Chapter VI
Inductive Definitions

"Let X be the smallest set containing ... and closed under ---." A definition expressed in this form is called an inductive definition. We have used this method of definition repeatedly in the previous chapters; for example, in defining the notions of Δ_0 formula, Σ formula, infinitary formula, provable using the $\mathfrak{M}$-rule, etc. In this chapter we turn method into object by studying inductive definitions in their own right. We will see that their frequent appearance is more than an accident.

1. Inductive Definitions as Monotonic Operators

Let A be an arbitrary set. An n-ary *inductive definition* on A is simply a mapping Γ from n-ary relations on A to n-ary relations on A which is *monotone increasing*; i.e. for all n-ary relations R, S on A

$$R \subseteq S \quad \text{implies} \quad \Gamma(R) \subseteq \Gamma(S).$$

If $\Gamma(R) = R$ then R is a *fixed point of* Γ.

1.1 Theorem. *Every inductive definition on A has a smallest fixed point. Indeed, there is a relation R such that:*

(i) $\Gamma(R) = R$,

(ii) *for any relation S on A, if* $\Gamma(S) \subseteq S$ *then* $R \subseteq S$.

Proof. Let $\mathbf{C} = \{S \subseteq A^n \mid \Gamma(S) \subseteq S\}$. Since $A^n \in \mathbf{C}$, $\mathbf{C}$ is non-empty. Let $R = \bigcap \mathbf{C}$. Since (ii) now holds by definition it remains to prove (i), that is, that $\Gamma(R) = R$. Let S be an arbitrary member of $\mathbf{C}$. Since $R \subseteq S$ and Γ is monotone we have $\Gamma(R) \subseteq \Gamma(S)$, but $\Gamma(S) \subseteq S$, so $\Gamma(R) \subseteq S$. Since S was an arbitrary member of $\mathbf{C}$, and $R = \bigcap \mathbf{C}$, we have $\Gamma(R) \subseteq R$. To show that $R \subseteq \Gamma(R)$ it suffices to prove that $\Gamma(R) \in \mathbf{C}$. But since $\Gamma(R) \subseteq R$ we have, by monotonicity, $\Gamma(\Gamma(R)) \subseteq \Gamma(R)$ so $\Gamma(R) \in \mathbf{C}$. ☐

The proof of 1.1, while correct, tells us next to nothing about the smallest fixed point of Γ and is certainly not the way we mentally justify a typical inductive definition. Let us look at an example.

1.2 Example. Our very first use of an inductive definition was the definition of the class of Δ_0 formulas. We defined it as the smallest class containing the atomic formulas and closed under $\wedge, \vee, \neg, \forall u \in v, \exists u \in v$. How do we convince ourselves that there is such a smallest set? We simply say: start with the atomic formulas and close under (i.e., iterate) the operations $\wedge, \vee, \neg, \forall u \in v, \exists u \in v$. We can turn this process into a much more instructive proof of Theorem 1.1. (By the way, to make the class of Δ_0 formulas fall under 1.1 we let A be the class of formulas of L^* and define the 1-ary Γ by

$\Gamma(U) = \{\varphi \in A \mid \varphi$ *is atomic or* $\varphi = (\psi \wedge \theta)$ *for some* $\psi, \theta \in U$ *or* ... *or* $\varphi = \exists u \in v \psi$ *for some* $\psi \in U\}$.)

Motivated by the above example we make the following definitions.

1.3 Definition. Let Γ be any n-ary inductive definition on a set A.

(i) The α^{th}-*iterate* of Γ, denoted by I_Γ^α, is the n-ary relation defined by

$$I_\Gamma^\alpha = \Gamma(\textstyle\bigcup_{\beta<\alpha} I_\Gamma^\beta).$$

(ii) $I_\Gamma = \bigcup_\alpha I_\Gamma^\alpha$, where the union is taken over all ordinals.

We will show that I_Γ is the smallest fixed point of Γ referred to in Theorem 1.1. We use the notation

$$I_\Gamma^{<\alpha} = \textstyle\bigcup_{\beta<\alpha} I_\Gamma^\beta$$

to simplify some equations.

1.4 Lemma. *Let Γ be any n-ary inductive definition on a set A.*

(i) $I_\Gamma^0 = \Gamma(0)$,

(ii) $I_\Gamma^\alpha = \Gamma(I_\Gamma^{<\alpha})$ *for all* α,

(iii) $\alpha \leqslant \beta$ *implies* $I_\Gamma^\alpha \subseteq I_\Gamma^\beta$, *and*

(iv) $I_\Gamma^{\alpha+1} = \Gamma(I_\Gamma^\alpha)$ *for all* α.

Proof. Parts (i) and (ii) are immediate from the definitions. Part (iii) follows from monotonicity since

$$I_\Gamma^{<\alpha} = \textstyle\bigcup_{\zeta<\alpha} I_\Gamma \subseteq \bigcup_{\zeta<\beta} I_\Gamma = I_\Gamma^{<\beta}$$

implies

$$I_\Gamma^\alpha = \Gamma(I_\Gamma^{<\alpha}) \subseteq \Gamma(I_\Gamma^{<\beta}) = I_\Gamma^\beta .$$

Part (iv) follows from (ii) since

$$I_\Gamma^{\alpha+1} = \Gamma(\textstyle\bigcup_{\zeta\leqslant\alpha} I_\Gamma^\zeta) = \Gamma(I_\Gamma^\alpha). \quad \square$$

1.5 Theorem. *Let Γ be an n-ary inductive definition on a set A.*

(i) *There is an ordinal γ (of cardinality $\leqslant \operatorname{card}(A^n)$) such that*

$$I_\Gamma^\gamma = I_\Gamma^{<\gamma}$$

and hence

$$I_\Gamma = \bigcup_{\alpha<\gamma} I_\Gamma^\alpha .$$

(ii) *I_Γ is the smallest fixed point of Γ.*

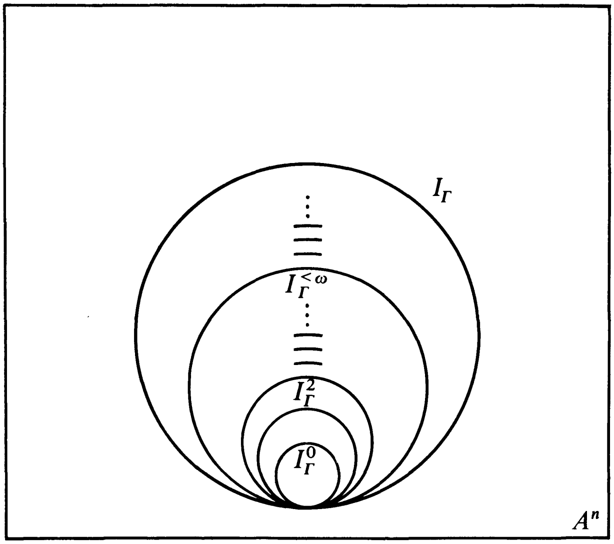

Fig. 1A. Building up the smallest fixed point I_Γ

Proof. First notice that the relations form an increasing sequence of subsets of A^n,

$$I_\Gamma^0 \subseteq I_\Gamma^1 \subseteq \cdots \subseteq I_\Gamma^\alpha \subseteq I_\Gamma^{\alpha+1} \subseteq \cdots ,$$

and hence the sequence must stop strictly increasing for some γ of cardinality $\leqslant \operatorname{card}(A^n)$, i.e.,

$$I_\Gamma^\gamma = \bigcup I_\Gamma^{<\gamma} .$$

But then $I_\Gamma^\alpha = I_\Gamma^{<\gamma}$ for all $\alpha \geqslant \gamma$ so $I_\Gamma = I_\Gamma^{<\gamma}$. To prove (ii), note that

$$\Gamma(I_\Gamma) = \Gamma(I_\Gamma^{<\gamma}) = I_\Gamma^\gamma = I_\Gamma^{<\gamma} = I_\Gamma$$

by using (i) repeatedly. Hence I_Γ is a fixed point and it remains to show that I_Γ is the smallest such. Let $\Gamma(S) \subseteq S$. We prove $I_\Gamma^\alpha \subseteq S$ for all α, by induction. The induction hypothesis asserts that $I_\Gamma^\beta \subseteq S$ for all $\beta < \alpha$ so $I_\Gamma^{<\alpha} \subseteq S$. By monotonicity we have

$$I_\Gamma^\alpha = \Gamma(I_\Gamma^{<\alpha}) \subseteq \Gamma(S) \subseteq S . \quad \square$$

1.6 Definition. Given an inductive definition Γ, the least ordinal γ such that $I_\Gamma^\gamma = I_\Gamma^{<\gamma}$ is called the *closure ordinal of* Γ and is denoted by $\|\Gamma\|$.

Most of the inductive definitions we have used in the previous chapters have had closure ordinal ω so that

$$I_\Gamma = \bigcup_{n<\omega} I_\Gamma^n .$$

One of the most important however, the set of sentences provable using the $\mathfrak{M}$-rule, will in general have closure ordinal greater than ω. (In fact, this inductive definition has closure ordinal $O(\mathfrak{M})$. See Exercise 3.19.)

Our interest in this chapter is in inductive definitions which are definable over an L-structure $\mathfrak{M}$ or over an admissible set $\mathbb{A}_{\mathfrak{M}}$. In order to insure the monotonicity condition on Γ we need the notion of an R-monotone formula.

1.7 Definition. Let $\mathfrak{N}$ be a structure for some language K (usually L or L^* in applications). A formula $\varphi(x_1, \ldots, x_n, \mathsf{R})$ of $\mathsf{K} \cup \{\mathsf{R}\}$ (possibly having parameters from $\mathfrak{N}$) is R-*monotone* on $\mathfrak{N}$ if for all $x_1, \ldots, x_n \in \mathfrak{N}$ and all relations $R_1 \subseteq R_2$ on $\mathfrak{N}$,

$$(\mathfrak{N}, R_1) \models \varphi(\mathsf{R})[x_1, \ldots, x_n]$$

implies

$$(\mathfrak{N}, R_2) \models \varphi(\mathsf{R})[x_1, \ldots, x_n] .$$

Recall the notion R-positive and corresponding notation $\varphi(\mathsf{R}_+)$ from V.2.1.

1.8 Lemma. *If* $\varphi(x_1, \ldots, x_n, \mathsf{R}_+)$ *is an* R-*positive formula of* K *then it is* R-*monotone for all* K-*structures* $\mathfrak{N}$.

Proof. Fix $\mathfrak{N}$ and prove the result by induction following the inductive definition of R-positive. $\square$

Most inductive definitions are actually given by R-positive formulas because most inductive definitions do not really depend on the particular structure $\mathfrak{N}$ and any formula which is R-monotone for all structures $\mathfrak{N}$ is equivalent to an R-positive formula (see Exercise 1.14).

1.9 Notation and restatement of results. Let $\mathfrak{N}$ be a structure for a language K. Let R be a new n-ary relation symbol and let $\varphi(x_1, \ldots, x_n, \mathsf{R})$ be R-monotone on $\mathfrak{N}$.

(i) The n-ary *inductive definition given by* φ, denoted by Γ_φ, is defined by

$$(x_1, \ldots, x_n) \in \Gamma_\varphi(R) \quad \text{iff} \quad (\mathfrak{N}, R) \models \varphi(\mathsf{R})[x_1, \ldots, x_n] .$$

(ii) We let I_φ denote I_{Γ_φ} and similarly for I_φ^α and $I_\varphi^{<\alpha}$. Thus I_φ is an n-ary relation on $\mathfrak{N}$ satisfying

$$(\mathfrak{N}, I_\varphi) \models \forall x_1, \ldots, x_n [\varphi(x_1, \ldots, x_n, \mathsf{R}) \leftrightarrow \mathsf{R}(x_1, \ldots, x_n)] .$$

Furthermore, if R is an n-ary relation on $\mathfrak{N}$ satisfying

$$(\mathfrak{N}, R) \models \forall x_1, \dots, x_n [\varphi(x_1, \dots, x_n, \mathsf{R}) \rightarrow \mathsf{R}(x_1, \dots, x_n)]$$

then $I_\varphi \subseteq R$. I_φ is called the *smallest fixed point* of the inductive definition Γ_φ and I_φ^α is called the α^{th} *stage* of Γ_φ. It satisfies

$$I_\varphi^\alpha = \{(x_1, \dots, x_n) \mid (\mathfrak{N}, I_\varphi^{<\alpha}) \models \varphi(\mathsf{R})[x_1, \dots, x_n]\}.$$

1.10 Proposition. *Let $\mathfrak{N}$ be any* K*-structure and let $\varphi(x_1, \dots, x_n, \mathsf{R})$ be* R*-monotone on $\mathfrak{N}$, where* R *is a new n-ary symbol. The fixed point I_φ is a Π_1^1 relation on $\mathfrak{N}$.*

Proof. By 1.9 we see that $(x_1, \dots, x_n) \in I_\varphi$ iff $\forall \mathsf{R}[\Gamma_\varphi(\mathsf{R}) \subseteq \mathsf{R} \rightarrow \mathsf{R}(x_1, \dots, x_n)]$ which becomes

$$\mathfrak{N} \models \forall \mathsf{R}[\forall y_1, \dots, y_n(\varphi(y_1, \dots, y_n, \mathsf{R}) \rightarrow \mathsf{R}(y_1, \dots, y_n)) \rightarrow \mathsf{R}(x_1, \dots, x_n)]$$

when written out in full. □

Let $\mathcal{N} = \langle \omega, 0, +, \cdot \rangle$. Spector [1959] observed that Kleene's analysis of Π_1^1 relations on $\mathcal{N}$ showed that every Π_1^1 relation or could be obtained by means of an inductive definition. This result will follow from more general results in § 3. We present the classical proof, nevertheless, since it is attractive and illustrates several important points.

1.11 Theorem. *Let $\mathcal{N} = \langle \omega, 0, +, \cdot \rangle$ and let S be an n-ary Π_1^1 relation on $\mathcal{N}$. There is a formula $\varphi(x_1, \dots, x_n, y, R_+)$ with R $n+1$-ary such that*

$$S(x_1, \dots, x_n) \Longleftrightarrow I_\varphi(x_1, \dots, x_n, 1)$$

for all $x_1, \dots, x_n \in \mathcal{N}$.

Proof. We prove the result for $n = 1$ and use the following normal form of Kleene for Π_1^1 sets S:

$$S(x) \quad \text{iff} \quad \forall f \, \exists n \, P(x, \bar{f}(n))$$

where the following are assumed:

P is recursive,
$\bar{f}(n)$ is a number s coding up the sequence $\langle f(0), \dots, f(n-1) \rangle$,
$s_1 \prec s_2$ means that s_1 is a sequence (code) properly extending s_2,
$P(x, s_2)$ and $s_1 \prec s_2$ implies $P(x, s_1)$,
1 codes the empty sequence,
if s codes $\langle x_1, \dots, x_n \rangle$ then $s \frown y$ codes $\langle x_1, \dots, x_n, y \rangle$.

The desired inductive definition φ is given by

$$s \text{ is a sequence code and, } P(x,s) \text{ or } \forall y\, \mathsf{R}(x, s^\frown y).$$

We first prove that

(1) $I_\varphi(x,s)$ implies $\forall f$ [*if* f *extends* s *then* $\exists n\, P(x,\bar{f}(n))$].

Let R be the set of pairs (x,s) satisfying the right side of (1). Note that $P(x,s)\to R(x,s)$. It suffices to prove that $\Gamma_\varphi(R)\subseteq R$. If $(x,s)\in\Gamma_\varphi(R)$ then either $P(x,s)$ or else $\forall y\, R(x,s^\frown y)$. But then $R(x,s)$ since every function extending s extends $s^\frown y$ for some y.

Next we prove the converse of (1), or rather, as much of it as we need:

(2) $\forall f\, \exists n\, P(x,\bar{f}(n))$ implies $I_\varphi(x,1)$.

If $P(x,1)$ then $(x,1)\in I_\varphi^0$ so we may assume $\neg P(x,1)$. Assuming the left side of (2) consider the set S of all s such that $\neg P(x,s)$. This set is well founded (under $\prec$) since any infinite descending sequence would produce an f with $\neg P(x,\bar{f}(n))$ holding for arbitrarily large n, and hence for all n. Let us write, in this proof, $\rho(s)$ for $\rho^{\prec\restriction S}(s)$; $\rho(s)$ is defined for all $s\in S$ since S is well founded. We prove by induction on ξ that if $\rho(s)=\xi$ then $(x,s)\in I_\varphi^{\xi+1}$. (Since $1\in S$ we then have $(x,1)\in I_\varphi^{\xi+1}$ where $\xi=\rho(1)$.) Observe that

$$\rho(s)=\sup\{\rho(s^\frown y)+1 \mid \neg P(x,s^\frown y),\, y\in\omega\}.$$

Now for each y, if $P(x,s^\frown y)$ then $(x,s^\frown y)\in I_\varphi^0$, and if $\neg P(x,s^\frown y)$ then $(x,s^\frown y)\in I_\varphi^{\beta+1}$ for some $\beta<\xi$ by the induction hypothesis. In either case

$$(x,s^\frown y)\in I_\varphi^\xi.$$

But then by the definition of φ,

$$(x,s)\in I_\varphi^{\xi+1}$$

as desired. Combining (1) and (2) yields the theorem. $\square$

One of our goals in this chapter is to prove some generalizations of this result to arbitrary structures. Looking at the above theorem and its proof, we are struck by three facts.

