $\left[\text{var} \bigcup_{\xi<\zeta} D_\xi\right]^n$ satisfies the $\aleph_1$-c.c. where $n < \omega$. So let $\underset{\sim}{\mathcal{J}}$ be a $R * \underset{\sim}{P}^{r(\zeta)}_{\alpha_r(\zeta)}$-name of a maximal antichain of $\left[\text{var} \bigcup_{<\zeta} D_\xi\right]^n$. We can find a $\xi < \zeta$, $\text{cf}(\xi) = \aleph_0$ such that $\mathcal{I}^*_\xi \overset{\text{def}}{=} \{x : x \in V^{R*\underset{\sim}{P}^{r(\zeta)}}_{\alpha_r(\zeta)}$ and every

$p \in (R * \underset{\sim}{P}^{r(\zeta)}_{\alpha_r(\zeta)})/(R * \underset{\sim}{P}^{r(\xi)}_{\alpha_r(\xi)})$ force x to be in $\mathcal{J}\}$ is pre-dense in $\left[\text{var} \bigcup_{\gamma<\xi} D_\gamma\right]^n$ (exists by (e) and assumption (i)). Check the rest.

Notice that we have used:

(a) *if* $Y \subseteq \bigcup_{\xi<\xi_0} D_\xi$, $\xi_0 \leq \zeta$, $Y \in V^{P_{\alpha(\xi_0)}}$ and Y is a pre-dense subset of $\bigcup_{\xi<\xi_0} D_\xi$ (it does not matter where but e.g., in V^P) *then* Y is a pre-dense subset of $\bigcup_{\xi<\zeta} D_\xi$; *because*

(a1) every $\tau \in D_{\xi_0}$ is included in a finite union of members of Y.

(a2) every $\tau \in \bigcup_{\xi<\varepsilon<\zeta} D_\varepsilon$ is included in some member of D_{ξ_0}.

4.2 Remark. 1) This argument works for many other forcing notions, e.g., Laver forcing.

2) The *var*(*Sacks*) was introduced by author to show Sacks forcing may not collapsed $\aleph_2$ (see Baumgartner and Laver [BL]).

3) In later work Velickovic get results for $> \aleph_1$ dense sets.

§5. Abraham's Second Problem – Iterating Changing Cofinality to ω

5.1 Definition. Let S be a subset of $\{2\} \cup \{\kappa : \kappa \text{ is regular cardinal}\}$, D a filter on a cardinal λ (or any other set). For any ordinal γ, we define a game $\Game^*(S, \gamma, D)$. It lasts γ moves. In the i-th move player I choose a cardinal $\kappa_i \in S$ and function F_i from λ to κ_i and then player II chooses $\alpha_i < \kappa_i$.

Player II wins a play if for every $i < \gamma$,

$$d(\langle \kappa_j, F_j, \alpha_j : j < i \rangle) \stackrel{\text{def}}{=} \{\zeta < \lambda : \text{ for every } j < i \text{ we have}$$

$$[\kappa_i = 2 \Rightarrow F_j(\zeta) = \alpha_j] \text{ and } [\kappa_j > 2 \Rightarrow F_j(\zeta) < \alpha_j]\} \neq \emptyset \mod D.$$

5.1A Remark.

(1) This is similar to the game of X4.9, but there we also demand $d(\langle \kappa_j, F_j, \alpha_j : j < \gamma) \neq \emptyset \bmod D$.

(2) If not said otherwise, we assume that $\lambda \setminus \{\zeta\} \in D$ for $\zeta < \lambda$.

(3) If D is an ultrafilter on λ which is κ^+-complete for each $\kappa \in S$ and $|\gamma|^+$-complete (if γ a cardinal γ-complete) *then* player II has a winning strategy (if γ is a cardinal, γ-completeness suffices).

(4) Of course only $F_i{\restriction}d(\langle \kappa_j, F_j, \alpha_j : j < i\rangle)$ matters so player I can choose only it.

5.2 Definition. For $\mathbf{F}$ a winning strategy for player II in $\partial^*(S, \gamma, D)$, D a filter on λ (we write $\lambda = \lambda(D)$), we define $Q = Q_{\mathbf{F},\lambda} = Q_{\mathbf{F},S,\gamma,D}$, with $Q = (|Q|, \leq, \leq_0)$ as follows.

Part A. Let $(T, H) \in Q$ iff

(i) T is a nonempty set of finite sequences of ordinals $< \lambda$.