The most prominent fact is that the proof uses a normal form for Π_1^1 predicates on $\mathcal{N}$ which has no generalization to Π_1^1 over arbitrary structures. If we can ignore this unsettling fact, however, we can go on to make two useful observations.

First, and very typical of the whole subject of inductive definitions, is that the Π_1^1 relation S was not defined as a fixed point but rather as a "section" of a fixed point:

$$S(\vec{x}) \Longleftrightarrow I_\varphi(\vec{x},1).$$

The proof makes it clear that the last coordinate of I_φ is where all the work is going on. It is only at the very last minute that we can set $s=1$. (To clinch matters, Feferman [1963] proves that not every Π_1^1 set over $\mathcal{N}$ is a fixed point.) This motivates the next definition.

1.12 Definition. Let K be a language, $\mathfrak{N}$ be any structure for K and let Φ be a set of formulas such that each $\varphi \in \Phi$ is of the form $\varphi(x_1,\ldots,x_n,\mathsf{R})$ for some n and some n-ary relation symbol R not in K) and is R-monotone on $\mathfrak{N}$.

(i) If $S=I_\varphi$ for some $\varphi\in\Phi$ then S is called a *Φ-fixed point.*

(ii) A relation S of m arguments is *Φ-inductive* if there is a Φ fixed point S' of $m+n$ arguments $(n\geqslant 0)$ and $y_1,\ldots,y_n\in\mathfrak{N}$ such that

$$S(x_1,\ldots,x_m) \quad \text{iff} \quad S'(x_1,\ldots,x_m,y_1,\ldots,y_n)$$

for all $x_1,\ldots,x_m\in\mathfrak{N}$. S is called a *section* of S'.

Combining 1.10 and 1.11 (and the triviality that a section of a Π_1^1 relation is Π_1^1) we see that a relation S on $\mathcal{N}$ is Π_1^1 iff it is first order inductive.

A final point on the proof of Theorem 1.11. We made heavy use of coding in the proof, coding of pieces of functions by sequences and sequences by numbers, not to mention the coding which goes into the proof of the normal form theorem. In an admissible set, coding presents no trouble. In an arbitrary structure $\mathfrak{M}$, however, we may be out of luck. In this case we have two options. One is to restrict ourselves to $\mathfrak{M}$ which have built in coding machinery (this amounts to Moschovakis [1974]'s use of "acceptable" structures). The second option, more natural in our context is to replace induction on $\mathfrak{M}$ by inductions on $\mathbb{HF}_{\mathfrak{M}}$. We study both approaches in the latter parts of this chapter.

1.13—1.19 Exercises

1.13. Let K be a language with only relation symbols. One form of the Lyndon Interpolation Theorem asserts that if $\varphi,\psi\in\mathsf{K}_{\omega\omega}$, if φ or ψ is R-positive, and if

$$\models(\varphi\rightarrow\psi)$$

then there is a θ which is R-positive and has symbols common to φ and ψ such that

$$\models(\varphi\rightarrow\theta) \quad\text{and}\quad \models(\theta\rightarrow\psi).$$

Prove a generalization of this to arbitrary countable, admissible fragments $\mathsf{K}_{\mathbb{A}}$.

1.14. Prove that if $\varphi(x_1,\ldots,x_n,\mathsf{R})$ is R-monotone for all models $\mathfrak{N}$ of some theory T of $\mathsf{K}_{\omega\omega}$ (T not involving R, of course) then there is an R-positive $\psi(x_1,\ldots,x_n,\mathsf{R}_+)$ of $\mathsf{K}_{\omega\omega}$ such that

$$T\vdash\forall x_1,\ldots,x_n\,(\varphi\leftrightarrow\psi).$$

[Use the $\mathsf{K}_{\omega\omega}$ version of 1.13.]

1.15. Let Γ be an inductive definition, i. e. a monotonic increasing operation on n-ary relations on some set A. Show that Γ has a largest fixed point.

1.16. Let Γ be an n-ary inductive definition on A and define

$$J_\Gamma^0 = \Gamma(A^n)$$

and for $\alpha > 0$,

$$J_\Gamma^\alpha = \Gamma(\bigcap_{\beta<\alpha} J_\Gamma^\beta).$$

Let

$$J_\Gamma = \bigcap_\alpha J_\Gamma^\alpha.$$

Show that J_Γ is the largest fixed point of Γ refered to in 1.15.

1.17. Let Γ be an n-ary inductive definition on A and let Γ' be defined by

$$\Gamma'(R) = A^n - \Gamma(A^n - R).$$

Prove that Γ' is an inductive definition. Prove that, for each α,

$$x \in I_{\Gamma'}^\alpha \quad \text{iff} \quad x \notin J_\Gamma^\alpha$$

and hence that

$$I_{\Gamma'} = A^n - J_\Gamma.$$

1.18. Let Φ_1, Φ_2 be classes of formulas R-monotone on a structure $\mathfrak{N}$, closed under logical equivalence and such that

$$\varphi(x_1,\ldots,x_k,\mathsf{R}) \in \Phi_1 \quad \text{iff} \quad (\neg\varphi(x_1,\ldots,x_k,\neg\mathsf{R})) \in \Phi_2.$$

A relation S on $\mathfrak{N}$ is Φ_1-*coinductive* iff for some $\varphi \in \Phi_1$ and some parameters $y_1,\ldots,y_n \in \mathfrak{N}$

$$S(x_1,\ldots,x_m) \quad \text{iff} \quad (x_1,\ldots,x_m,y_1,\ldots,y_n) \in J_\varphi$$

for all $x_1,\ldots,x_n \in \mathfrak{N}$. Show that S is Φ_1-coinductive iff $\neg S$ is Φ_2-inductive. (Hence every coinductive relation on $\mathfrak{N}$ is Σ_1^1. You can also prove this directly.)

1.19. Let G be an abelian p-group. Define $\Gamma(H)$, for $H \subseteq G$, by

$$\Gamma(H) = \{px \mid x \in H\}.$$

Show that J_Γ is the largest divisible subgroup of G. In this case the least ordinal α such that $J_\Gamma = \bigcap_{\beta<\alpha} J_\Gamma^\beta$ is usually called the length of the group G. It plays a key role in the study of p-groups.

1.20 Notes. We have built monotonicity of Γ into our definition of "inductive definition". There are also things called "non-monotonic inductive definitions" which have interesting relationships with admissible sets. For references on these operators, we refer the reader to Richter-Aczel [1974] and Moschovakis [1975].

All the results of § 1 are standard.

2. Σ Inductive Definitions on Admissible Sets

Let $\Sigma(\mathsf{R}_+)$ be the collection of R-positive Σ formulas of $\mathsf{L}^*(\mathsf{R})$ and let Σ_+ be the union of the $\Sigma(\mathsf{R}_+)$ as R ranges over all relation symbols not in L^*. Applying Definition 1.12 (with $\mathsf{K}=\mathsf{L}^*$, $\mathfrak{N}=\mathfrak{A}_{\mathfrak{M}}$ and $\Phi=\Sigma_+$) we have the companion notions of Σ_+ *fixed point* and Σ_+ *inductive relation.* These notions are the primary object of study of this section. The proofs, however, give information about a wider class of relations.

Let $\mathscr{K}$ be a class of L^*-structures and let $\Sigma(\mathsf{R}\uparrow\mathscr{K})$ be the collection of Σ formulas $\varphi(x_1,\dots,x_n,\mathsf{R})$ of $\mathsf{L}^*(\mathsf{R})$ which are monotone increasing on each structure in $\mathscr{K}$. We let $\Sigma(\uparrow\mathscr{K})$ be the union of the $\Sigma(\mathsf{R}\uparrow\mathscr{K})$ as R varies. (Read "Σ increasing on $\mathscr{K}$" for $\Sigma(\uparrow\mathscr{K})$.) Given a structure $\mathfrak{A}_{\mathfrak{M}}\in\mathscr{K}$ we have corresponding notions of $\Sigma(\uparrow\mathscr{K})$ *fixed point on* $\mathfrak{A}_{\mathfrak{M}}$ and $\Sigma(\uparrow\mathscr{K})$ *inductive relation on* $\mathfrak{A}_{\mathfrak{M}}$. If $\mathscr{K}=\{\mathfrak{A}_{\mathfrak{M}}\}$ then we write $\Sigma(\uparrow\mathfrak{A}_{\mathfrak{M}})$ for $\Sigma(\uparrow\mathscr{K})$.

Note that by Lemma 1.8, $\Sigma_+\subseteq\Sigma(\uparrow\mathscr{K})$ for all $\mathscr{K}$. If $\mathscr{K}$ is the class of all structures for L^* which are models of some theory T then Exercise 1.14 tells us that $\Sigma_+=\Sigma(\uparrow\mathscr{K})$, up to logical equivalence. In the results below, however, $\mathscr{K}$ is usually a single admissible set or a class of admissible sets.)

We have already studied the most important Σ_+ inductive definition at some length back in Chapter III. Let $\mathsf{K}_{\mathbb{A}}$ be an admissible fragment and let $Thm_{\mathbb{A}}$ be the set of theorems of $\mathsf{K}_{\mathbb{A}}$. By definition, $Thm_{\mathbb{A}}$ is the smallest set of formulas of $\mathsf{K}_{\mathbb{A}}$ containing the axioms (A 1)—(A 7) and closed under (R 1)—(R 3). This is, of course, a typical example of an inductive definition. Let Γ_0 be this inductive definition.

2.1 Proposition. *Using the notation just above we have*

(i) *Γ_0 is a Σ_+ inductive definition, and hence*

(ii) *$Thm_{\mathbb{A}}$ is a Σ_+ fixed point.*

Proof. We simply write out the definition of Γ_0 to see that it is in fact Σ_+. Let R be a new unary symbol and recall that

$$x\in\Gamma_0(R) \quad \text{iff} \quad x\in\mathsf{K}_{\mathbb{A}}\wedge[(\mathrm{A})\vee(\mathrm{R}\,1)\vee(\mathrm{R}\,2)\vee(\mathrm{R}\,3)]$$

where we have used

(A) "x is an instance of (A 1)—(A 7)".

(R 1) $\exists y\,[y\in R\wedge(y\to x)\in R]$.

(R 2) "x is of the form $(\psi\to\forall v\,\theta(v))$ where v is not free in ψ and $(\psi\to\theta(v))\in R$".

(R 3) "x is of the form $(\psi\to\bigwedge\Phi)$ and, for each $\varphi\in\Phi$, $(\psi\to\varphi)\in R$".

We can rewrite this schematically in the form

$$x \in \Gamma_0(R) \quad \text{iff} \quad \Delta_1 \wedge [\Delta_1 \vee \Sigma_1(\mathsf{R}_+) \vee \Delta_1(\mathsf{R}_+) \vee \Delta(\mathsf{R}_+)],$$

so Γ_0 is indeed a $\Sigma(\mathsf{R}_+)$ inductive definition. □

Now one of the primary aims of § III.5 was to prove that $Thm_{\mathbb{A}}$ was in fact Σ_1 *definable* on $\mathbb{A}$. In this section we use this fact to prove that every Σ_+ inductive relation on an admissible set is Σ_1 on that admissible set. For $\mathbb{A}$ countable, even more is true.

2.2 Theorem. *Let $\mathbb{A}$ be a countable admissible set. Every $\Sigma(\uparrow\mathbb{A})$ inductive relation on $\mathbb{A}$ is Σ_1 on $\mathbb{A}$.*

Proof. It clearly suffices to prove that every $\Sigma(\uparrow\mathbb{A})$ fixed point on $\mathbb{A}$ is Σ_1 since the Σ_1 relations are closed under sections. Let $\varphi(x_1,\ldots,x_n,\mathsf{R}) \in \Sigma(\uparrow\mathbb{A})$. The proof goes back to the Extended Completeness Theorem for countable admissible fragments and, hence, to our analysis of Γ_0 carried out in § III.5. Let K be the formalized version of $\mathsf{L}^*(\mathsf{R}) \cup \{\overline{\mathsf{x}} \mid x \in \mathbb{A}\}$ and let $\mathsf{K}_{\mathbb{A}}$ be the fragment given by $\mathbb{A}$ $(=\mathbb{A}_{\mathfrak{M}})$. Let T be the $\mathsf{K}_{\mathbb{A}}$ theory:

$$\text{Diagram}(\mathbb{A}),$$

$$\forall v\,[v \in \overline{\mathsf{a}} \leftrightarrow \textstyle\bigvee_{x \in a} v = \overline{\mathsf{x}}] \quad \text{for all} \quad a \in \mathbb{A},$$

$$\forall v_1,\ldots,v_n\,[\varphi(v_1,\ldots,v_n,\mathsf{R}) \rightarrow \mathsf{R}(v_1,\ldots,v_n)].$$

We claim that

(1) $(x_1,\ldots,x_n) \in I_\varphi$ iff $T \models \mathsf{R}(\overline{\mathsf{x}}_1,\ldots,\overline{\mathsf{x}}_n)$

from which the conclusion follows by the Extended Completeness Theorem. The ($\Leftarrow$) half of (1) follows from the observation that

$$(\mathbb{A}, I_\varphi) \models T$$

when R is interpreted by I_φ. To prove ($\Rightarrow$) suppose that $(\mathfrak{B}_{\mathfrak{N}}, R)$ is an arbitrary model of T. We need to prove that whenever $(x_1,\ldots,x_n) \in I_\varphi$, we have

$$(\mathfrak{B}_{\mathfrak{N}}, R) \models \mathsf{R}(\overline{\mathsf{x}}_1,\ldots,\overline{\mathsf{x}}).$$

If we let $R_0 = R \upharpoonright \mathbb{A}_{\mathfrak{M}}$ then we note that (up to isomorphism)

$$(\mathbb{A}_{\mathfrak{M}}, R_0) \subseteq_{\text{end}} (\mathfrak{B}_{\mathfrak{N}}, R)$$

so what we need to prove is that $I_\varphi \subseteq R_0$. This will follow (from 1.5 (ii)) if we prove that $\Gamma_\varphi(R_0) \subseteq R_0$; i.e., that

(2) $(\mathbb{A}_{\mathfrak{M}}, R_0) \models \forall y_1,\ldots,y_n\,[\varphi(y_1,\ldots,y_n,\mathsf{R}) \rightarrow \mathsf{R}(y_1,\ldots,y_n)].$

So suppose that $y_1,\dots,y_n \in \mathbb{A}_{\mathfrak{M}}$ and

$$(\mathbb{A}_{\mathfrak{M}}, R_0) \models \varphi(y_1,\dots,y_n,\mathsf{R}).$$

Since φ is a Σ formula and $(\mathfrak{B}_{\mathfrak{N}}, R)$ is an end extension of $(\mathbb{A}_{\mathfrak{M}}, R_0)$ we have

$$(\mathfrak{B}_{\mathfrak{N}}, R) \models \varphi(y_1,\dots,y_n,\mathsf{R}),$$

and so, by the last axiom of T, $R(y_1,\dots,y_n)$ holds, and hence $R_0(y_1,\dots,y_n)$. This establishes (2) and hence the theorem. ▯

Let $\varphi(x_1,\dots,x_n,v_1,\dots,v_k,\mathsf{R})$ be a fixed Σ formula of $\mathsf{L}^*(\mathsf{R})$. The following remarks are intended to lift much of Theorem 2.2 to arbitrary admissible sets by means of the Absoluteness Principle.

2.3 Remark. The Σ_1 formula defining I_φ in Theorem 2.2 is independent of $\mathbb{A}$ except for the parameters occuring in φ. More fully, let

$$I_\varphi(\mathbb{A}, y_1,\dots,y_k)$$

denote the smallest fixed point defined on $\mathbb{A}$ by Γ_φ when $v_1=y_1,\dots,y_k=y_k$ (provided $\varphi(x_1,\dots,x_n,y_1,\dots,y_k,\mathsf{R}) \in \Sigma(\mathsf{R}{\restriction}\mathbb{A})$). *There is a Σ_1 formula $\psi(x_1,\dots,x_n,v_1,\dots,v_k)$ of L^* such that for all countable, admissible $\mathbb{A}$ and all $y_1,\dots,y_k \in \mathbb{A}$,*

(3) *if $\varphi(x_1,\dots,x_n,y_1,\dots,y_k,\mathsf{R})$ is R-monotone on $\mathbb{A}$ then for all $x_1,\dots,x_n \in \mathbb{A}$,*
$(x_1,\dots,x_n) \in I_\varphi(\mathbb{A}, y_1,\dots,y_k)$ *iff* $\mathbb{A} \models \psi(x_1,\dots,x_n,y_1,\dots,y_k)$.