(ii) $\eta \in T \Rightarrow \eta{\restriction}\ell \in T$, and for some (unique) n and η of length n we have: $T \cap {}^{n\geq}\lambda = \{\eta{\restriction}\ell : \ell \leq n\}$, $|T \cap {}^{n+1}\lambda| \geq 2$; we call η the trunk of T, $\eta = \mathrm{tr}(T) = \mathrm{tr}(T, H)$ (it is unique).

(iii) H is a function, $T \setminus \{\mathrm{tr}(T){\restriction}\ell : \ell < \ell g(\mathrm{tr}(T))\} \subseteq \mathrm{Dom}(H) \subseteq {}^{\omega>}\lambda$.

(iv) for each $\eta \in \mathrm{Dom}(H)$, $H(\eta)$ is a proper initial segment of a play of the game $\partial^*(S, \gamma, D)$ in which player II use his strategy $\mathbf{F}$ so $H(\eta) = \langle \lambda_i^{H(\eta)}, F_i^{H(\eta)}, \alpha_i^{H(\eta)} : i < i^{H(\eta)}\rangle$ and $i^{H(\eta)} < \gamma$.

(v) if $\mathrm{tr}(T) \trianglelefteq \eta \in \mathrm{Dom}(H) \cap T$ we have $\{\zeta < \lambda : \eta\hat{\ }\langle\zeta\rangle \in T\} = d(H(\eta))$ (see Definition 5.1).

(vi) convention: if $p = (T, H)$ we may write $\eta \in p$ for $\eta \in T$.

Part B. $(T_1, H_1) \leq (T_2, H_2)$ (where both belong to Q) iff $T_2 \subseteq T_1$ and for each $\eta \in T_2$, if $\mathrm{tr}(T_2) \trianglelefteq \eta$ then $H_1(\eta)$ is an initial segment of $H_2(\eta)$.

Part C. $(T_1, H_1) \leq_0 (T_2, H_2)$ (where both belong to Q) if $(T_1, H_1) \leq (T_2, H_2)$ and $\mathrm{tr}(T_1) = \mathrm{tr}(T_2)$.

5.2A Remark . (1) So if $(T, H) \in Q_{\mathbf{F},\lambda}$ and $\mathbf{F}, \lambda, D, \gamma, S$ are as above, $\eta \in T, \eta \trianglerighteq \mathrm{tr}(T)$ *then* $d(H(\eta)) \neq \emptyset \bmod D$. (So this forcing is similar to Namba forcing, but here we have better control of the sets $Suc_T(\eta)$.)

(2) We can of course generalize this to cases where we have different strategies (and even different λ's and D's) in different nodes.

(3) If $(T, H_\ell) \in Q_{F,\lambda}$ for $\ell = 1,2$ $H_1 \restriction T = H_2 \restriction T$ then $(T, H_1), (T, H_2)$ are equivalent (see Chapter II).

5.2B Notation . For $p = (T,\ H) \in Q_{\mathbf{F},\lambda}$ and $\eta \in T$ let $p^{[\eta]} = (T^{[\eta]},\ H)$, where $T^{[\eta]} = \{\nu \in T : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\}$. Clearly $p \le p^{[\eta]} \in Q_{\mathbf{F},\lambda}$.

5.3 Lemma. If $Q = Q_{\mathbf{F},S,\gamma,D}$ and D is a uniform filter on $\lambda(D)$ *then*

$$\Vdash_Q \text{“cf}[\lambda(D)] = \aleph_0\text{”}.$$

Proof. Let $\underset{\sim}{\eta}_Q = \bigcup\{\mathrm{tr}(p) : p \in \underset{\sim}{G}_Q\}$.

Clearly if $(T_\ell,\ H_\ell) \in \underset{\sim}{G}_Q$ for $\ell = 1,2$ then for some $(T,\ H) \in \underset{\sim}{G}_Q$, $(T_\ell,\ H_\ell) \le (T,\ H)$; hence $\mathrm{tr}(T_\ell) \trianglelefteq \mathrm{tr}(T)$, hence $\mathrm{tr}(T_1,\ H_1) \cup \mathrm{tr}(T_2, H_2)$ is in ${}^{\omega>}\lambda$. Hence $\underset{\sim}{\eta}_Q$ is a sequence of ordinals of length $\le \omega$.

For every $p = (T,\ H) \in Q$, and n, there is $\eta \in T \cap {}^n\lambda$, hence $p \le p^{[\eta]} \in Q$ (see 5.2B), and $p^{[\eta]} \Vdash$ “$\ell g(\underset{\sim}{\eta}_Q) \ge n$” because $\eta \trianglelefteq \mathrm{tr}(p^{[\eta]})$ and for every $q \in Q$ we have $q \Vdash_Q$ “$\mathrm{tr}(q) \trianglelefteq \underset{\sim}{\eta}_Q$”. So $\Vdash_Q$ “$\underset{\sim}{\eta}_Q$ has length $\ge n$” hence $\Vdash_Q$ “$\underset{\sim}{\eta}_Q$ has length ω”.

Obviously, $\Vdash_Q$ “$\mathrm{Rang}(\underset{\sim}{\eta}_Q) \subseteq \lambda$”. Why $\Vdash_Q$ “$\sup \mathrm{Rang}(\underset{\sim}{\eta}_Q) = \lambda$”? Because for every $(T,\ H) \in Q$ and $\alpha < \lambda$, letting $\eta \overset{\text{def}}{=} \mathrm{tr}(T)$, clearly $d(H(\eta)) \ne \emptyset \bmod D$ (see Definition 5.2) but D is uniform, hence there is $\beta \in d(H(\eta))$, $\beta > \alpha$, so $\eta\,\hat{}\,\langle\beta\rangle \in T$, and $(T,\ H) \le (T,\ H)^{[\eta\,\hat{}\,\langle\beta\rangle]} \in Q$ and $(T,\ H)^{[\eta\,\hat{}\,\langle\beta\rangle]} \Vdash_Q$ “$\eta\,\hat{}\,\langle\beta\rangle \le \underset{\sim}{\eta}_Q$” hence $(T,\ H)^{[\eta\,\hat{}\,\langle\beta\rangle]} \Vdash$ “$\sup\mathrm{Rang}(\underset{\sim}{\eta}_Q) \ge \beta$”, as $\alpha < \beta$ we finish. $\square_{5.3}$

5.4 Lemma. If $\lambda, S,\ \gamma,\ D$ are as in Definition 5.1, $\aleph_0 \notin S, \mathbf{F}$ a winning strategy of player II in $\eth^*(S,\gamma,D)$ and $\mathrm{cf}(\gamma) > \aleph_0$, *then* $Q_{F,\lambda}$ satisfies $(S, \mathrm{cf}(\gamma)) - \mathrm{Pr}_1^+$ (see Definition 2.1(2)). [So if $2 \in S$, then forcing by $Q_{F,\lambda}$ add no bounded subsets of γ].

Proof. In Definition 2.1, parts (i), (ii), (iii), (iv) and part (vi) are clear. So let us check part (v). Let $\kappa \in S$, $\underset{\sim}{\tau}$ be a Q-name, $\Vdash_Q$ “$\underset{\sim}{\tau} \in \kappa$” and $p = (T,\ H) \in Q$. We define by induction on n, $p_n = (T_n,\ H_n)$ and $\langle \alpha_\eta : \eta \in T_n \cap {}^n\lambda\rangle$ such that:

(α) $p_0 = p$, $p_n \leq_0 p_{n+1}$, $T_n \cap {}^{n\geq}\lambda = T_{n+1} \cap {}^{n\geq}\lambda$

(β) if $\eta \in T_n \cap {}^{n}\lambda$, and there are $q \in Q$ and $\alpha < \kappa$ satisfying

($*$) If $p_n^{[\eta]} \leq_0 q \in Q$, $\alpha < \kappa$ and $q \Vdash$ "if $\kappa = 2$, $\underset{\sim}{\tau} = \alpha$, if $\kappa > \aleph_0$, $\underset{\sim}{\tau} < \alpha$"

then $p_{n+1}^{[\eta]}$, α_η satisfy this.