Proof. Let ψ be the formula which expresses

$$\exists p\ [p \textit{ is a proof of } \sigma \to \mathsf{R}(\overline{\mathsf{x}}_1,\dots,\overline{\mathsf{x}}_n) \textit{ where } \sigma \textit{ is a conjunction of members of } T],$$

where T is as in the proof of 2.2, and examine the proof of Theorem 2.2. ▯

2.4 Remark. The operation $I_\varphi^\alpha(\mathbb{A}, y_1,\dots,y_k)$, is a Σ operation of $\mathbb{A}, y_1,\dots,y_k$, since it is defined by Σ Recursion on α. In ZF we proved the existence of an α (depending on $\mathbb{A}, y_1,\dots,y_k$) such that

$$I_\varphi^\alpha(\mathbb{A}, y_1,\dots,y_k) = I_\varphi^{<\alpha}(\mathbb{A}, y_1,\dots,y_k).$$

(This step takes us outside KPU since it requires some form of Σ_1 Separation.) Thus, in ZF, *the predicate*

$$(x_1,\dots,x_n) \in I_\varphi(\mathbb{A}, y_1,\dots,y_k)$$

is a Δ_1 predicate of $\mathbb{A}$, $x_1,\dots,x_n$, $y_1,\dots,y_k$. It is expressed by the Σ_1 formula

$$\exists \alpha\,[(x_1,\dots,x_n) \in I_\varphi^\alpha(\mathbb{A}, y_1,\dots,y_k)]$$

and the Π_1 formula

$$\forall\alpha\,[I^{\alpha}_{\varphi}(\mathbb{A},y_1,\ldots,y_k)=I^{<\alpha}_{\varphi}(\mathbb{A},y_1,\ldots,y_k)\rightarrow(x_1,\ldots,x_n)\in I^{\alpha}_{\varphi}(\mathbb{A},y_1,\ldots,y_k)]\,.$$

(The characterization of $I_{\varphi}(\mathbb{A},y_1,\ldots,y_k)$ as smallest fixed point of Γ_{φ} gives another Π_1 definition.)

2.5 Remark. *The conclusion of line* (3) *above is a* Δ_1 *predicate of* $\mathbb{A}$, $y_1,\ldots,y_k$. The hypothesis, however, is a Π_1 predicate of $\mathbb{A}$, $y_1,\ldots,y_k$ which makes (3) of the form $\Pi_1\rightarrow\Delta_1$ and hence a Σ_1 predicate of $\mathbb{A}$, $y_1,\ldots,y_k$. To apply the Absoluteness Principle we would need the result to be Π_1.

We are now ready to lift Theorem 2.2 to the uncountable. We give two proofs because each contains information not available in the other (see the two corollaries 2.7 and 2.8).

2.6 Gandy's Theorem. *Let* $\mathbb{A}$ *be any admissible set. Every* Σ_+ *inductive relation on* $\mathbb{A}$ *is* Σ_1 *on* $\mathbb{A}$.

First Proof of Theorem 2.6. Fix $\varphi(x_1,\ldots,x_n,v_1,\ldots,v_k,\mathsf{R})\in\Sigma(\mathsf{R}_+)$. Since φ is R-positive it is R-monotone for all structures for L^* and hence for all admissible sets. The troublesome hypothesis of line (3) is thus superfluous and we see that we have proved for all countable $\mathbb{A}$:

if $\mathbb{A}$ *is admissible then for all* $y_1,\ldots,y_k\in\mathbb{A}$ *and all* $x_1,\ldots,x_n\in\mathbb{A}$

$$(x_1,\ldots,x_n)\in I_{\varphi}(\mathbb{A},y_1,\ldots,y_k)\leftrightarrow\mathbb{A}\models\psi(x_1,\ldots,x_n,y_1,\ldots,y_k)\,.$$

The displayed part is Δ_1 so by the Lévy Absoluteness Principle, the result holds for all $\mathbb{A}$. $\square$

2.7 Corollary. *Let* $\mathscr{K}$ *be a class of admissible sets which is* Σ_1 *definable in* ZFC. *Then for any* $\mathbb{A}\in\mathscr{K}$, *every* $\Sigma(\uparrow\mathscr{K})$ *inductive relation on* $\mathbb{A}$ *is* Σ_1 *on* $\mathbb{A}$.

Proof. The hypothesis asserts that there is a Σ_1 formula $\theta(x)$ without parameters such that

$$\mathbb{A}\in\mathscr{K}\quad\text{iff}\quad\theta(\mathbb{A}),$$

$$\mathrm{ZFC}\vdash\theta(\mathbb{A})\rightarrow\mathbb{A}\quad\text{is admissible.}$$

Replace "$\mathbb{A}$ is admissible" by "$\theta(\mathbb{A})$" in the above proof. $\square$

For example, the $\mathscr{K}$ in 2.6 might be the class of all admissible sets or the class of $L(\alpha)$ where α is recursively inaccessible or nonprojectible.

Second Proof of Theorem 2.6. This proof is more traditional in that it uses the Second Recursion Theorem. For simplicity we let $n=1$ and we suppress parameters $y_1,\dots,y_k$ entirely since they are held constant in this proof. To simplify notation, whenever S is a relation on $\mathbb{A}$ and $\varphi(x,\mathsf{R})\in\Sigma(\mathsf{R}_+)$, we write

$$\mathbb{A}\models\varphi(x,S)$$

instead of the more accurate

$$(\mathbb{A},S)\models\varphi(\mathsf{R})\,[x].$$

Now let $\varphi(x,\mathsf{R})\in\Sigma(\mathsf{R}_+)$. Use the Second Recursion Theorem to define a Σ_1 formula ψ of L^* such that

$$\mathrm{KPU}\vdash\psi(x,\beta)\leftrightarrow\varphi(x,\exists\gamma<\beta\,\psi(\cdot\,,x)).$$

(More precisely,

$$\mathrm{KPU}\vdash\psi(x,\beta)\leftrightarrow\varphi(x,\lambda y\;\exists\gamma<\beta\,\psi(y,\gamma)).$$

To fit thus into Second Recursion Theorem, first let S be a new binary symbol and let $\varphi'(x,\beta,\mathsf{S})$ be $\varphi(x,\exists\gamma<\beta\,\mathsf{S}(\cdot\,,\gamma))$ and then apply the Second Recursion Theorem.) We claim that

(4) for $\beta<o(\mathbb{A})$

$$x\in I_\varphi^\beta \quad\text{iff}\quad \mathbb{A}\models\psi(x,\beta).$$

The proof proceeds by induction on β. The induction hypothesis gives us, for $\gamma<\beta$,

$$I_\varphi^\gamma=\{x\mid\mathbb{A}\models\psi(x,\gamma)\}$$

so, taking unions,

$$I_\varphi^{<\beta}=\{x\mid\mathbb{A}\models\exists\gamma<\beta\,\psi(x,\gamma)\}.$$

Then for any $x\in\mathbb{A}$ we have

$$\begin{aligned}
x\in I_\varphi^\beta \quad&\text{iff}\quad x\in\Gamma_\varphi(I_\varphi^{<\beta})\\
&\text{iff}\quad \mathbb{A}\models\varphi(x,I_\varphi^{<\beta})\\
&\text{iff}\quad \mathbb{A}\models\varphi(x,\exists\gamma<\beta\,\psi(\cdot\,,\gamma))\\
&\text{iff}\quad \mathbb{A}\models\psi(x,\beta).
\end{aligned}$$

Let $\alpha=o(\mathbb{A})$. From (4) we obtain

(5) $I_\varphi^{<\alpha}=\{x\mid\mathbb{A}\models\exists\beta\,\psi(x,\beta)\}.$

Now we claim that

(6) $\Gamma_\varphi(I_\varphi^{<\alpha}) = I_\varphi^{<\alpha}$.

It suffices to prove $\Gamma_\varphi(I_\varphi^{<\alpha}) \subseteq I_\varphi^{<\alpha}$, so suppose $x \in \Gamma_\varphi(I_\varphi^{<\alpha})$, i.e., that

$$\mathbb{A} \models \varphi(x, I_\varphi^{<\alpha}).$$

By (5) this becomes

$$\mathbb{A} \models \varphi(x, \exists\beta\, \psi(\cdot, \beta)).$$

By the Σ Reflection Theorem and Lemma V.2.2 there is a $\delta < \alpha$ such that

$$\mathbb{A} \models \varphi(x, \exists\beta < \delta\, \psi(\cdot, \beta))$$

which, by (4), is equivalent to

$$\mathbb{A} \models \varphi(x, I_\varphi^{<\delta}).$$

Thus $x \in \Gamma_\varphi(I_\varphi^{<\delta}) = I_\varphi^\delta$. But $I_\varphi^\delta \subseteq I_\varphi^{<\alpha}$ so $x \in I_\varphi^{<\alpha}$ as desired. But (6) immediately implies that $I_\varphi = I_\varphi^{<\alpha}$, so $\|\Gamma_\varphi\| \leqslant \alpha$ and

$$I_\varphi = \{x \mid \mathbb{A} \models \exists\beta\, \psi(x, \beta)\},$$

which proves that I_φ is Σ_1 on $\mathbb{A}$. ☐

2.8 Corollary (Second half of Gandy's Theorem). *Let $\mathbb{A}$ be admissible and let $\varphi(x_1, \ldots, x_n, \mathsf{R}_+)$ be a Σ formula with parameters from $\mathbb{A}$. Let $\alpha = o(\mathbb{A})$.*

(i) $\|\Gamma_\varphi\| \leqslant \alpha$.

(ii) *For all β, I_φ^β is Σ_1 on $\mathbb{A}$.*

Proof. Part (i) was explicitly mentioned in the second proof of 2.6. For (ii) we have the result for $\beta \geqslant \alpha$ by 2.6 and for $\beta < \alpha$ by line (4) above. ☐

The results mentioned in 2.8 also hold for arbitrary R-monotone $\varphi(x, \mathsf{R})$ if the admissible set $\mathbb{A}$ is countable. The proof of this, however, must await a stronger reflection principle, the $s-\Pi_1^1$ Reflection Principle.

For sets of the form $\mathrm{L}(\alpha)$ the conclusions of 2.6 and 2.8 are actually equivalent to the hypothesis of admissibility. This will follow from Theorem 3.17 in the next section.

2.9—2.11 Exercises

2.9. Let $\mathfrak{A}_{\mathfrak{M}}$ be a nonstandard model of KPU. Show that $\mathscr{W}\!\!\mathscr{f}(\mathfrak{A}_{\mathfrak{M}})$ is a Σ_+ fixed point which is not first order definable over $\mathfrak{A}_{\mathfrak{M}}$. What is the length of the inductive definition?

2.10 (Stavi). Show that there are pure transitive sets which are not admissible but such that every Σ_+ inductive relation is Σ_1. [Hint: Let $\mathbb{A} = \mathrm{L}(\tau_1) \cap \mathrm{V}(\alpha)$ for suitably nice $\alpha, \omega < \alpha < \tau_1$.]

2.11. Let $\varphi(x_1, \ldots, x_n, \mathrm{R}_+)$ be a Π formula and let J_φ be the largest fixed point of Γ_φ on an admissible set $\mathbb{A}$. Show that J_φ is Π_1^0.

2.12 Notes. The fact that, over an admissible set $\mathbb{A}$, a Σ_+ inductive definition Γ_φ has a Σ_1 fixed point and closure ordinal $\|\Gamma_\varphi\| \leqslant o(\mathbb{A})$ is usually called Gandy's Theorem. He proved this theorem in lectures at the UCLA Logic year in 1968 by adapting the proof-theoretic approach used to prove the Barwise Completeness Theorem. A similar approach is taken in Gandy [1974]. We have given two new proofs for this theorem, one which shows that the result can be derived from the Barwise Completeness Theorem, the other a much more standard recursion theoretic approach using the Second Recursion Theorem.

The recursion theoretic approach to Gandy's Theorem suggests an alternate approach to the material in this book. One could prove Gandy's Theorem (by means of the Second Recursion Theorem) and then quote it to prove that the set $Thm_{\mathbb{A}}$ of theorems of an admissible fragment $\mathsf{K}_{\mathbb{A}}$ is Σ_1 on $\mathbb{A}$. This would suffice for many applications of the Completeness Theorem, but not all. Some applications actually need the notion of $\mathsf{K}_{\mathbb{A}}$-*proof* used in § III.5, since there is important information coded inside the proof.

The approach taken here also has the advantage of stressing the interplay of all branches of mathematical logic, which is one of the attractive features of admissible set theory.

3. *First Order Positive Inductive Definitions and* $\mathbb{HYP}_{\mathfrak{M}}$

We have seen various ways in which $\mathbb{HYP}_{\mathfrak{M}}$ is a mini-universe of set theory above $\mathfrak{M}$. For countable $\mathfrak{M}$, we have seen that the relations on $\mathfrak{M}$ which are elements of $\mathbb{HYP}_{\mathfrak{M}}$ are exactly the Δ_1^1 relations. This characterization breaks down for uncountable $\mathfrak{M}$ (see Exercise VII.1.16) so we are left with two problems in the general case:

To characterize the relations on $\mathfrak{M}$ which are elements of $\mathbb{HYP}_{\mathfrak{M}}$, and
To characterize the Δ_1^1 relations on $\mathfrak{M}$ in terms of $\mathbb{HYP}_{\mathfrak{M}}$.

The first of these two problems is solved by Theorem 3.6 below. The second problem is solved at the end of § VIII.2.

3.1 Definition (Moschovakis). Let K be a language and let Φ be the set of all finitary formulas of the form $\varphi(\mathsf{R}_+)$, for any new relation symbol R. Let $\mathfrak{N}$ be a structure for K and let S be a relation on $\mathfrak{N}$.

(i) S is a *(first order positive) fixed point* on $\mathfrak{N}$ if S is a Φ-fixed point (in the sense of 1.12) on $\mathfrak{N}$.
(ii) S is *inductive* on $\mathfrak{N}$ of S is Φ-inductive on $\mathfrak{N}$.
(iii) S is *coinductive* on $\mathfrak{N}$ if $\neg S$ is Φ-inductive on $\mathfrak{N}$.
(iv) S is *hyperelementary* on $\mathfrak{N}$ if S is inductive and coinductive on $\mathfrak{N}$.

(For more intuition into the notion of coinductive, the student should do Exercises 1.15—1.18.)

The theorems of this section are suggested by the following classical result.

3.2 Theorem. *Let $\mathcal{N} = \langle \omega, 0, +, \cdot \rangle$ and let S be a relation on $\mathcal{N}$.*
(i) *S is Π_1^1 on $\mathcal{N}$ iff S is inductive on $\mathcal{N}$.*
(ii) *S is Δ_1^1 on $\mathcal{N}$ iff S is hyperelementary on $\mathcal{N}$.*

Proof. We proved (i) in 1.10 and 1.11; (ii) is immediate from (i). □

Thus we see that for relations on $\mathcal{N}$,

$$\Sigma_1 \textit{ on } \mathbb{HYP}_{\mathcal{N}} = \textit{inductive on } \mathcal{N},$$

$$\textit{element of } \mathbb{HYP}_{\mathcal{N}} = \textit{hyperelementary on } \mathcal{N}.$$

We would like to generalize these equations from $\mathcal{N}$ to an arbitrary structure $\mathfrak{M}$. We would like to, but we can't because the generalization works only for $\mathfrak{M}$ which have some built in coding machinery. We discuss just how much coding is needed in the next section. For now we simply state one special case where all goes smoothly, and then take a different tack.

3.3 Theorem. *Let $\mathbb{A}$ be an admissible set and let S be a relation on $\mathbb{A}$.*
(i) *S is Σ_1 on $\mathbb{HYP}(\mathbb{A})$ iff S is inductive on $\mathbb{A}$.*
(ii) *S is an element of $\mathbb{HYP}(\mathbb{A})$ iff S is hyperelementary on $\mathbb{A}$.*

Proof. We merely sketch a proof since this result is a special case of Theorem 3.8 and the results of the next section. The proof sketched here is more direct. As usual, (ii) follows trivially from (i). We first show that if S is inductive on $\mathbb{A}$ then S is Σ_1 on $\mathbb{HYP}(\mathbb{A})$. It clearly suffices to prove the result for the case where S is a fixed point I_φ of some first order positive inductive definition Γ_φ. Since φ is first order over $\mathbb{A}$ it is Δ_0 in $\mathbb{HYP}(\mathbb{A})$ so Γ_φ is, in particular, a Σ_+ inductive definition over $\mathbb{HYP}(\mathbb{A})$, hence by Gandy's Theorem, I_φ is Σ_1 on $\mathbb{HYP}(\mathbb{A})$. (A more direct proof which works here but not in 3.8 is to observe that I_φ^β is a $\mathbb{HYP}(\mathbb{A})$-recursive function of β, for $\beta < o(\mathbb{HYP}(\mathbb{A}))$, and use Σ Reflection to prove that $\|\Gamma_\varphi\| \leqslant o(\mathbb{HYP}(\mathbb{A}))$. This would give the following Σ_1 definition of S:

$$S(\vec{x}) \quad \text{iff} \quad \mathbb{HYP}(\mathbb{A}) \models \exists \beta\, (\vec{x} \in I_\varphi^\beta).)$$

To prove the other half, suppose $S \subseteq \mathbb{A}$ is Σ_1 on $\mathbb{HYP}(\mathbb{A})$. By Theorem IV.7.3 (or, more precisely, Corollary 3.14 below) S is weakly representable in KPU′ using the $\mathbb{A}$-rule, where KPU′ is the theory

$$\text{KPU},$$
$$\text{diagram}(\mathbb{A}),$$
$$\overline{\mathsf{x}} \in \overline{\mathsf{A}} \quad (\text{all } x \in \mathbb{A}),$$
$$\exists a\, \forall v\, [v \in a \leftrightarrow \overline{\mathsf{A}}(v)].$$

But the set $C_{\mathbb{A}}(\text{KPU}')$ of consequences of KPU′ using the $\mathbb{A}$-rule is clearly an inductive subset of $\mathbb{A}$. Thus we have

$$S(\vec{x}) \quad \text{iff} \quad f(\vec{x}) \in C_{\mathbb{A}}(\text{KPU}')$$

for some $\mathbb{A}$-recursive function f. An easy exercise (Exercise 3.20) establishes that S is inductive on $\mathbb{A}$. □

We have been deliberately sketchy in the above proof to give the student a feel for the main idea. This must be gone into in more detail to prove Theorem 3.8 below, the main result of this section. First, though, let's draw some easy corollaries of Theorem 3.3.