Let p_ω be the limit of $\langle p_n : n < \omega \rangle$, i. e., $p_\omega = (T_\omega, H_\omega)$ where $T_\omega \stackrel{\text{def}}{=} \bigcap_{n<\omega} T_n$ and $H_\omega(\eta)$ is the limit of the sequences $H_n(\eta)$ (for $\eta \in T_\omega \setminus \{\text{tr}(T){\restriction}\ell : \ell < \ell g(\text{tr}T)\}$). It is well defined as $\text{cf}(\gamma) > \aleph_0$ and $p_n \leq_0 p_\omega \in Q$. We now prove two facts:

5.4A Fact . If $p = (T, H) \in Q$ and $f : T \cap {}^{n+1}\lambda \to \kappa$ and $\kappa \in S$, *then* there is $p' = (T', H') \in Q$ and $\langle \beta_\eta : \eta \in {}^{n}\lambda \cap T \rangle$ with $\beta_\eta < \kappa$, such that:

(a) $p \leq_0 p'$

(b) $T_n \cap {}^{n\geq}\lambda = T' \cap {}^{n\geq}\lambda$

(c) for every $\eta \in T' \cap {}^{n}\lambda$ we have: $\kappa = 2$ and $f{\restriction}\text{Suc}_{T'}(\eta)$ is constantly $= \beta_\eta$ *or* $\kappa > \aleph_0$ and $\text{Rang}(f{\restriction}\text{Suc}_{T'}(\eta)) \subseteq \beta_\eta$.

Note that we may allow f to be a partial function; now if $\kappa = 2$ then $f{\restriction}\text{Suc}_{T'}(\eta)$ is defined on all or undefined on all. If $\kappa \geq \aleph_0$, $f{\restriction}\text{Suc}_{T'}(\eta)$ may be a partial function. Similarly in 5.4B.

Proof. For each $\eta \in T \cap {}^{n}\lambda$ we have: $H(\eta)$ is a proper initial segment of a play of the game $\eth^*(S, \gamma, D)$, and it lasts $i^{H(\eta)}$ moves. Player I could choose in his $i^{H(\eta)}$-th move the cardinal κ and the function $f_\eta : \lambda \to \kappa$,

$$f_\eta(\zeta) = f(\eta\hat{}\langle\zeta\rangle) \quad \text{(which is } < \kappa) \text{ if } \eta\hat{}\langle\zeta\rangle \in T$$
$$f_\eta(\zeta) = 0 \qquad \text{if otherwise.}$$

So, for some β_η, $H(\eta)\hat{}\langle\alpha, f_\eta, \beta_\eta\rangle$ is also a proper initial segment of a play of $\eth^*(S, \gamma, D)$ in which player II uses the strategy $\mathbf{F}$. So there is $p' = (T', H')$ such that $H'{\restriction}(T' \setminus {}^{n}\lambda) = H{\restriction}(T' \setminus {}^{n}\lambda)$ and $H'(\eta) = H(\eta)^\wedge\langle(\kappa, f_\eta, \beta_\eta)\rangle$ for $\eta \in T \cap {}^{n}\lambda$. (If is partial for $\kappa = 2$ we should do this twice: for definability and for value.)

$\square_{5.4A}$

We can easily show

5.4B Fact . If $p = (T, H) \in Q$, $m < \omega$, and $\kappa \in S$ and $f : T \to \kappa$, *then* for some $p_1 = (T_1, H_1) \in Q$, $p \leq_0 p_1$, and for every $k \leq m$ we have $[\kappa = 2 \;\&\; f{\restriction}(T_1 \cap {}^k\lambda)$ is constant$]$ or $[\kappa \geq \aleph_1 \;\&\; f{\restriction}(T_1 \cap {}^k\lambda)$ is bounded below $\kappa]$.

Proof. W.l.o.g. $m > \ell g(\mathrm{tr} p)$. We define by downward induction on $n \in [\ell g(\mathrm{tr} T), m]$ the condition r^n, $p \leq_0 r^{n+1} \leq_0 r^n \in Q$, r^n satisfying the conclusion of 5.4B for $p^{[\eta]}$ for every $\eta \in p$ of length n. For $n = m$ this is trivial. For $n < m$, use Fact 5.4A $m - n$ times, for $k \in (n, m]$ for the function $f_k^{n+1} : T^{r_{n+1}} \cap {}^{n+1}\lambda \to \kappa$ defined by: $f_k^{n+1}(\eta)$ is γ if

$$(\forall \nu)[\nu \in T^{r_{n+1}} \cap {}^k\lambda \to (\kappa = 2 \;\&\; f(\nu) = \gamma) \;\vee\; (\kappa > \aleph_0 \;\&\; f(\nu) < \gamma)];$$

now $r^{\ell g(\mathrm{tr} p)}$ is as required]. $\square_{5.4B}$

Continuation of the proof of 5.4. By repeated application of 5.4B we can define by induction on n, $q_n \in Q$ such that $q_0 = p_\omega$ (see before 5.4A) and $q_n \leq_0 q_{n+1}$ and β_η^n for $\eta \in T^{q_n}$, $\ell g(\eta) \leq n$ such that:

(a) $\beta_\eta^0 = \alpha_\eta$ if this is well-defined, β_η^0 =-1 otherwise (on α_η see (β) above).

(b) when $\kappa > \aleph_0$: $\ell g(\eta) \leq n \;\&\; \eta\hat{\ }\langle\zeta\rangle \in T^{q_{n+1}} \Rightarrow \beta_\eta^{n+1} \geq \beta_{\eta\hat{\ }<\zeta>}^n$ whenever the later is well defined.

(c) when $\kappa = 2$: $\ell g(\eta) \leq n \;\&\; \eta\hat{\ }\langle\zeta\rangle \in T^{q_{n+1}} \Rightarrow \beta_\eta^{n+1} = \beta_{\eta\hat{\ }<\zeta>}^n$ (so both are defined or both not defined).

Lastly let $q_\omega \in Q$ be such that $q_n \leq_0 q_\omega$ for $n < \omega$.

Now if $\kappa > \aleph_0$ (is regular), we claim

$$q_\omega \Vdash_Q \text{“}\underset{\sim}{\tau} \leq \bigcup_{n<\omega} \beta_{<>}^n\text{”}$$

Clearly $p \leq_0 q_\omega \in Q$ and $\bigcup_{n<\omega} \beta_{<>}^n < \kappa$ so this suffices. Why does this hold? If not, then for some q' and β, $q_\omega \leq q' \in Q$, $q' \Vdash_Q$ “$\underset{\sim}{\tau} = \beta$” and $\kappa > \beta > \bigcup_n \beta_{<>}^n$. Let $\eta = \mathrm{tr}(q')$, so $\eta \in T^{q_\omega}$, and α_η is well defined, and $> \beta$. But as $\eta \in \bigcap_{n<\omega} T^{q_n}$ and $\beta_{<>}^{\ell g(\eta)} \geq \beta_\eta^0 = \alpha_\eta$, and we get a contradiction.

If $\kappa = 2$, we note just that for some $\eta \in T^{q_\omega}$, the number α_η is well defined, hence $\beta_\eta^{\ell g(\eta)}$ is defined hence β_η^ℓ is defined for $\ell \in [\mathrm{tr}(q_\omega), \ell g(\eta)]$. $\square_{5.4}$

Remark. We can rephrase much of this lemma as a partition theorem on trees as in [RuSh:117].

5.5 Lemma . Suppose $\bar{Q} = \langle P_i, Q_i : i < \lambda \rangle$ is a $\kappa - \mathrm{Sp}_2$-iteration, $|P_i| < \lambda$ for $i < \lambda, \gamma \leq \kappa$, each Q_i has $(S, \gamma) - Pr_1^+$ and κ regular and even successor, $S \subseteq \{2\} \cup \{\theta : \theta \text{ regular uncountable } \leq \kappa\}$ and in V, D is a normal ultrafilter on λ (so λ is a measurable cardinal). *Then* $\Vdash_{P_\lambda}$ "player II wins $\partial^*(S, \gamma, D)$".

Proof: Let $A = \{\mu < \lambda : (\forall i < \mu)[|P_i| < \mu]$, μ strongly inaccessible Mahlo cardinal $> \kappa\}$.

Let $G_\lambda \subseteq P_\lambda$ be generic over V and for $\alpha < \lambda$ let $G_\alpha = G \cap P_\alpha$.

W.l.o.g. player I choose P_λ-names of functions and cardinals in S. Now we work in V and describe player II's strategy there (see proof of XIII 1.9). For each $\mu \in A$ the forcing notion P_λ/P_μ has $(S, \gamma) - \mathrm{Pr}_1^+$; hence, player II has a winning strategy $\underset{\sim}{\mathbf{F}}(P_\lambda/\underset{\sim}{G}_\mu) \in V[\underset{\sim}{G}_\mu]$ for the game from 2.1(1)(vi), so $\underset{\sim}{\mathbf{F}}(P_\lambda/\underset{\sim}{G}_\mu)$ is a P_μ-name, $\langle \underset{\sim}{\mathbf{F}}(P_\lambda/\underset{\sim}{G}_\mu) : \mu \rangle$ a P_λ-name. Let us describe a winning strategy for player II (for the game $\partial^*(S, \gamma, D)$).