3.4 Corollary. *If $\mathbb{A}$ is a countable admissible set then*

$$\Pi_1^1 \text{ on } \mathbb{A} = \text{inductive on } \mathbb{A},$$
$$\Delta_1^1 \text{ on } \mathbb{A} = \text{hyperelementary on } \mathbb{A}.$$

Proof. This is an immediate consequence of Theorem 3.3 and the results of § IV.3. □

3.5 Lemma. *Let $\mathbb{A}$ be admissible.*

(i) *There is an $(n+1)$-ary inductive relation on $\mathbb{A}$ which parametrizes the class of n-ary inductive relations on $\mathbb{A}$.*

(ii) *There is an inductive subset of $\mathbb{A}$ which is not hyperelementary.*

Proof. By V.5.3, $\mathbb{HYP}(\mathbb{A})$ is projectible into $\mathbb{A}$. Thus the lemma is just a restatement using 3.3. □

Using these results we can show just exactly how one gets from one admissible ordinal τ_α to the next admissible ordinal $\tau_{\alpha+1}$. Namely

$$\tau_{\alpha+1} = \sup\{\|\Gamma_\varphi\| : \Gamma_\varphi \text{ is a first order positive inductive definition over } \mathrm{L}(\tau_\alpha)\},$$

and this sup is actually obtained. This is a special case of the following result.

3.6 Corollary. *Let $\mathbb{A}$ be admissible and let $\alpha = o(\mathbb{HYP}(\mathbb{A}))$. Then α is equal to the* sup *of all $\|\Gamma_\varphi\|$ where Γ_φ is a first order positive inductive definition over $\mathbb{A}$, and this* sup *is actually attained.*

Proof. We know that any first order positive inductive definition Γ_φ over A is Σ_+ over $\mathbb{HYP}(\mathbb{A})$ (in fact "Δ_{0+}") so $\|\Gamma_\varphi\| \leqslant \alpha$ by the second half of Gandy's Theorem. To show that α is such an ordinal $\|\Gamma_\varphi\|$, use 3.5(ii) to choose an inductive subset $S \subseteq \mathbb{A}$ which is not hyperelementary. Then S is a section of some fixed point I_φ. Clearly I_φ is not hyperelementary either. We claim that $\|\Gamma_\varphi\| = \alpha$. As mentioned in the proof of Theorem 3.3, I_φ^β is a $\mathbb{HYP}(\mathbb{A})$ recursive function of β, for $\beta < \alpha$. Hence $I_\varphi^\beta \in \mathbb{HYP}(\mathbb{A})$ for all $\beta < \alpha$. But then, if $\|\Gamma_\varphi\| = \beta < \alpha$, $I_\varphi = I_\varphi^\beta \in \mathbb{H}YP(\mathbb{A})$ which makes I_φ hyperelementary, a contradiction. $\square$

As we'll see in the next section, the hypothesis that $\mathbb{A}$ is admissible is far too strong for the above results. All we really need is a reasonable amount of coding apparatus.

What we are really after, though, is a characterization of the relations on $\mathfrak{M}$ in $\mathbb{HYP}_\mathfrak{M}$ which works for *all* structures $\mathfrak{M}$, not just those with built in coding machinery. The best way around this is to slightly strengthen the notion of inductive definition, so that one can do the coding needed in the inductive definition itself.

3.7 Definition. Let Φ be the set of extended first order formulas $\varphi(\mathsf{R}_+)$ of $\mathsf{L}^*(\mathsf{R})$ as defined in II.2.7, p. 50. Let $\mathfrak{M}$ be a structure for L and let S be a relation on $\mathfrak{M}$ (or even $\mathbb{HF}_\mathfrak{M}$).

(i) S is *extended inductive* (written *inductive**) on $\mathfrak{M}$ iff S is Φ inductive on $\mathbb{HF}_\mathfrak{M}$.

(ii) S is *extended hyperelementary* (written *hyperelementary**) on $\mathfrak{M}$ iff S and $\neg S$ are inductive* on $\mathfrak{M}$.

Our second, and principal, generalization of Theorem 3.2 is the following result.

3.8 Theorem. *Let $\mathfrak{M} = \langle M, R_1, \dots, R_l\rangle$ be a structure for L and let S be a relation on $\mathfrak{M}$ (or even on $\mathbb{HF}_\mathfrak{M}$).*

(i) *S is Σ_1 on $\mathbb{HYP}_\mathfrak{M}$ iff S is inductive* on $\mathfrak{M}$.*

(ii) *S is Δ_1 on $\mathbb{HYP}_\mathfrak{M}$ iff S is hyperelementary* on $\mathfrak{M}$.*

Its corollaries are analogous to those of 3.3.

3.9 Corollary. *Let $\mathfrak{M} = \langle M, R_1, \dots, R_l\rangle$ be a countable structure for L.*

(i) *Π_1^1 on $\mathfrak{M}$ = inductive* on $\mathfrak{M}$.*

(ii) *Δ_1^1 on $\mathfrak{M}$ = hyperelementary* on $\mathfrak{M}$.* $\square$

3.10 Lemma. *Let $\mathfrak{M} = \langle M, R_1, \dots, R_l\rangle$ be a structure for L.*

(i) *There is an $(n+1)$-ary inductive* relation on $\mathbb{HF}_\mathfrak{M}$ which parameterizes the class of n-ary inductive* relations on $\mathbb{HF}_\mathfrak{M}$.*

(ii) *There is an inductive* relation on $\mathbb{HF}_\mathfrak{M}$ which is not hyperelementary*.*

Proof. $\mathbb{H}YP_{\mathfrak{M}}$ is projectible into $\mathbb{H}F_{\mathfrak{M}}$, again by V.5.3, so the results follows from V.5.6 and 3.8. □

We use these corollaries to get the most intelligible description yet of $O(\mathfrak{M})$ (and hence of $\mathbb{H}YP_{\mathfrak{M}}$ since $\mathbb{H}YP_{\mathfrak{M}} = L(\alpha)_{\mathfrak{M}}$ where $\alpha = O(\mathfrak{M})$).

3.11 Theorem. *If* $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ *is a structure for* L *then*

$$O(\mathfrak{M}) = \sup\{\|\Gamma_\varphi\| \mid \Gamma_\varphi \textit{ is an extended inductive definition over } \mathfrak{M}\}$$

and this sup *is actually attained.*

Proof. The proof of 3.11 is exactly like the proof of 3.6 when $O(\mathfrak{M}) > \omega$ for then $\mathbb{H}F_{\mathfrak{M}} \in \mathbb{H}YP_{\mathfrak{M}}$. Suppose $\mathbb{H}YP_{\mathfrak{M}}$ has ordinal ω. Let Γ_φ be an extended first order inductive definition on $\mathfrak{M}$. As we will see in the proof of Theorem 3.8, Γ_φ is Σ_+ on $\mathbb{H}YP_{\mathfrak{M}}$, so $\|\Gamma_\varphi\| \leqslant \omega$ by the second half of Gandy's Theorem. It is simple to give an example of extended first order inductive definitions of length ω, e. g.,

$$x \in \Gamma(R) \quad \text{iff} \quad \text{“}x \text{ is a natural number } \wedge \forall y < x\, R(y)\text{”}$$

defines ω in $\mathbb{H}F_{\mathfrak{M}}$ with

$$I_\Gamma^n = \{0, \ldots, n\}$$

so $\|\Gamma\| = \omega$. □

It is worthwhile digressing to compare 3.8 with the following consequence of 3.3, just to make sure the student is not confusing two distinct things.

3.12 Corollary. *Let* $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ *be a structure for* L *which is not recursively saturated. Let S be a relation on* $\mathfrak{M}$ *(or even* $\mathbb{H}F_{\mathfrak{M}}$*).*

(i) *S is* Σ_1 *on* $\mathbb{H}YP_{\mathfrak{M}}$ *iff S is inductive on* $\mathbb{H}F_{\mathfrak{M}}$.

(ii) $S \in \mathbb{H}YP_{\mathfrak{M}}$ *iff S is hyperelementary on* $\mathbb{H}F_{\mathfrak{M}}$.

Proof. Since $\mathfrak{M}$ is not recursively saturated, $o(\mathbb{H}YP_{\mathfrak{M}}) > \omega$ so $\mathbb{H}F_{\mathfrak{M}} \in \mathbb{H}YP_{\mathfrak{M}}$. But then $\mathbb{H}YP(\mathbb{H}F_{\mathfrak{M}}) = \mathbb{H}YP_{\mathfrak{M}}$ since $\mathbb{H}YP(\mathbb{H}F_M)$ is the smallest admissible set with $\mathbb{H}F_{\mathfrak{M}}$ as an element. Thus 3.10 is a special case of 3.3. □

The student must be clear about the difference between inductive* definitions on $\mathfrak{M}$ and inductive definitions on $\mathbb{H}F_{\mathfrak{M}}$. The latter are, in general, much more powerful since they allow unbounded universal quantification over sets in $\mathbb{H}F_{\mathfrak{M}}$ in addition to the unbounded existential allowed by inductive* definitions.

We have already done most of the work for proving Theorem 3.8 back in § III.3, the section on $\mathfrak{M}$-logic and the $\mathfrak{M}$-rule.

In the discussion below we let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ be a fixed L-structure and we let L$^+$ be an expansion of L with a new unary symbol $\overline{\mathsf{M}}$ and symbols $\overline{\mathsf{p}}$

for each $p \in \mathfrak{M}$, just as in our discussion of $\mathfrak{M}$-logic in § III.3. We assume that L^+ is coded up in an effective way on $\mathbb{HF}_{\mathfrak{M}}$.

3.13 Proposition. *Let T be a set of sentences of $\mathsf{L}^+_{\omega\omega}$ which is Σ_1 on $\mathbb{HF}_{\mathfrak{M}}$. Let $C_{\mathfrak{M}}(T)$ be the set of formulas of $\mathsf{L}^+_{\omega\omega}$ which are provable from T using the $\mathfrak{M}$-rule. Then $C_{\mathfrak{M}}(T)$ is inductive*.*

Proof. We simply write out the original definition Γ of $C_{\mathfrak{M}}(T)$ given in III.3.4 and observe that it has the correct form. Let R be a new unary symbol and define Γ by

$$x \in \Gamma(R) \quad \text{if} \quad x \in \mathsf{L}^+_{\omega\omega} \wedge [(1) \vee \cdots \vee (5)]$$

where (1)...(5) are given below.

(1) (Logical Axioms) "x is an axiom of first order logic";
(2) (Nonlogical Axioms) $x \in T$;
(3) (Modus Ponens) $\exists y\, [y \in R \wedge (y \to x) \in R]$;
(4) (Generalization) "x is of the form $(\psi \to \forall v\, \theta(v))$ where v is not free in ψ and $(\psi \to \theta(v)) \in R$";
(5) ($\mathfrak{M}$-rule) "x is of the form $\forall v_0\, [\overline{\mathsf{M}}(v_0) \to \theta(v_0)]$ and for all $p \in M$, $\theta(\overline{\mathsf{p}}/v_0) \in R$".

Clearly Γ defines $C_{\mathfrak{M}}(T)$, i. e., $C_{\mathfrak{M}}(T) = I_\Gamma$ so that $C_{\mathfrak{M}}(T)$ is actually a fixed point. Γ is definable over $\mathbb{HF}_{\mathfrak{M}}$ by an R-positive formula; the only unbounded universal quantifier is in (5) and it is a quantifier over M. □

The reader may remember that we left a couple of proofs unfinished in § IV.7, the section on representability using the $\mathfrak{M}$-rule. We proved IV.7.3 and IV.7.4 in the countable case but left the absoluteness of those results until later. Proposition 3.13 allows us to finish these proofs.

3.14 Corollary. *Assume the notation of Proposition* 3.13.

(i) *$x \in C_{\mathfrak{M}}(T)$ is a Δ_1 predicate of x, T and $\mathfrak{M}$, Δ_1 in the theory* ZF.

(ii) *Consequently, the proofs given in* § IV.7 *of* IV.7.3 *and* IV.7.4 *for the countable case, together with Lévy's Absoluteness Principle, yield the general results.*

Proof. Part (i) is a consequence of Remark 2.4. For (ii), the proofs of IV.7.3 and IV.7.4 are quite similar. Since IV.7.3 is the more important for us here (we apply it in the next proof) let us treat it in some detail. Again 7.3 (i) and 7.3 (ii) are similar so we prove (i). Suppose, as in the proof of (i), that $\varphi(x_1, \ldots, x_n, p_1, \ldots, p_k, M)$ is a Σ_1 formula with the property that for all $q_1, \ldots, q_n \in M$

$$\mathbb{HYP}_{\mathfrak{M}} \models \varphi(q_1, \ldots, q_n, \vec{p}, M) \quad \text{iff} \quad \mathrm{KPU}^+ \models_{\mathfrak{M}} \varphi(\overline{\mathsf{q}}_1, \ldots, \overline{\mathsf{q}}_n, \overline{\mathsf{p}}, \overline{\mathsf{M}}).$$

Now, if $\mathfrak{M}$ is countable we use the $\mathfrak{M}$-completeness theorem to write

$$\mathbb{HYP}_{\mathfrak{M}} \models \varphi(q_1, \ldots, q_n, \vec{p}, M) \quad \text{iff} \quad \mathrm{KPU}^+ \vdash_{\mathfrak{M}} \varphi(\overline{\mathsf{q}}_1, \ldots, \overline{\mathsf{q}}_n, \overline{\mathsf{p}}, \overline{\mathsf{M}}).$$

I. e., we have for all *countable* $\mathfrak{M}$ and all $q_1,\ldots,q_n \in M$:

$$\mathbb{HYP}_{\mathfrak{M}} \models \varphi(q_1,\ldots,q_n,\vec{p},M) \quad \text{iff} \quad \varphi(\bar{\mathsf{q}}_1,\ldots,\bar{\mathsf{q}}_n,\bar{\mathsf{p}},\bar{\mathsf{M}}) \in C_{\mathfrak{M}}(\mathrm{KPU}^+).$$

We claim that this is a Δ_1 predicate of $\mathfrak{M}$, Δ_1 in ZF. The right hand side of the iff is Δ_1 by (i), and the left hand side is Δ_1 since satisfaction is Δ_1 and since $\mathbb{HYP}_{\mathfrak{M}}$ is a Σ_1 operation of $\mathfrak{M}$ by the argument given in IV.3.5. By Lévy Absoluteness, the result holds for *all* $\mathfrak{M}$. □

Theorem 3.8 will follow from Proposition 3.13 given the next lemma. It is a special case of the Combination Lemma of Moschovakis [1974].

3.15 Lemma. *Let* $U \subseteq \mathbb{HF}_{\mathfrak{M}}$ *be inductive*, let* $f\colon \mathbb{HF}_{\mathfrak{M}}^n \to \mathbb{HF}_{\mathfrak{M}}$ *be* Σ_1 *on* $\mathbb{HF}_{\mathfrak{M}}$ *and let* P *be defined by*

$$P(x_1,\ldots,x_n) \quad \textit{iff} \quad f(x_1,\ldots,x_n) \in U.$$

Then P *is inductive* on* $\mathfrak{M}$.

Proof. Suppose U is a section of the fixed point I_φ where $\varphi(v_1,v_2,\mathsf{R}_+)$ is extended first order positive on $\mathfrak{M}$, say

$$U(y) \leftrightarrow ((y,z_0) \in I_\varphi).$$

We define an $n+3$-ary inductive* definition Γ_ψ so that a section of I_ψ^ξ (with $i=0$) imitates I_φ^α and the section with $i=1$ takes care of f. Define $\psi(i,x_1,\ldots,x_n,v_1,v_2,\mathsf{S}_+)$, where S is $n+3$-ary, by the following, where $t_1,\ldots,t_n,z_1,z_2$ are arbitrary but fixed elements of $\mathbb{HF}_{\mathfrak{M}}$:

$$i=0 \wedge \vec{x}=\vec{t} \wedge \varphi(v_1,v_2,\lambda w_1 w_2\, \mathsf{S}(0,t_1,\ldots,t_n,w_1,w_2)/\mathsf{R}), \quad \text{or}$$
$$i=1 \wedge v_1,v_2=z_1,z_2 \wedge S(0,t_1,\ldots,t_n,f(x_1,\ldots,x_n),z_0).$$

A simple proof by induction shows that

(6) $I_\varphi^\alpha(v_1,v_2)$ iff $I_\psi^\alpha(0,t_1,\ldots,t_n,v_1,v_2)$

so that

$$U(y) \quad \text{iff} \quad I_\psi(0,t_1,\ldots,t_n,y,z_0).$$

Another proof by induction, using (6), shows that

$$(f(x_1,\ldots,x_n),z_0) \in I_\varphi^\alpha \quad \text{iff} \quad (1,x_1,\ldots,x_n,z_1,z_2) \in I_\psi^{\alpha+1}.$$

Thus

$$P(x_1,\ldots,x_n) \quad \text{iff} \quad (1,x_1,\ldots,x_n,z_1,z_2) \in I_\psi$$

so P is a section of I_ψ. The only universal quantifiers in ψ are those in φ so ψ is extended first order positive. $\square$

We now return to prove the main theorem of this section, Theorem 3.8.