So in the i-th move player I chooses $\underset{\sim}{\theta}_i \in S$ and $\underset{\sim}{f}_i : \lambda \to \underset{\sim}{\theta}_i$. Player II chooses in his i-th move not only $\underset{\sim}{\alpha}_i < \underset{\sim}{\theta}_i$ but also A_i, $\underset{\sim}{f}_i$, γ_i, $\langle \langle \underset{\sim}{p}_j^\mu : j \leq i \rangle : \mu \in A_i \rangle$ such that

(0) γ_i is an ordinal $< \lambda$,

(1) $j < i \Rightarrow \gamma_j < \gamma_i$

(2) $A_i \in D$, $A_i \in V$, $A_i \subseteq \bigcap_{j<i} A_j$ and $A_\delta = \bigcap_{j<\delta} A_j$

(3) $\Vdash$ "$\underset{\sim}{f}_i : \lambda \to \underset{\sim}{\theta}_i$, $\underset{\sim}{\theta}_i \in S$"

(4) for $\mu \in A_i$, $\langle \underset{\sim}{p}_j^\mu : j \leq 2i + 2 \rangle$ is a P_μ-name of an initial segment of a play as in (vi) of 2.1(1) for the forcing P_λ/G_μ, $p_{2j+1}^\mu \Vdash_{P_\lambda/G_\mu}$ "$\underset{\sim}{f}_j(\mu) = \underset{\sim}{\alpha}_j^\mu$ if $\theta_j = 2$, $f_j(\mu) < \alpha_j^\mu$ if $\theta_j \geq \aleph_0$", $\underset{\sim}{\alpha}_j^\mu$ a P_μ-name.

In the i-th stage clearly $A_i^0 \stackrel{\text{def}}{=} \bigcap_{j<i} A_j \cap A$ is in D, and let $\gamma_i^0 = \sup_{j<i}\gamma_i$, so $\gamma_i^0 < \lambda$ and choose $\gamma_i' \in (\gamma_i^0, \lambda)$ such that $\underset{\sim}{\theta}_i$ is a $P_{\gamma_i'}$-name. For every $\mu \in A_i^0$, $\mu > \gamma_i'$, we can define P_μ-names $\underset{\sim}{p}_{2i}^\mu$, $\underset{\sim}{p}_{2i+1}^\mu$, α_i^μ such that:

(a) $\Vdash_{P_\mu}$ "$\langle \underset{\sim}{p}_j^\mu : j < 2i+2\rangle$ is an initial segment of a part as in (v) of 1.1(1) for P_λ/P_μ in which player II uses his winning strategy $\underset{\sim}{\mathbf{F}}(P_\lambda/\underset{\sim}{G}_\mu)$.

(b) $p_{2i+1}^\mu \Vdash_{P_\lambda/P_\mu}$ "$\underset{\sim}{f}_i(\mu) = \underset{\sim}{\alpha}_i^\mu$ if $\underset{\sim}{\theta} = 2$, $\underset{\sim}{f}_i(\mu) < \alpha_i^\mu$ if $\underset{\sim}{\theta}_i \geq \aleph_0$".

Now as $\underset{\sim}{\alpha}_i^\mu$ is a P_μ-name of an ordinal $< \kappa \leq \mu$, it is $P_{\beta(\mu)}$-name for some $\beta[\mu] < \mu$ (as P_μ satisfies the μ-c.c. see 2.17(2)). By the normality of the ultrafilter D, on some $A_i^1 \subseteq A_i^0$, $\beta[\mu] = \beta_i$ for every $\mu \in A_i^1$. Let $\gamma_i = \gamma_i^1 + \beta_i$.

Easily for each $i < \sigma$, $\Vdash_{P_\lambda}$ "$\{\mu \in A_i : p_{2i+1}^\mu \in \underset{\sim}{G}_\lambda\} \neq \emptyset \bmod D$", so we finish.

$\square_{5.5}$

5.5A Comment.

We can present it (and the proof of XIII 1.4) slightly differently.

In V let

$$\mathbf{W}_i = \{\langle \bar{p}^\mu : \mu \in A\rangle : A \in D, \bar{p}^\mu = \langle \underset{\sim}{p}_j^\mu : j \leq i\rangle,$$
$$\Vdash_{P_\mu} \text{"}\langle \underset{\sim}{p}_j^\mu : j < i\rangle \text{ is a } \leq_0 \text{-increasing sequence in } P_\lambda/P_\mu\text{"}\}$$

and let

$$\mathbf{W} = \bigcup_i \mathbf{W}_i$$

We define on $\mathbf{W}$ a relation $\leq$ by:

$$\langle \bar{p}^{\mu,1} : \mu \in A_1\rangle \leq \langle \bar{p}^{\mu,2} : \mu \in A_2\rangle \text{ iff}$$
$$A_1 \supseteq A_2 \text{ and } \mu \in A_2 \Rightarrow \bar{p}^{\mu,1} \text{ is an initial segment of } \bar{p}^{\mu,2}$$

Clearly $\leq$ is a partial order on $\mathbf{W}$, and for $\mathbf{p} = \langle \bar{p}_j^\mu : j < i\rangle$ let

$$\underset{\sim}{B}(\mathbf{p}) = \{\mu \in A : \{p_j^\mu : j < i\} \subseteq \underset{\sim}{G}_\lambda\},$$

so clearly

(a) $\Vdash_{P_\lambda}$ "$\mathbf{p}^1 \le \mathbf{p}^2$ implies $\underset{\sim}{B}(\mathbf{p}^1) \supseteq B(\mathbf{p}^2)$"

(b) $\Vdash_{P_\lambda}$ "$\underset{\sim}{B}(\mathbf{p}) \ne \emptyset \mod D$"

(c) For every $\alpha < \lambda$, $\mathbf{p} \in \mathbf{W}_i$ and P_λ-names $\underset{\sim}{\theta} \in S$ and $\underset{\sim}{f} : \lambda \to \underset{\sim}{\theta}$ *there is* $\mathbf{q}$, $\mathbf{p} \le \mathbf{q} \in \mathbf{W}_{i+1}$ and a P_λ name $\underset{\sim}{\tau}$ of an ordinal $< \theta$ such that:

  (∗) if $\mathbf{q} = \langle \bar{q}^\mu : \mu \in A^q \rangle$, and $\mu \in A^q$, then $\underset{\sim}{q}^\mu_{i+1}$ forces

  $(\underset{\sim}{\theta} = 2 \ \&\ \underset{\sim}{f}(\mu) = \underset{\sim}{\tau}) \ \vee\ (\underset{\sim}{\theta} \ge \aleph_0 \ \&\ \underset{\sim}{f}(\mu) < \underset{\sim}{\tau})$

Now we can solve the second Abraham problem. (See also X 5.5.)

5.6 Conclusion . Suppose λ is strongly inaccessible $\{\mu < \lambda : \mu \text{ measurable}\}$ is stationary, $\kappa = \mathrm{cf}(\kappa) < \lambda$ a successor cardinal, $\kappa^+ \cap \mathrm{RUCar} \subseteq S \subseteq \{2\} \cup \{\theta : \theta \le \kappa \text{ regular uncountable}\}$. *Then* for some forcing notion P we have: $|P| = \lambda, P$ satisfies λ-c.c. and $(S,\kappa) - Pr_1^+$ and $\Vdash_P$ "$\lambda = |\kappa|^+$" (so $\Vdash_{P_\mu} 2^{|\kappa|} = \lambda$): and for every $A \subseteq \lambda$, for some $\delta < \lambda$, there is a countable set $a \subseteq \delta$, which is not in $V[A \cap \delta]$. We can also get suitable axiom (see 3.5).

Proof. Should be clear (see 3.4 Case I (and 5.4)). $\square_{5.6}$

5.6A Remark . 1) We can also prove (by the same forcing) the consistency of "there is a normal filter on λ to which $\{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ belongs which is precipitous" if in addition there is a normal ultrafilter on λ concentrating on measurables.

2) We can use $(S, < \kappa) - \mathrm{Pr}_1-$ forcing notions.

5.7 Discussion. Can we weaken the assumption $\mathrm{cf}(\gamma) > \aleph_0$ in 5.4 to $\mathrm{cf}(\gamma) \ge \aleph_0$ and/or allow $\kappa = \aleph_0$? The answer is yes if $\{2, \kappa\} \subseteq S$.

As in 5.4A, 5.4B we can assume $p = (T, H)$ satisfies

(∗) for $\eta \in p$, $\ell g(\eta) \ge \mathrm{tr}(p)$ and there are q and $\alpha < \kappa$ such that $p^{[\eta]} \le_0 q \in Q$ and $q \Vdash$ "$\kappa = 2 \ \&\ \underset{\sim}{\tau} = \alpha$ or $\kappa > \aleph_0 \ \&\ \underset{\sim}{\tau} < \alpha$" then $p^{[\eta]}$, α_η satisfies this.