3.16 Proof of Theorem 3.8. (i) Let Γ_φ be an extended first order inductive definition over $\mathfrak{M}$. Since $\mathbb{HF}_{\mathfrak{M}}$ is a Σ_1 subset of $\mathbb{HYP}_{\mathfrak{M}}$, relativizing the unbounded (existential) set quantifiers in Γ_φ to $\mathbb{HF}_{\mathfrak{M}}$ and relativizing the unbounded quantifiers over $\mathfrak{M}$ to the set M turns Γ_φ into a Σ_+ inductive definition over $\mathbb{HYP}_{\mathfrak{M}}$ and hence Γ_φ has a Σ_1 fixed point I_φ, by Gandy's Theorem.

To prove the other half, let us consider a relation S on $\mathfrak{M}$ which is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. By Theorem IV.7.3, S is weakly representable in KPU^+ using the $\mathfrak{M}$-rule. Thus there is a formula $\varphi(v_1,\ldots,v_n)$ of L^* such that for all $x_1,\ldots,x_n \in M$,

$$S(x_1,\ldots,x_n) \quad \text{iff} \quad \varphi(\overline{\mathsf{x}}_1,\ldots,\overline{\mathsf{x}}_n) \in C_{\mathfrak{M}}(\mathrm{KPU}^+).$$

Now, by 3.13, $C_{\mathfrak{M}}(\mathrm{KPU}^+)$ is inductive* over $\mathfrak{M}$. Let $f(x_1,\ldots,x_n) = \varphi(\overline{\mathsf{x}}_1/v_1,\ldots,\overline{\mathsf{x}}_n/v_n)$. Then

$$S(x_1,\ldots,x_n) \quad \text{iff} \quad f(x_1,\ldots,x_n) \in C_{\mathfrak{M}}(\mathrm{KPU}^+)$$

so S is inductive* by Lemma 3.15. The same proof works if $S \subseteq \mathbb{HF}_{\mathfrak{M}}$ except that Exercise IV.7.5 replaces Theorem IV.7.3. Part (ii) follows from (i) as usual. $\square$

The final results of this section show that for nonadmissible sets of the form $\mathrm{L}(\alpha)_{\mathfrak{M}}$ (for example), Σ_+ inductive definitions are just as strong as arbitrary first order inductive definitions, and that they are just as long. The results thus yield partial converses to the results of § 2 by showing how necessary the assumption of admissibility was for those results.

3.17 Theorem. *Let $M \subseteq a$ where a is transitive in $\mathbb{V}_M$ and let β be any limit ordinal such that*

$$\mathbb{A}_{\mathfrak{M}} = (\mathfrak{M}; \mathrm{L}(a,\beta) \cap \mathbb{V}_{\mathfrak{M}}, \in)$$

is not admissible.

(i) *A relation S on $\mathbb{A}_{\mathfrak{M}}$ is Σ_1 on $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$ iff S is Σ_+ inductive on $\mathbb{A}_{\mathfrak{M}}$.*

(ii) *The ordinal $o(\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}}))$ is equal to*

$$\sup\{\|\Gamma_\varphi\| \mid \Gamma_\varphi \text{ is a } \Sigma_+ \text{ inductive definition on } \mathbb{A}_{\mathfrak{M}}\}$$

and the sup *is actually attained.*

3.18 Corollary. *Let $M \subseteq a$ where a is transitive in $\mathbb{V}_{\mathfrak{M}}$ and let β be any limit ordinal. Let*

$$\mathbb{A}_{\mathfrak{M}} = (\mathfrak{M}; \mathrm{L}(a,\beta) \cap \mathbb{V}_{\mathfrak{M}}, \in).$$

The following are equivalent, where $\alpha = o(\mathbb{A}_{\mathfrak{M}})$.

(i) $\mathbb{A}_{\mathfrak{M}}$ *is admissible.*

(ii) *Every* Σ_+ *inductive set on* $\mathbb{A}_{\mathfrak{M}}$ *is* Σ_1 *on* $\mathbb{A}_{\mathfrak{M}}$.

(iii) *For every* Σ_+ *inductive definition* Γ_φ *on* $\mathbb{A}_{\mathfrak{M}}$, $\|\Gamma_\varphi\| \leqslant \alpha$.

Proof. By the results of the previous section, (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii). To prove (ii)$\Rightarrow$(i), suppose $\mathbb{A}_{\mathfrak{M}}$ is not admissible. Let S be a subset of $\mathbb{A}_{\mathfrak{M}}$ which is Σ_1 on $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$ but not $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$-finite; such an S exists since $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$ is projectible into $\mathbb{A}_{\mathfrak{M}}$. But then S is Σ_+ inductive on $\mathbb{A}_{\mathfrak{M}}$ by 3.17. S cannot be Σ_1 on $\mathbb{A}_{\mathfrak{M}}$ for then it would be Δ_0 on $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$, hence in $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$. Thus $\neg$(i)$\Rightarrow\neg$(ii). For the same reason, the length $\|\Gamma_\varphi\|$ of an inductive definition of S could not be $\leqslant\alpha$ so $\neg$(i)$\Rightarrow\neg$(iii). $\square$

The proof of Theorem 3.17 uses ideas similar to those used in the proofs of Theorem 3.3 and 3.8. We leave a few of the details to the student.

Proof of Theorem 3.17. We prove (i) assuming $\mathbb{A}_{\mathfrak{M}}$ is countable, leaving the extension (via Lévy's Absoluteness Principle) to the student. The ($\Leftarrow$) half of (i) is obvious, so let S be a relation on $\mathbb{A}_{\mathfrak{M}}$ which is Σ_1 on $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$. Every $x \in \mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$ has a good Σ_1 definition with parameters from $L(a,\beta) \cup \{L(a,\beta)\}$ by II.5.14. Since $\mathbb{A}_{\mathfrak{M}}$ is not admissible, β and hence $L(a,\beta)$ also have Σ_1 definitions on $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$ with parameters from $L(a,\beta)$ by the last step in the proof of II.5.14. Thus every $x \in \mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$ has a Σ_1 definition with parameters from $L(a,\beta)$. But then S has a Σ_1 definition (as a subset, now, not an element) with parameters from $L(a,\beta)$ since the other parameters can be defined away. Thus suppose that for all $x \in L(a,\beta)$

$$S(x) \quad \text{iff} \quad \mathbb{HYP}(\mathbb{A}_{\mathfrak{M}}) \models \varphi(x,y)$$

where $y \in L(a,\beta)$ and φ is Σ_1. By the Truncation Lemma $S(x)$ is equivalent to

$$(7) \text{ for all } \quad \mathfrak{B}_{\mathfrak{M}} \supseteq_{\text{end}} \mathbb{A}_{\mathfrak{M}}, \quad \text{if} \quad \mathfrak{B}_{\mathfrak{M}} \models \text{KPU} \quad \text{then} \quad \mathfrak{B}_{\mathfrak{M}} \models \varphi(x,y).$$

Since β is a limit, $L(a,\beta)$ is closed under pairs, union and Δ_0 Separation so we may code up $\mathsf{K} = \mathsf{L}^* \cup \{\overline{\mathsf{x}} \mid x \in \mathbb{A}_{\mathfrak{M}}\}$ on $\mathbb{A}_{\mathfrak{M}}$. Let $\mathsf{K}_{\mathbb{A}}$ be the (nonadmissible) fragment of $\mathsf{K}_{\infty\omega}$ given by $\mathbb{A}_{\mathfrak{M}}$. Let $T \subseteq \mathsf{K}_{\mathbb{A}}$ the the theory

KPU

Diagram$(\mathbb{A}_{\mathfrak{M}})$

$\forall v\,[v \in \overline{\mathsf{a}} \leftrightarrow \bigvee_{x\in a} v = \overline{\mathsf{x}}]$, for all $a \in \mathbb{A}_{\mathfrak{M}}$,

$\forall p\,[p \in \overline{\mathsf{M}}]$.

Every model of T is isomorphic to some $\mathfrak{B}_{\mathfrak{M}} \supseteq_{\text{end}} \mathbb{A}_{\mathfrak{M}}$ so (7) is equivalent to

$$T \models \varphi(\overline{\mathsf{x}}, \overline{\mathsf{y}}).$$

By Theorem III.4.5 (really III.4.6) this is equivalent to saying that $\varphi(\overline{\mathsf{x}},\overline{\mathsf{y}})$ is in the smallest set of sentences of $\mathsf{K}_{\mathbb{A}}$ containing T and (A 1)—(A 7) which is closed under (R 1)—(R 3). This clearly amounts to a Σ_+ inductive definition Γ_φ such that

$$R(x) \quad \text{iff} \quad \varphi(\overline{\mathsf{x}},\overline{\mathsf{y}}) \in I_\Gamma .$$

Therefore R is Σ_+ inductive by Exercise 3.21.

To prove (ii) we need only find a Σ_+ inductive definition on $\mathbb{A}_{\mathfrak{M}}$ with length $o(\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}}))$. Let $R \subseteq \mathbb{A}_{\mathfrak{M}}$ be $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$-r.e. but not an element of $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$. There is such an R since $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$ is projectible into $\mathbb{A}_{\mathfrak{M}}$ by V.5.4. Then R is a section of I_φ, where Γ_φ is some Σ_+ inductive definition. But now the argument used earlier, in the proof of 3.6 for example, shows that $\|\Gamma_\varphi\| = o(\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}}))$. □

3.19—3.22 Exercises

3.19. Let $C_{\mathfrak{M}}(\mathrm{KPU}^+) = I_\varphi$, where Γ_φ is extended inductive, by 3.13. Show that $O(\mathfrak{M}) = \|\Gamma_\varphi\|$. Thus, for example, $O(\mathfrak{M})$ is just the least ordinal not assigned to a proof using the $\mathfrak{M}$-rule, under the usual assignment of ordinals to proofs.

3.20. Let $\mathfrak{A}$ be a structure, let U be inductive on $\mathfrak{A}$ and let $f\colon A^n \to A$ be first order definable. Modify the proof of 3.15 to show that

$$P(\vec{x}) \quad \text{iff} \quad U(f(\vec{x}))$$

defines an inductive relation on $\mathfrak{A}$.

3.21. Let $\mathfrak{A}$ be a structure, let $U \subseteq \mathfrak{A}$ be Σ_+ inductive on $\mathfrak{A}$, let $f\colon A^n \to A$ have a Σ_1 graph and define P by

$$P(x_1,\ldots,x_n) \quad \text{iff} \quad f(x_1,\ldots,x_n) \in U .$$

Show that P is Σ_+ inductive on $\mathfrak{A}$. [Mimic the proof of 3.15.]

3.22. Give the absoluteness argument for lifting Theorem 3.17 from the countable to the uncountable.

3.23 Notes. The main results of this section are from Barwise-Gandy-Moschovakis [1971], at least in the case of pure admissible sets. Theorem 3.17 and its corollaries are new here.

4. *Coding* $\mathbb{HF}_{\mathfrak{M}}$ *on* $\mathfrak{M}$

A *pairing function* on a set M is simply a one-one function mapping $M \times M$ into M. An n-ary function f on a structure $\mathfrak{M}$ is inductive (or hyperelementary) if its graph is an $(n+1)$-ary inductive (or hyperelementary, respectively) relation on $\mathfrak{M}$. In this section we show how to code $\mathbb{HF}_{\mathfrak{M}}$ on $\mathfrak{M}$ using an inductive pairing function on $\mathfrak{M}$. Our goal is to prove the following theorem.

4.1 Theorem. *Let $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ be a structure with an inductive pairing function. The inductive and inductive* relations on $\mathfrak{M}$ coincide.*

We give the applications of this theorem (and a couple of related results obtained along the way) in the next section by showing how a great many results on inductive relations on $\mathfrak{M}$ can be obtained in a simple fashion by projecting the recursion theory of $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}$. In so doing, we tie up the theory of admissible sets with the theory of inductive relations as developed in Moschovakis [1974]. Since our aim in these sections is to relate our theory to Moschovakis' theory, we feel only mildly apologetic for using without proof two results (4.2 and 4.3 below) from Chapter 1 of Moschovakis [1974]. The proofs are sketched in Exercises 4.17 and 4.18.

A relation P on $\mathfrak{M}$ is defined from Q by *hyperelementary substitution* if there are hyperelementary functions $f_1,\ldots,f_k$ so that

$$P(x_1,\ldots,x_n) \quad \text{iff} \quad Q(f_1(x_1,\ldots,x_n),\ldots,f_k(x_1,\ldots,x_n))$$

for all $x_1,\ldots,x_n \in \mathfrak{M}$.

4.2 Theorem. *The inductive relations on $\mathfrak{M}$ contain all first order relations and are closed under $\wedge$, $\vee$, $\exists$, $\forall$ and hyperelementary substitution. Hence, the hyperelementary relations on $\mathfrak{M}$ contain all first order relations on $\mathfrak{M}$ and are closed under $\neg$, $\wedge$, $\vee$, $\exists$, $\forall$ and hyperelementary substitution.*

Proof. This result follows easily from 4.3. See Theorem 1D.1 of Moschovakis [1974] or Exercise 4.18. □

The inductive relations on $\mathfrak{M}$ are closed under induction in a sense made precise by 4.3.

4.3 Theorem. *Let $S_1,\ldots,S_k$ be relations on $\mathfrak{M}$ and consider an inductive definition Γ_φ over the expanded structure $(\mathfrak{M},S_1,\ldots,S_k)$, where φ is of the form $\varphi(x_1,\ldots,x_n,\mathsf{R}_+,\mathsf{S}_1,\ldots,\mathsf{S}_k)$ in $\mathsf{L}\cup\{\mathsf{R},\mathsf{S}_1,\ldots,\mathsf{S}_k\}$.*

(i) *If $S_1,\ldots,S_k$ are hyperelementary on $\mathfrak{M}$ then the fixed point I_φ defined on $(\mathfrak{M},S_1,\ldots,S_k)$ is inductive on the original structure $\mathfrak{M}$.*

(ii) *If $S_1,\ldots,S_k$ are inductive on $\mathfrak{M}$ then the conclusion of* (i) *still holds provided φ is S_i-positive for $i=1,\ldots,k$.*

(iii) *In either case* (i) *or* (ii), *I_φ is a section of a fixed point I_ψ for some $\psi(x_1,\ldots,x_m,\mathsf{R}_+)\in\mathsf{L}\cup\{\mathsf{R}\}$ with $\|\Gamma_\psi\|\geqslant\|\Gamma_\varphi\|$.*

Proof. See Theorem 1C.3 of Moschovakis [1974] or Exercise 4.17. □

There is one simple consequence of 4.2 that deserves mention. If f is an inductive function on $\mathfrak{M}$ and if its domain D is hyperelementary (e. g., if f is total) then f is hyperelementary, since

$$f(x_1,\ldots,x_n)\neq y \quad \text{iff} \quad (x_1,\ldots,x_n\notin D \vee \exists z\,[f(x_1,\ldots,x_n)=z \wedge z\neq y]\,.$$

Thus, if $\mathfrak{M}$ has an inductive pairing function p, p is actually hyperelementary since p is total.

The plan for the proof of Theorem 4.1 is simple. Fix an inductive pairing function p on $\mathfrak{M}$. We are going to use p to assign notations to the elements of $\mathbb{HF}_M$. The set T of notations will be inductive on $\mathfrak{M}$ but not, in general, hyperelementary. An extended first order formula of the form

$$\exists a \in \mathbb{HF}_{\mathfrak{M}}(\ldots)$$

will translate into

$$\exists x\,(x \in T \wedge \cdots)$$

which will keep us within the class of inductive relations since the inductive set T occurs positively. On the other hand, a quantifier of the form

$$\forall a \in \mathbb{HF}_{\mathfrak{M}}(\ldots)$$

would translate into

$$\forall x\,(x \notin T \vee \cdots)$$

which is not permitted since T occurs negatively. The only complications in the proof are caused by the following two facts. Since $\{p,q\}=\{q,p\}$ we are not going to be able to have unique notations for the elements of $\mathbb{HF}_{\mathfrak{M}}$. Secondly, we must find some way to handle bounded universal quantifiers in a positive way. (This accounts for the relation $\check{\mathscr{E}}$ used below and most of the other complications.)

The notation system used is based upon the fact that $\mathbb{HF}_{\mathfrak{M}}$ is the closure of $M \cup \{0\}$ under the operation

$$S(x,y)=x \cup \{y\}.$$

Define a hierarchy $\mathrm{HF}_{\mathfrak{M}}^{(n)}$ as follows:

$$\mathrm{HF}_{\mathfrak{M}}^{(0)}=\{0\},$$
$$\mathrm{HF}_{\mathfrak{M}}^{(n+1)}=\mathrm{HF}_{\mathfrak{M}}^{(n)} \cup \{a \cup \{x\} : a, x \in \mathrm{HF}_{\mathfrak{M}}^{(n)} \cup M\}.$$

This hierarchy grows more slowly than the $\mathrm{HF}_{\mathfrak{M}}(n)$ hierarchy used in § II.2 but it eventually gets the job done.

4.4 Lemma. $\mathbb{HF}_{\mathfrak{M}}=\bigcup_{n<\omega} \mathrm{HF}_{\mathfrak{M}}^{(n)}$.

Proof. Suppose there were some set $a \in \mathbb{HF}_{\mathfrak{M}}$ which did not appear at any stage of our new hierarchy. Among such sets a choose one of least rank and, among those of least rank, choose one of smallest cardinality. Since $0 \in \mathrm{HF}_{\mathfrak{M}}^{(0)}$, a is non-empty so we may write

$$a=\{x_1,\ldots,x_{k+1}\}.$$

Let $a_0 = \{x_1, \ldots, x_k\}$. Since $\mathrm{rk}(a_0) \leqslant \mathrm{rk}(a)$ and $\mathrm{card}(a_0) < \mathrm{card}(a)$, a_0 is formed in our new hierarchy, by choice of a. Since $\mathrm{rk}(x_{k+1}) < \mathrm{rk}(a)$, x_{k+1} is also formed. Pick n so that both a_0 and x_{k+1} are in $\mathrm{HF}_{\mathfrak{M}}^{(n)}$. Then $a = S(a_0, x_{k+1})$ is in $\mathrm{HF}_{\mathfrak{M}}^{(n+1)}$. □

Let M be an infinite set with pairing function $p: M \times M \to M$. Let x_0, x_1, x_2 be distinct elements of M. We use the following notational conventions.

$$\begin{array}{lll} \emptyset & \text{for} & p(x_0, x_0), \\ \dot{x} & \text{for} & p(x_1, x), \\ x \,\dot{\sigma}\, y & \text{for} & p(x_2, p(x, y)). \end{array}$$

4.5 Lemma. *The functions f_1, f_2 defined below are one-one, they have disjoint ranges and $\emptyset$ is in the range of neither. They are $\mathbb{H}\mathrm{F}_{(\mathfrak{M},p)}$-recursive and hyper-elementary on $(\mathfrak{M}, p)$:*

$$f_1(x) = \dot{x} \qquad f_2(x, y) = x \,\dot{\sigma}\, y.$$

Proof. This is immediate since p is one-one and x_0, x_1, x_2 are distinct. □

We use these functions to define two sets of closed terms: the ur-terms denote elements of M; the set-terms denote hereditarily finite sets over M.

4.6 Definition. (i) For each $x \in M$, $\dot{x}$ is an ur-term and $\dot{x}$ denotes x, written

$$|\dot{x}| = x.$$

The set of ur-terms is called T_u.

(ii) The set T_s of set-terms and the function $|\cdot|$ mapping T_s onto $\mathbb{H}\mathrm{F}_M$ are defined inductively:

a) $\emptyset$ is in T_s and $\emptyset$ is a notation for 0, i. e.,

$$|\emptyset| = 0.$$

b) If x is in T_s and y is in $T_u \cup T_s$ and if $|y| \notin |x|$ then $x \,\dot{\sigma}\, y$ is in T_s and

$$|x \,\dot{\sigma}\, y| = |x| \cup \{|y|\}.$$

(iii) The set T of all notations is $T_u \cup T_s$.

We require $|y| \notin |x|$ to keep the set of notations of each $a \in \mathbb{H}\mathrm{F}_M$ finite.

The definition of T_s is an inductive definition, not over $(\mathfrak{M}, p)$ but rather over $\mathbb{H}\mathrm{F}_{(\mathfrak{M},p)}$. One of our tasks is to show that T_s is actually inductive over $(\mathfrak{M}, p)$ after all.

Note that by Lemma 4.4, every $a \in \mathbb{HF}_M$ is $|x|$ for some $x \in T_s$. Define the following relations on M:

$$x \mathscr{E} y \quad \text{iff} \quad x, y \in T \text{ and } |x| \in |y|;$$

$$x \check{\mathscr{E}} y \quad \text{iff} \quad y \in T \text{ and if } x \in T \text{ then } |x| \notin |y|;$$

$$x \approx y \quad \text{iff} \quad x, y \in T \text{ and } |x| = |y|;$$

$$x \check{\approx} y \quad \text{iff} \quad y \in T \text{ and if } x \in T \text{ then } |x| \neq |y|.$$

4.7 Main Lemma. *The sets T_s, T and the relations $\mathscr{E}$, $\check{\mathscr{E}}$, $\approx$, and $\check{\approx}$ are all inductive on (M,p). The set T_u is definable on (M,p).*

Proof. It is clear that T_u is definable on (M,p) since

$$y \in T_u \quad \text{iff} \quad \exists x (y = \dot{x}).$$

We will give an informal simultaneous inductive definition of the six other relations as well as two auxillary relations R and $\check{R}$. First, however, let N be the smallest subset of M containing $\emptyset$ and closed under

$$\text{if} \quad x \in N \quad \text{then} \quad (x \,\delta\, x) \in N.$$

Thus N is inductive on (M,p) and N contains a unique notation for each natural number. We will confuse a natural number with its notation in this proof. Define

$$R(n,x) \quad \text{iff} \quad n \in N \text{ and } x \in T_s \text{ and } |x| \in \mathrm{HF}_M^{(n)};$$

$$\check{R}(n,x) \quad \text{iff} \quad n \in N \text{ and if } x \in T_s \text{ then } |x| \notin \mathrm{HF}_M^{(n)}.$$

The following clauses constitute a simultaneous inductive definition of all the above relations. It should be pretty obvious to the reader how one could turn this into one giant inductive definition over (M,p) and then extract the given relations as sections. (If he needs help, the student can consult the Simultaneous Induction Lemma on p. 12 of Moschovakis [1974].)

(1) $x \in T_s$ iff $x = \emptyset$ or there is a $y \in T_s$ and a $z \in T_u \cup T_s$ such that $z \check{\mathscr{E}} y$ and x is $y \,\delta\, z$.

(2) $x \in T$ iff $x \in T_u$ or $x \in T_s$.

(3) $x \mathscr{E} y$ iff $y \in T_s$ and y is of the form $u \,\delta\, v$ and $x \mathscr{E} u$ or $x \approx v$.

(4) $x \check{\mathscr{E}} y$ iff $y \in T$ and y is $\emptyset$ or $y \in T_u$ or y is of the form $u \,\delta\, v$ and $x \check{\mathscr{E}} u$ and $x \check{\approx} v$.

(5) $x \approx y$ iff $x, y \in T$ and $x = y$ or $x, y \in T_s$ and for every z $(z \check{\mathscr{E}} x \vee z \mathscr{E} y)$ and $(z \check{\mathscr{E}} y \vee z \mathscr{E} x)$.

(6) $R(0,x)$ iff $x = \emptyset$;

$R(n+1,x)$ iff $x \in T_s$ and $R(n,x)$ or else x is of the form $y \check{\sigma} z$ where $R(n,y)$ and $(z \in T_u \vee R(n,z))$.

(7) $\breve{R}(0,x)$ iff $x \neq \emptyset$;

$\breve{R}(n+1,x)$ iff $\breve{R}(n,x)$ and either x is not of the form $u \check{\sigma} v$ (for all u, v) or else x is of the form $u \check{\sigma} v$ but one of the following holds:

$$v \mathscr{E} u, \qquad \breve{R}(n,u), \qquad \breve{R}(n,v).$$

(8) $x \breve{\approx} y$ iff there is an $n \in N$ such that $\breve{R}(n,x)$ but $R(n,y)$ or there is an $n \in N$ such that $R(n,x)$ and $R(n,y)$ (in which case x is in T_s) and there is a z such that

$$((z \mathscr{E} x \wedge z \breve{\mathscr{E}} y) \vee (z \mathscr{E} y \wedge z \breve{\mathscr{E}} x)).$$

It takes a bit of checking to see that in each case the induction is pushed back, but this checking is best done on scratch paper. ▯

The relations $R, \breve{R}$ used above are needed only to prove the Main Lemma. They should not be confused with other relations R used later on.

We are now ready to fill in the outline of the proof of Theorem 4.1. For simplicity of notation let us suppose our language L has only one binary symbol Q. Let R be a new relation symbol for use in inductive definitions. We consider $\mathsf{L}^*(\mathsf{R}) = \mathsf{L}(\in, \mathsf{R})$ as a single sorted language with unary symbols U (for urelements) and S (for sets) with bounded quantification as a primitive. We let K be a new language with atomic symbols

$$\mathsf{Q}, \mathsf{U}, \mathsf{S}, \mathsf{R}, \mathscr{E}, \breve{\mathscr{E}}, \approx, \breve{\approx}.$$

We define a mapping $\hat{}$ from $\mathsf{L}^*(\mathsf{R})$ into K as follows: given $\varphi \in \mathsf{L}^*(\mathsf{R})$, first push the negations inside as far as possible so that the only negative subformulas in φ are negated atomic. Replace each positive occurrence of $x \in y$ by $x \mathscr{E} y$, each occurrence of $\neg(x \in y)$ by $x \breve{\mathscr{E}} y$, each positive occurrence of $x = y$ by $x \approx y$, each occurrence of $\neg(x = y)$ by $x \breve{\approx} y$, each bounded quantifer

$$\forall x \in y(\ldots) \quad \text{by} \quad \forall x\, (x \breve{\mathscr{E}} y \vee \ldots),$$

$$\exists x \in y(\ldots) \quad \text{by} \quad \exists x\, (x \mathscr{E} y \wedge \ldots).$$

Thus, in $\hat{\varphi}$, all occurrences of $\mathscr{E}, \breve{\mathscr{E}}, \approx, \breve{\approx}$ are positive. If φ is extended first order then S also occurs positively in $\hat{\varphi}$ since it only appears in the contexts

$$\exists x\, (\mathsf{S}(x) \wedge \ldots)$$

and

$$\exists x\, ((\mathsf{U}(x) \vee \mathsf{S}(x)) \wedge \ldots).$$

Let M be the infinite set with pairing function p used above. Let Q be any binary relation on M. Define $\tilde{Q}$ on T_u by

$$\tilde{Q}(\dot{p}, \dot{q}) \quad \text{iff} \quad Q(p,q)$$

for all $p, q \in M$ so that map $t \to |t|$ gives an isomorphism of $(T_u, \tilde{Q})$ onto $\mathfrak{M} = (M, Q)$. We let $\tilde{\mathfrak{M}}$ be the structure for K with universe M and with interpretations given by

symbol:	U	S	Q	$\mathscr{E}$	$\check{\mathscr{E}}$	$\approx$	$\check{\approx}$
interpretation:	T_u	T_s	$\tilde{Q}$	$\mathscr{E}$	$\check{\mathscr{E}}$	$\approx$	$\check{\approx}$

Thus U, Q are interpreted by (hyper)elementary relations; the other symbols (which will occur positively in $\hat{\varphi}$ whenever φ is extended first order) are interpreted by inductive relations so things are set up to apply Theorem 4.3 (i), (ii).

Given an n-ary relation R on $\mathbb{HF}_{\mathfrak{M}}$ we define $\tilde{R}$ on T by

$$\tilde{R}(t_1, \ldots, t_n) \quad \text{iff} \quad R(|t_1|, \ldots, |t_n|), \quad \text{for} \quad t_1, \ldots, t_n \in T.$$

4.8 Lemma. *For any formula* $\varphi(v_1, \ldots, v_k, \mathsf{R}) \in \mathsf{L}^*(\mathsf{R})$, *any relation* R *on* $\mathbb{HF}_{\mathfrak{M}}$, *and any* $t_1, \ldots, t_k \in T$ *we have*

$$(\mathbb{HF}_{\mathfrak{M}}, R) \models \varphi[|t_1|, \ldots, |t_k|] \quad \textit{iff} \quad (\tilde{\mathfrak{M}}, \tilde{R}) \models \hat{\varphi}[t_1, \ldots, t_k].$$

Proof. By induction on formulas $\varphi \in \mathsf{L}^*(\mathsf{R})$. For atomic and negated atomic formulas, it follows by the definitions. The induction step is immediate since every $x \in \mathbb{HF}_{\mathfrak{M}}$ is denoted by some term t. ☐

4.9 Lemma. *Let* $\varphi(x_1, \ldots, x_n, \mathsf{R}_+) \in \mathsf{L}^*(\mathsf{R})$. *For each* α *and each* $t_1, \ldots, t_n \in T$ *we have*

$$(|t_1|, \ldots, |t_n|) \in I_\varphi^\alpha \quad \textit{iff} \quad (t_1, \ldots, t_n) \in I_{\hat{\varphi}}^\alpha,$$

where the induction on the left is over $\mathbb{HF}_{\mathfrak{M}}$, *that on the right over* $\tilde{\mathfrak{M}}$.

Proof. By induction, of course. The induction hypothesis asserts that

$$(|t_1|, \ldots, |t_n|) \in I_\varphi^{<\alpha} \quad \text{iff} \quad (t_1, \ldots, t_n) \in I_{\hat{\varphi}}^{<\alpha},$$

i. e., that $(\widetilde{I_\varphi^{<\alpha}}) = I_{\hat{\varphi}}^{<\alpha}$. But then

$$\begin{aligned}(|t_1|, \ldots, |t_n|) \in I_\varphi^\alpha \quad & \text{iff} \quad (\mathbb{HF}_{\mathfrak{M}}, I_\varphi^{<\alpha}) \models \varphi(|t_1|, \ldots, |t_n|, \mathsf{R}_+) \\ & \text{iff} \quad (\tilde{\mathfrak{M}}, I_{\hat{\varphi}}^{<\alpha}) \models \hat{\varphi}(t_1, \ldots, t_n, \mathsf{R}_+) \quad \text{(by 4.8)} \\ & \text{iff} \quad (t_1, \ldots, t_n) \in I_{\hat{\varphi}}^\alpha. \quad ☐\end{aligned}$$

We are now ready to prove Theorem 4.1. The following result comes out of the proof.

4.10 Corollary. *Let* $\mathfrak{M}$ *be a structure for* L *with an inductive pairing function. If* Γ_φ *is an extended first order inductive definition over* $\mathfrak{M}$ *then there is a first order inductive definition* Γ_ψ *over* $\mathfrak{M}$ *with* $\|\Gamma_\psi\| \geq \|\Gamma_\varphi\|$.

Proof of Theorem 4.1 and Corollary 4.10. Let $\mathfrak{M}=\langle M,Q\rangle$ be an L-structure and let p be an inductive, hence hyperelementary, pairing function on $\mathfrak{M}$. By 4.2 (i), $\mathfrak{M}$ and the expanded structure $(\mathfrak{M},p)$ have exactly the same inductive and hyperelementary relations. Thus T_u, $\tilde{Q}$ are hyperelementary on $\mathfrak{M}$, and T_s, $\mathscr{E}$, $\breve{\mathscr{E}}$, $\approx$, and $\breve{\approx}$ are inductive on $\mathfrak{M}$. Let $S\subseteq M^n$ be inductive*. Choose an extended first order inductive definition Γ_φ and parameters $y_1,\ldots,y_k\in M\cup\mathbb{H}\mathrm{F}_{\mathfrak{M}}$ such that

$$S(x_1,\ldots,x_n) \quad \text{iff} \quad (x_1,\ldots,x_n,\vec{y})\in I_\varphi .$$

Now consider the inductive definition $\Gamma_{\hat{\varphi}}$ over $\tilde{\mathfrak{M}}$. By the above lemma $\|\Gamma_\varphi\|=\|\Gamma_{\hat{\varphi}}\|$ and, for any $t_1,\ldots,t_{n+k}\in T$,

$$(t_1,\ldots,t_{n+k})\in I_{\hat{\varphi}} \quad \text{iff} \quad (|t_1|,\ldots,|t_{n+k}|)\in I_\varphi .$$

By Theorem 4.3 (ii) and the remarks above about the relations T_s, $\mathscr{E}$, $\breve{\mathscr{E}}$, $\approx$ and $\breve{\approx}$ all occuring positively in $\hat{\varphi}$, $I_{\hat{\varphi}}$ is inductive over the original $\mathfrak{M}$. Choose $t_1,\ldots,t_k$ with $|t_1|=y_1,\ldots,|t_k|=y_k$. Then, for all $x_1,\ldots,x_n\in M$,

$$S(x_1,\ldots,x_n) \quad \text{iff} \quad (\dot{x}_1,\ldots,\dot{x}_n,t_1,\ldots,t_k)\in I_{\hat{\varphi}}$$

so S is obtained from the inductive set $I_{\hat{\varphi}}$ by hyperelementary substitution and, hence, is inductive. By 4.3 (iii) there is an inductive definition Γ_ψ over $\mathfrak{M}$ with $\|\Gamma_\psi\|\geqslant\|\Gamma_{\hat{\varphi}}\|=\|\Gamma_\varphi\|$, so this also proves the corollary. □

The notation system we have been using can be seen to be a notation system in the precise sense of § V.5. This follows from the next lemma. We assume the notation from above.

4.11 Lemma. *Define a function π on $\mathbb{H}\mathrm{F}_{(M,p)}$ by*

$$\pi(x)=\{y\in T\mid |y|=x\}.$$

Then π is a total $\mathbb{H}\mathrm{F}_{(\mathfrak{M},p)}$-recursive function.