Let for $\eta \in T^p$ of length $\geq \ell g(\mathrm{tr}(p))$:

$$\mathcal{P}_p(\eta) \stackrel{\mathrm{def}}{=} \{A \subseteq \lambda : \text{for some initial segment } y \text{ of the game continuing } H^p(\eta), \text{ we have } d(y) \subseteq A\}$$

(where d is from Definition 5.1).

Note that as $2 \in S$ we have

(**) $A \subseteq \lambda \Rightarrow A \in \mathcal{P}_p(\eta) \ \vee \ (\lambda \setminus A) \in \mathcal{P}_p(\eta)$

(* * *) if $A \in \mathcal{P}_p(\eta)$, $A \subseteq B \in \lambda$ then $B \in \mathcal{P}_p(\eta)$.

We define a function $\mathrm{rk}_p : T^* = \{\eta : \mathrm{tr}(p) \trianglelefteq \eta \in T\} \to \mathrm{Ord} \cup \{\infty\}$ by defining by induction on the ordinal ζ when $\mathrm{rk}_p(\eta) \geq \zeta$, the definition is splited to cases.

Case A. ζ limit

$\mathrm{rk}_p(\eta) \geq \zeta$ *iff* $(\forall \xi < \zeta)[\mathrm{rk}_p(\eta) \geq \xi]$.

Case B. $\zeta = 1$

$\mathrm{rk}_p(\eta) \geq 1$ iff α_η is not well defined (and $\eta \in T^*$)

Case C. $\zeta = \varepsilon + 1 > 1$

so $\varepsilon > 0$; let $\mathrm{rk}_p(\eta) \geq \zeta$ *iff*: $\mathrm{tr}(p) \trianglelefteq \eta \in T^p$ and the set $\{\beta < \lambda : \mathrm{rk}_p(\eta\char`^\langle\beta\rangle) \geq \varepsilon\}$ belongs to $\mathcal{P}_p(\eta)$.

So $\mathrm{rk}(\eta) = 0$ if α_η is well defined and $\mathrm{rk}(\eta) = \zeta > 0$ if $\neg(\mathrm{rk}(\eta) \geq \zeta + 1)$, ζ minimal, and $\mathrm{rk}(\eta) = \infty$ if $\mathrm{rk}(\eta) \geq \zeta$ for every $\zeta \geq 1$. Now the proof is splited:

Subcase C1. $\mathrm{rk}_p(\mathrm{tr}(p)) = \infty$.

Clearly for $\eta \in T^*$, if $\mathrm{rk}_p(\eta) = \infty$ then $\{\beta < \lambda : \mathrm{rk}_p(\eta\char`^\langle\beta\rangle) = \infty\} \in \mathcal{P}_p(\eta)$. Hence we can find q such that $p \leq_0 q \in Q$ such that:

$$\mathrm{tr}(p) \trianglelefteq \eta \in T^q \Rightarrow \mathrm{rk}_p(\eta) = \infty.$$

There is r such that $q \leq r \in Q$ and r forces a value to $\underset{\sim}{\tau}$, so $\alpha_{\mathrm{tr}(r)}$ is well defined but $\mathrm{tr}(p) \trianglelefteq \mathrm{tr}(r) \in T^r \subseteq T^q$ hence $\mathrm{rk}_p(\mathrm{tr}(r)) = \infty$ hence $\alpha_{\mathrm{tr}(r)}$ is not well defined, contradiction.

Subcase C2. $\mathrm{rk}_p(\mathrm{tr}(p)) < \infty$.

So choose $\eta \in T^q$, $\mathrm{tr}(p) \trianglelefteq \eta \in T^p$ such that α_η is not defined and, under those restrictions, $\mathrm{rk}_p(\eta)$ is minimal.

Let

$$A = \{\gamma < \lambda : \eta\hat{\ }\langle\gamma\rangle \in p \text{ and } \alpha_{\eta\hat{\ }\langle\gamma\rangle} \text{ is not defined}\}.$$

We can find q, $p^{[\eta]} \leq_0 q$ and $d(H^q(\eta))$ is included in A or disjoint to it. In the second case we can easily get "α_η well defined", contradiction. So assume $d(H^q(\eta)) \subseteq A$, and neccessarily there is $\nu \in d(H^Q(\eta))$ such that $\mathrm{rk}_p(\nu) < \mathrm{rk}_p(\eta)$ by the definition of rank. We get easy contradiction.