Proof. Given a set a of cardinality $\geqslant 1$, we call a pair (a_0,x) a *splitting* of a if $a=a_0\cup\{x\}$ but $x\notin a_0$. Let

$$\mathrm{Spl}(a)=\{(a_0,x)\mid (a_0,x) \text{ is a splitting of } a\}$$

for all $a\in\mathbb{H}\mathrm{F}_M$. It is a simple matter to check that Spl is $\mathbb{H}\mathrm{F}_{\mathfrak{M}}$-recursive. We first define π more explicitly and then discuss the method used to see that the definition is $\mathbb{H}\mathrm{F}_{(\mathfrak{M},p)}$-recursive. The definition of π parallels the proof of 4.4.

$$\pi(p)=\{\dot{p}\} \quad \text{for} \quad p\in M,$$

$$\pi(0)=\{\emptyset\}.$$

For nonempty sets a, $\pi(a)$ is defined by a double induction, first on $\mathrm{rk}(a)$ and, among sets of the same rank, on $\mathrm{card}(a)$. So suppose $\pi(x)$ is defined for all $x \in a$ and all $x \subseteq a$ with $\mathrm{card}(x) < \mathrm{card}(a)$. If $a = \{x_1, \ldots, x_n\}$ with $n \geqslant 1$ then we look at any splitting (a_0, x) of a. Now $\pi(a_0)$, $\pi(x)$ are defined and, for $t_0 \in \pi(a_0)$, $|t_0| = a_0$ and for $t_1 \in \pi(x)$, $|t_1| = x$ so $|t_0 \,\delta\, t_1| = a_0 \cup \{x\} = a$. Thus we may define

$$\pi(a) = \{t_0 \,\delta\, t_1 \colon \text{for some } (a_0, x) \in \mathrm{Spl}(a),\ t_0 \in \pi(a_0) \text{ and } t_1 \in \pi(x)\}.$$

With this definition π is clearly $\mathbb{HF}_{(\mathfrak{M}, p)}$-recursive by the Second Recursion Theorem. ☐

4.12 Theorem. *Let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ be a structure for* L.

(i) *If $\mathfrak{M}$ has an $\mathbb{HF}_{\mathfrak{M}}$-recursive pairing function then $\mathbb{HF}_{\mathfrak{M}}$ is projectible into $\mathfrak{M}$.*

(ii) *If $\mathfrak{M}$ has a $\mathbb{HYP}_{\mathfrak{M}}$-recursive pairing function then $\mathbb{HYP}_{\mathfrak{M}}$ is projectible into $\mathfrak{M}$.*

Proof. (i) The sets in $\mathbb{HF}_{\mathfrak{M}}$ depend only on M, not on the whole structure $\mathfrak{M}$, so if we add a pairing function p to $\mathfrak{M}$, $\mathbb{HF}_{(\mathfrak{M}, p)}$ has the same sets as $\mathbb{HF}_{\mathfrak{M}}$. By Lemma 4.11, $\mathbb{HF}_{(\mathfrak{M}, p)}$ is projectible into $\mathfrak{M}$; i.e., there is an $\mathbb{HF}_{(\mathfrak{M}, p)}$ recursive notation system π with $D_\pi \subseteq M$. But then, if p is $\mathbb{HF}_{\mathfrak{M}}$-recursive, π is also $\mathbb{HF}_M$-recursive. The proof of (ii) is similar. Let p be a $\mathbb{HYP}_{\mathfrak{M}}$-recursive pairing function so that $\mathbb{HYP}_{\mathfrak{M}}$ and $\mathbb{HYP}_{(\mathfrak{M}, p)}$ have the same universe of sets. By V.5.3 we have a notation system π_0 for $\mathbb{HYP}_{\mathfrak{M}}$ with $D_\pi \subseteq \mathbb{HF}_{\mathfrak{M}}$. By 4.11, there is a $\mathbb{HYP}_{(M, p)}$-recursive map π_1 on $\mathbb{HF}_{\mathfrak{M}}$ with $\pi_1(x) \subseteq M$, $\pi_1(x) \cap \pi_1(y) = 0$ for $x \neq y$. Let π be defined by

$$\pi(x) = \bigcup \{\pi_1(y) \mid y \in \pi_0(x)\}.$$

Then π is a notation system for $\mathbb{HYP}_{\mathfrak{M}}$ with $D_\pi \subseteq M$. ☐

The following special case of 4.12 (ii) will be of great use to us in the next section.

4.13 Corollary. *Let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ be a structure for* L *with an inductive pairing function. Then $\mathbb{HYP}_{\mathfrak{M}}$ is projectible into $\mathfrak{M}$.*

Proof. If p is an inductive pairing function on $\mathfrak{M}$ then it is hyperelementary and hence an element of $\mathbb{HYP}_{\mathfrak{M}}$. Thus 4.12 (ii) applies. ☐

4.14—4.18 Exercises

4.14. Let $\mathfrak{M} = \langle M, \sim \rangle$ where $\sim$ is an equivalence relation on M which exactly one equivalence class of each finite cardinality. Define

$$x < y \quad \text{iff} \quad \mathrm{card}(x/\!\sim) < \mathrm{card}(y/\!\sim).$$

(i) Prove that $<$ is Σ_1 on $\mathbb{HF}_{\mathfrak{M}}$ and hence is extended inductive on $\mathfrak{M}$.
(ii) (Kunen). Prove that $<$ is not inductive on $\mathfrak{M}$.
(iii) Prove that $o(\mathbb{HYP}_{\mathfrak{M}})>\omega$.

4.15. This exercise introduces the Moschovakis [1974] notions of *acceptable* and *almost acceptable* structures. A *coding scheme* $\mathscr{C}$ for a structure $\mathfrak{M}$ consists of:
(a) a subset $N^{\mathscr{C}}$ of M and a linear ordering $<^{\mathscr{C}}$ of $N^{\mathscr{C}}$ such that

$$\langle N^{\mathscr{C}}, <^{\mathscr{C}}\rangle \cong \langle \omega, <\rangle, \quad \text{and}$$

(b) an injection $\langle\ \rangle^{\mathscr{C}}$ of the set of all finite sequences from M into M.
Given a fixed coding scheme $\mathscr{C}$ we use $\dot{0}, \dot{1}, \dot{2}, \ldots$ to indicate the appropriate members of $N^{\mathscr{C}}$ as ordered by $<^{\mathscr{C}}$. Associated with a coding scheme $\mathscr{C}$ there are some natural relations and functions.

$Seq^{\mathscr{C}}(x)$ iff $x=\langle\ \rangle^{\mathscr{C}}$ or $x=\langle x_1,\ldots,x_n\rangle^{\mathscr{C}}$ for some n and some $x_1,\ldots,x_n$.

$$\begin{aligned} lh^{\mathscr{C}}(x) &= \dot{0} \quad \text{if} \quad \neg Seq^{\mathscr{C}}(x)\\ &= \dot{n} \quad \text{if} \quad Seq^{\mathscr{C}}(x) \text{ and } x=\langle x_1,\ldots,x_n\rangle^{\mathscr{C}}. \end{aligned}$$

$$\begin{aligned} q^{\mathscr{C}}(x,\dot{m}) &= x_m \text{ if for some } x_1,\ldots,x_n,\ x=\langle x_1,\ldots,x_n\rangle^{\mathscr{C}} \text{ and } 1\leqslant m\leqslant n\\ &= \dot{0} \text{ otherwise.} \end{aligned}$$

A structure $\mathfrak{M}$ is *almost acceptable* (or *acceptable*) if M has a coding scheme $\mathscr{C}$ with all of $N^{\mathscr{C}}$, $<^{\mathscr{C}}$, $Seq^{\mathscr{C}}$, $lh^{\mathscr{C}}$, $q^{\mathscr{C}}$ hyperelementary (or first order, resp.).
(i) Show that every almost acceptable structure has an inductive pairing function.
(ii) Let $\mathfrak{M}$ be a structure with an inductive pairing function. Show that M is almost acceptable iff M is not recursively saturated. [It is easy to see that if $\mathfrak{M}$ is almost acceptable then $o(\mathbb{HYP}_{\mathfrak{M}})>\omega$. To prove the converse use Corollary 4.10.]

4.16. Show that all models of Peano arithmetic, KPU and ZF have definable pairing functions, even the recursively saturated ones.

4.17. Let $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ be an infinite structure and let Γ_ψ be an inductive definition over $\mathfrak{M}$, say $\psi=\psi(u_1,\ldots,u_4,\mathsf{S}_+)$. Now let $\mathfrak{M}'=(\mathfrak{M},S)$ where S is defined by:

$$S(x_1,x_2) \quad \text{iff} \quad (x_1,x_2,a_1,a_2)\in I_\psi.$$

Let $\varphi(v_1,\ldots,v_3,\mathsf{S}_+,\mathsf{T}_+)\in \mathsf{L}\cup\{\mathsf{S},\mathsf{T}\}$, where S is binary (to denote S) and T is 3-ary (to be used in an induction) and let Γ_φ be the natural inductive definition over $(\mathfrak{M},S)$ given by φ. We are going to outline the proof from Moschovakis [1974] that I_φ is inductive over the original structure $\mathfrak{M}$, thus proving Theorem 4.3.

Let $0, 1, \bar{u}_1, \ldots, \bar{u}_4, \bar{v}_1, \ldots, \bar{v}_3$ be constants from M with $0 \neq 1$. Let Q be a new 8-ary $(8=1+4+3)$ relation symbol and define $\theta(i, u_1, \ldots, u_4, v_1, \ldots, v_3, \mathsf{Q}_+)$ by

$$[i=0 \wedge \psi(u_1, \ldots, u_4, \mathsf{Q}(0, \cdot, \cdot, \cdot, \cdot, \bar{v}_1, \bar{v}_2, \bar{v}_3)/\mathsf{R}] \vee$$
$$[i=1 \wedge \varphi(v_1, v_2, v_3, \mathsf{Q}(0, \cdot, \cdot, a_1, a_2, \bar{v}_1, \bar{v}_2, \bar{v}_3)/\mathsf{S}, \mathsf{Q}(1, \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \cdot, \cdot, \cdot)/\mathsf{T}].$$

Consider the induction definition Γ_θ over $\mathfrak{M}$.

(i) Prove that for each α,

$$(u_1, \ldots, u_4) \in I_\psi^\alpha \quad \text{iff} \quad (0, u_1, \ldots, u_4, \bar{v}_1, \ldots, \bar{v}_3) \in I_\theta^\alpha$$

and hence

$$(u_1, \ldots, u_4) \in I_\psi \quad \text{iff} \quad (0, u_1, \ldots, u_4, \bar{v}_1, \ldots, \bar{v}_3) \in I_\theta.$$

(ii) Prove that if $(1, \bar{u}_1, \ldots, \bar{u}_4, v_1, \ldots, v_3) \in I_\theta^\alpha$ then $(v_1, \ldots, v_3) \in I_\varphi^\alpha$.

(iii) Prove that if $(v_1, \ldots, v_3) \in I_\varphi^\alpha$ then for some β, $(1, \bar{u}_1, \ldots, \bar{u}_4, v_1, \ldots, v_3 \in I_\theta^\beta$, by induction on α, using (i).

(iv) Use (ii), (iii) to conclude that I_φ is a section of I_θ and hence is inductive on $\mathfrak{M}$.

(v) Show that $\|\Gamma_\theta\| \geqslant \|\Gamma_\varphi\|$.

(vi) Prove Theorem 4.3.

4.18 Use Theorem 4.3 to prove Theorem 4.2 [For example, show that if S_1, S_2 are inductive on $\mathfrak{M}$ then $S_1 \cup S_2$ is inductive on $(\mathfrak{M}, S_1, S_2)$ with an inductive definition in which S_1, S_2 occur positively.]

4.19 Notes. The fact that an inductive pairing function suffices for coding $\mathbb{HF}_{\mathfrak{M}}$ on $\mathfrak{M}$ goes back, indirectly, to Aczel [1970]. The proof of Theorem 4.1 given above owes much to ideas of Aczel and Nyberg.

5. Inductive Relations on Structures with Pairing

Inductive and coinductive definitions appear in most branches of mathematics. Spector [1961] was the first to focus attention on them as objects worthy of study in their own right, but then only over the structure $\mathcal{N}$ of the natural numbers. The development over an absolutely arbitrary structure $\mathfrak{M}$ was not carried out until Moschovakis [1974] produced his attractive and coherent picture. Our object in this section is to view portions of Moschovakis' picture as projections of $\mathbb{HYP}_{\mathfrak{M}}$.

Let us summarize the results at our disposal.

5.1 Theorem. *Let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ be a structure with an inductively definable pairing function. Let S be a relation on $\mathfrak{M}$.*

(i) *S is inductive on $\mathfrak{M}$ iff S is Σ_1 on* $\mathbb{HYP}_{\mathfrak{M}}$.
(ii) *S is hyperelementary on $\mathfrak{M}$ iff* $S \in \mathbb{HYP}_{\mathfrak{M}}$.
(iii) *$O(\mathfrak{M})$ is equal to*

$$\sup\{\|\Gamma_\varphi\| \mid \Gamma_\varphi \text{ is first order positive inductive on } \mathfrak{M}\}$$

and this sup *is attained.*
(iv) $\mathbb{HYP}_{\mathfrak{M}}$ *is projectible into* $\mathfrak{M}$.

Proof. Part (i) follows from Theorems 3.8 and 4.1; (ii) follows from (i). Part (iii) follows from Theorem 3.11 and Corollary 4.10. Part (iv) is Theorem 4.12 (ii). □

We want to use this theorem to obtain some of the results in Moschovakis [1974]. In order to facilitate comparison we use the same names for theorems as in Moschovakis, even when our theorem is a little more or a little less general.

5.2 Corollary (The Abstract Kleene Theorem). *If $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ is a countable structure with an inductively definable pairing function then the Π_1^1 relations coincide with the inductive relations on $\mathfrak{M}$.*

Proof. Both classes of relations coincide with the class of relations on $\mathfrak{M}$ which are Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ by 5.1 and § IV.3. □

Notice that this result makes no reference to admissible sets; it is only in the proof that they appear. The same remark applies to many of the results below. In order to make this more obvious we use Moschovakis' notation

$$\kappa^{\mathfrak{M}} = \sup\{\|\Gamma_\varphi\| \mid \Gamma_\varphi \text{ is a first order positive inductive definition over } M\}.$$

Thus $\kappa^{\mathfrak{M}} = O(\mathfrak{M})$ if $\mathfrak{M}$ has an inductive pairing function. In this section $\mathfrak{M}$ always denotes a structure $\langle M, R_1, \ldots, R_l \rangle$ for the language L.

5.3 Proposition (The Closure Theorem). *Let $\mathfrak{M}$ have an inductive pairing function and let $\varphi(x_1, \ldots, x_n, \mathsf{R}_+)$ define Γ_φ over $\mathfrak{M}$.*
(i) *For each $\alpha < \kappa^{\mathfrak{M}}$, I_φ^α is hyperelementary on $\mathfrak{M}$.*
(ii) *I_φ is hyperelementary iff $\|\Gamma_\varphi\| < \kappa^{\mathfrak{M}}$.*

Proof. I_φ^α is a $\mathbb{HYP}_{\mathfrak{M}}$-recursive function of α, for $\alpha \in \mathbb{HYP}_{\mathfrak{M}}$. Hence each $I_\varphi^\alpha \in \mathbb{HYP}_{\mathfrak{M}}$ for $\alpha \in \mathbb{HYP}_{\mathfrak{M}}$ and is thus hyperelementary by 5.1 (ii). This proves (i) and the ($\Leftarrow$) half of (ii). Consider the map ρ_φ defined on I_φ by

$$\rho_\varphi(x) = \text{least } \beta (x \in I_\varphi^\beta).$$

This is clearly $\mathbb{HYP}_{\mathfrak{M}}$-recursive. If $I_\varphi \in \mathbb{HYP}_{\mathfrak{M}}$ then, by Σ Replacement

$$\|\Gamma_\varphi\| = \sup\{\rho_\varphi(x) \mid x \in I_\varphi\}$$

exists in $\mathbb{HYP}_{\mathfrak{M}}$ and is thus less than $\kappa^{\mathfrak{M}}$. □

One of the awkward points in the theory of inductive definitions (when not done in the context of admissible sets) is that one needs to deal with ordinals but the ordinals are not in your structure. To get around this difficulty, Moschovakis introduces the concept of an inductive norm. A norm on a set S is simply a mapping ρ of S onto some ordinal λ. We use

$$\rho: S \twoheadrightarrow \lambda$$

to indicate that ρ is a norm mapping S onto λ. Given $\rho: S \twoheadrightarrow \lambda$, define

$$x \leqslant_\rho y \quad \text{iff} \quad x \in S \wedge (y \notin S \vee \rho(x) \leqslant \rho(y)),$$
$$x <_\rho y \quad \text{iff} \quad x \in S \wedge (y \notin S \vee \rho(x) < \rho(y)).$$

A *norm* $\rho: S \twoheadrightarrow \lambda$ *is inductive* on $\mathfrak{M}$ if the relations $\leqslant_\rho$ and $<_\rho$ are inductive on $\mathfrak{M}$. Notice that if $\rho: S \twoheadrightarrow \lambda$ is inductive then S is inductive since $S(x)$ iff $x \leqslant_\rho x$.

The most natural inductive norms are those on fixed point I_φ defined by

$$\rho_\varphi(x) = \text{least}\, \beta(x \in I_\varphi^\beta).$$

(To see that this norm $\rho = \rho_\varphi$ is inductive observe that

$$x \leqslant_\rho y \quad \text{iff} \quad x \in I_\varphi \wedge y \notin I_\varphi^{<\rho(x)},$$
$$x <_\rho y \quad \text{iff} \quad x \in I_\varphi \wedge y \notin I_\varphi^{\rho(x)}$$

and the relations on the right are clearly Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$, hence inductive on $\mathfrak{M}$.)

One of the most useful lemmas on inductive definitions is the Prewellordering Theorem which asserts that every inductive set has an inductive norm. In terms of admissible sets, this is a consequence of the fact that $\mathbb{HYP}_{\mathfrak{M}}$ is resolvable, in fact

$$\mathbb{HYP}_{\mathfrak{M}} = \mathrm{L}(\alpha)_{\mathfrak{M}}$$

where $\alpha = O(\mathfrak{M})$. Most of the consequences of the Prewellordering Theorem in Moschovakis [1974] are actually obtained more easily from this equation. See for example, Exercise 5.19 for the Reduction and Separation Theorems.

5.4 The Prewellordering Theorem. *Let $\mathfrak{M}$ have an inductively definable pairing function. Every inductive relation S on $\mathfrak{M}$ has an inductive norm.*

Proof. Let S be Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$, say

$$S(x) \quad \text{iff} \quad \mathrm{L}(\alpha)_{\mathfrak{M}} \models \exists z\, \varphi(x, z)$$

where φ is Δ_0 and $\alpha = o(\mathbb{HYP}_{\mathfrak{M}}) = \kappa^{\mathfrak{M}}$. Let R be the $\mathbb{HYP}_{\mathfrak{M}}$-recursive predicate given by

$$R(\beta, x) \quad \text{iff} \quad \exists z \in \mathrm{L}(\beta)_{\mathfrak{M}}\, \varphi(x, z)$$

so

$$S(x) \quad \text{iff} \quad \exists \beta\, R(\beta, x).$$

Now the map f on S defined by

$$f(x) = \text{least}\, \beta\; R(\beta, x)$$

is not onto an ordinal so it is not a norm. Define p on S by

$$p(x) = \{y \in M \mid \exists \gamma < f(x)\, R(\gamma, y)\}.$$

Now

$$y < x \quad \text{iff} \quad y \in p(x)$$

is a well-founded relation so its associated rank function $\rho = \rho^{<}$ is a norm. We claim it is inductive on $\mathfrak{M}$. To see this observe that

$$y <_\rho x \quad \text{iff} \quad y \in S \quad \text{and} \quad \forall \beta \leqslant f(y)\, \neg R(\beta, x),$$

$$y \leqslant_\rho x \quad \text{iff} \quad y \in S \quad \text{and} \quad x \notin p(y)$$

so both relations are Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$, hence inductive on $\mathfrak{M}$. $\square$

The Closure Theorem shows that every fixed point I_φ is the uniform limit of hyperelementary sets, the I_φ^β. The Prewellordering Theorem allows us to extend this from fixed points to arbitrary inductive sets. If $\rho: S \twoheadrightarrow \lambda$ then ρ endows S with stages S_ρ^β in a natural way:

$$S_\rho^\beta = \{x \in S \mid \rho(x) \leqslant \beta\}.$$

The Boundedness Theorem, Corollary 5.6, is the natural generalization of the Closure Theorem.

5.5 Theorem. *Let $\mathfrak{M}$ be a structure with an inductive pairing function. Let $\rho: S \twoheadrightarrow \lambda$ be an inductive norm on a relation S.*

(i) *$\lambda \leqslant o(\mathbb{HYP}_{\mathfrak{M}})$ and ρ is $\mathbb{HYP}_{\mathfrak{M}}$-recursive.*

(ii) *For each $\alpha < o(\mathbb{HYP}_{\mathfrak{M}})$, $S_\rho^\alpha \in \mathbb{HYP}_{\mathfrak{M}}$ and, as a function of α, S_ρ^α is a $\mathbb{HYP}_{\mathfrak{M}}$-recursive function.*

Proof. Define a function p with domain S by

$$p(x) = \{y \in M \mid \rho(y) < \rho(x)\}.$$

For $x \in S$,

$$p(x) = \{y \in M \mid y <_\rho x\} = \{y \in M \mid \neg(x \leqslant_\rho y)\}$$

so $p(x) \in \mathbb{HYP}_{\mathfrak{M}}$ by Δ_1 Separation. Further, p is $\mathbb{HYP}_{\mathfrak{M}}$-recursive since its graph is Σ_1 definable:

$$\rho(x) = z \quad \text{iff} \quad x \in S \wedge \forall y \in z\, (y <_\rho x) \wedge \forall y \in M - z\, (x \leqslant_\rho y).$$

Now we may apply V.3.1 to p. Define

$$y \prec x \quad \text{iff} \quad y \in p(x);$$

and note that $\prec$ is well founded since $y \prec x$ implies $\rho(y) < \rho(x)$. But then ρ is $\mathbb{HYP}_{\mathfrak{M}}$-recursive by V.3.1 since

$$\rho(x) = \sup\{\rho(y)+1 \mid y \prec x\}.$$

This proves (i). To prove (ii) first define $Q(\beta, x)$ by

$$Q(\beta, x) \quad \text{iff} \quad \beta < \lambda \quad \text{and} \quad \rho(x) \leqslant \beta.$$

We claim Q is $\mathbb{HYP}_{\mathfrak{M}}$-recursive. The clause $\beta < \lambda$ causes no trouble since either $\lambda = o(\mathbb{HYP}_{\mathfrak{M}})$ in which case the clause is redundant or else $\lambda < o(\mathbb{HYP}_{\mathfrak{M}})$ in which case "$\beta < \lambda$" is Δ_0. But for $\beta < \lambda$

$$Q(\beta, x) \quad \text{iff} \quad \exists \gamma\,[\rho(x) = \gamma \wedge \gamma \leqslant_\rho \beta],$$
$$\neg Q(\beta, x) \quad \text{iff} \quad \exists y\,[\rho(y) = \beta \wedge y <_\rho x]$$

so Q is Δ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. But

$$S_\rho^\beta = \{x \in M \mid Q(\beta, x)\}$$

so $S_\rho^\beta \in \mathbb{HYP}_M$ by Δ_1 Separation. The graph $z = S_\rho^\beta$ is Σ_1 since it is equivalent to

$$\forall x \in z\, Q(x, \beta) \wedge \forall x \in M\, [Q(x, \beta) \to x \in z]$$

so (ii) holds. $\square$

5.6 Corollary (The Boundedness Theorem). *Let $\mathfrak{M}$ be a structure with an inductive pairing function. Let $\rho: S \twoheadrightarrow \lambda$ be any inductive norm.*

(i) $\lambda \leqslant \kappa^{\mathfrak{M}}$.

(ii) *For each $\alpha < \kappa^{\mathfrak{M}}$, S_ρ^α is hyperelementary.*

(iii) *S is hyperelementary iff $\lambda < \kappa^{\mathfrak{M}}$.*

Proof. The only part left to prove, after Theorem 5.5, is that if S is hyperelementary then every inductive norm $\rho: S \twoheadrightarrow \lambda$ has $\lambda < \kappa^{\mathfrak{M}}$. This follows by Σ Replacement since

$$\lambda = \sup\{\rho(x) \mid x \in S\}$$

and ρ is $\mathbb{HYP}_{\mathfrak{M}}$-recursive. $\square$

The next result, the Covering Theorem, is one of the most useful consequences of the Boundedness Theorem. We state only the special case that we need in the Exercises.

5.7 Corollary (The Covering Theorem). *Let $\mathfrak{M}$ be a structure with an inductive pairing function. Let S be an inductive subset of $\mathfrak{M}$ and let $T \subseteq S$ be coinductive on $\mathfrak{M}$. Let $\rho: S \twoheadrightarrow \lambda$ be any inductive norm on S. Then T is a subset of one of the hyperelementary resolvents S_ρ^β for $\beta < \kappa^{\mathfrak{M}}$.*

Proof. Suppose that the conclusion failed. Then we could write

$$M - S = \{x \in M \mid \forall y \in M (y \in T \rightarrow y <_\rho x)\}$$

which makes $M - S$ a Σ_1 subset of $\mathbb{HYP}_{\mathfrak{M}}$ and hence $S \in \mathbb{HYP}_{\mathfrak{M}}$ since S is also Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. But then $S = S_\rho^\lambda$ and $\lambda < \kappa^{\mathfrak{M}}$ by 5.6, so T is, after all, a subset of the hyperelementary resolvent S_ρ^λ. ☐

We now return to more familiar matters.

5.8 Theorem. *Let $\mathfrak{M}$ be a structure with an inductive pairing function. For each $n \geqslant 1$ there is an inductive relation of $n+1$ arguments that parametrizes the class of n-ary inductive relations.*

Proof. In view of 5.1 (iv), this is just a restatement of V.5.6. ☐

As always, we have the following corollary, to be compared with 5.13 below.

5.9 Corollary. *If $\mathfrak{M}$ is a structure with an inductive pairing function, then not every inductive relation is hyperelementary.* ☐

Some further uses of $\mathbb{HYP}_{\mathfrak{M}}$ in the study of inductive relations are sketched in the exercises, see especially 5.19, 5.23 and 5.24.

We can get an excellent feeling for the inductive, coinductive and hyperelementary relations on a structure by returning to infinitary logic.

Let α be an admissible ordinal, let $\mathbb{A} = L(\alpha)$ and let $\mathsf{L}_\mathbb{A}$ be the admissible fragment of $\mathsf{L}_{\infty\omega}$ given by $\mathbb{A}$. We refer to the elements of $\mathsf{L}_\mathbb{A}$ as the *α-finite formulas.*

Let $\mathfrak{M}$ be a structure for L. A relation S on $\mathfrak{M}$ is *defined by an α-finite formula* if there is an α-finite $\varphi(x_1,\dots,x_n,y_1,\dots,y_k)$ and there are $q_1,\dots,q_k \in \mathfrak{M}$ such that

(1) $$S(x_1,\dots,x_n) \quad \text{iff} \quad \mathfrak{M} \models \varphi[x_1,\dots,x_n,q_1,\dots,q_k]$$

for all $x_1,\dots,x_n \in \mathfrak{M}$. S is *defined by an α-recursive n-type* if there is an α-recursive set $\Phi(x_1,\dots,x_n,y_1,\dots,y_k)$ of α-finite formulas and there are $q_1,\dots,q_k \in \mathfrak{M}$ such that

(2) $$S(x_1,\dots,x_n) \quad \text{iff} \quad \mathfrak{M} \models \textstyle\bigwedge_{\varphi\in\Phi} \varphi[x_1,\dots,x_n,q_1,\dots,q_k]$$

for all $x_1,\dots,x_n \in \mathfrak{M}$. Replace the infinite conjunction in (2) by an infinite disjunction

(3) $$S(x_1,\dots,x_n) \quad \text{iff} \quad \mathfrak{M} \models \textstyle\bigvee_{\varphi\in\Phi} \varphi[x_1,\dots,x_n,q_1,\dots,q_k]$$

and we say that S is *defined by an α-recursive n-cotype*. Notice that S is defined by an α-recursive type iff $\neg S$ is defined by an α-recursive cotype. The student should compare 5.10 with Theorem II.7.3. (Another version holds without the pairing function assumption; see Exercise 5.29.)

5.10 Theorem. *Let $\mathfrak{M}$ be a structure for L with an inductive pairing function and let $\alpha = O(\mathfrak{M})$.*

(i) *A relation S on $\mathfrak{M}$ is hyperelementary on $\mathfrak{M}$ iff S is defined by an α-finite formula.*

(ii) *A relation S on $\mathfrak{M}$ is inductive on $\mathfrak{M}$ iff S is defined by an α-recursive cotype; S is coinductive on $\mathfrak{M}$ iff S is defined by an α-recursive type.*

Proof. We first prove the ($\Leftarrow$) parts of (i) and (ii). Since $\kappa^{\mathfrak{M}} = o(\mathbb{HYP}_{\mathfrak{M}})$, $\mathrm{L}(\alpha) \subseteq \mathbb{HYP}_{\mathfrak{M}}$, so every α-finite formula is in $\mathbb{HYP}_{\mathfrak{M}}$. Thus any relation defined by an α-finite formula is in $\mathbb{HYP}_{\mathfrak{M}}$ by Δ_1 Separation and, hence, is hyperelementary. It suffices to prove either half of (ii) so suppose that Φ is an α-recursive (or even α-r.e.) set of α-finite formulas and S is defined by (3) above. Then $S(x_1,\ldots,x_n)$ iff the following is true in $\mathbb{HYP}_{\mathfrak{M}}$:

$$\exists\psi\,[\psi\in\Phi \wedge \mathfrak{M}\models\psi[x_1,\ldots,x_n, q_1,\ldots,q_k]].$$

This makes S a Σ_1 set on $\mathbb{HYP}_{\mathfrak{M}}$ so S is inductive on $\mathfrak{M}$ by 5.1.

We now prove the ($\Rightarrow$) parts of (ii) and (i). Suppose S is inductive on $\mathfrak{M}$, say

$$S(x) \quad \text{iff} \quad (x, q_0)\in I_\varphi$$

where $\varphi(v_1, v_2, q, \mathsf{R}_+)$ has R binary and has an extra parameter q. Since $\alpha = \kappa^{\mathfrak{M}}$,

$$I_\varphi = \bigcup_{\beta<\alpha} I_\varphi^\beta.$$

We define formulas ψ_β by recursion on β as follows, where $\theta(\mathfrak{f}/\mathsf{R})$ denotes the result of replacing $\mathsf{R}(t_1,t_2)$ by $t_1\neq t_1 \wedge t_2\neq t_2$:

$$\psi_0(v_1,v_2,v_3) \quad \text{is} \quad \varphi(v_1,v_2,v_3,\mathfrak{f}/\mathsf{R}),$$
$$\psi_\beta(v_1,v_2,v_3) \quad \text{is} \quad \varphi(v_1,v_2,v_3, \textstyle\bigvee_{\gamma<\beta}\psi\,(\cdot,\cdot,v_3)/\mathsf{R}).$$

A simple proof by induction shows that

$$(x,y)\in I_\varphi^\beta \quad \text{iff} \quad \mathfrak{M}\models\psi_\beta(x,y,q)$$

and, hence,

$$(x,y)\in I_\varphi \quad \text{iff} \quad \mathfrak{M}\models\textstyle\bigvee_{\beta<\alpha}\psi_\beta(x,y,q).$$

Then we have

$$S(x) \quad \text{iff} \quad \mathfrak{M}\models\textstyle\bigvee_{\beta<\alpha}\psi_\beta(x,q_0,q).$$

Thus it remains to check that the set

$$\Phi = \{\psi_\beta \mid \beta < \alpha\}$$

is an α-recursive set. The function $f(\beta) = \psi_\beta$ is clearly definable by Σ Recursion in $L(\alpha)$ so Φ is at least α-r.e. Define a measure of complexity of formulas, say $c(\theta)$, by recursion as follows:

$$c(\theta) = 1 \quad \text{if} \quad \theta \text{ is atomic,}$$

$$c(\theta) = c(\psi) + 1 \quad \text{if} \quad \theta \text{ is } \neg\psi,\ \exists v\, \psi \text{ or } \forall v\, \psi,$$

$$c(\theta) = \sup\{c(\psi) + 1 \mid \psi \in \Theta\} \quad \text{if} \quad \theta \text{ is } \bigwedge\Theta \text{ or } \bigvee\Theta.$$

Then $c(\psi_\beta) \geqslant \beta$ so

$$\theta \in \Phi \quad \text{iff} \quad \exists \beta \leqslant c(\theta)\, [\theta = \psi_\beta]$$

which shows that Φ is α-recursive. This finishes the proof of (ii), but what happens if S is actually hyperelementary? Then $S \in \mathbb{HYP}_{\mathfrak{M}}$ and we can define a function $g \in \mathbb{HYP}_{\mathfrak{M}}$ with $\mathrm{dom}(g) = S$ by

$$g(x) = \text{least}\, \beta\, (\mathfrak{M} \models \psi_\beta(x, q_0, q)).$$

Let $\gamma = \sup(\mathrm{rng}(g))$. Then $\gamma < \alpha$ by Σ Replacement in $\mathbb{HYP}_{\mathfrak{M}}$. Then

$$S(x) \quad \text{iff} \quad \mathfrak{M} \models \bigvee_{\beta \leqslant \gamma} \psi_\beta(x, q_0, q)$$

so S is defined by an α-finite formula. $\square$

The converses of Theorem 5.10 (i), (ii) also hold. We prove the converse of (i) and leave the other as Exercise 5.22. First a lemma.

5.11 Lemma. *Let $\mathfrak{M}$ be an* L*-structure with an inductive pairing function, let* $\mathsf{L}_\mathbb{A}$ *be an admissible fragment which is an element of* $\mathbb{HYP}_{\mathfrak{M}}$, *and let*

$$\begin{aligned} \mathbf{S}^n = \{S \subseteq M^n \mid & \textit{ for some } \varphi \in \mathsf{L}_\mathbb{A}, \textit{ and some } q_1, \ldots, q_k \in M, \\ & \mathfrak{M} \models \varphi[x_1, \ldots, x_n, q_1, \ldots, q_k] \textit{ iff } S(x_1, \ldots, x_n) \\ & \textit{for all } x_1, \ldots, x_n \in M\}. \end{aligned}$$

(i) *The collection $\mathbf{S}^n$ can be parametrized by an $n+1$-are hyperelementary relation, with indices from M.*

(ii) *There is a hyperelementary set which is not in $\mathbf{S}^1$.*

Proof. (ii) follows from (i) by the usual diagonalization argument. The proof of (i) is a routine modification of Theorem V.5.7 since $\mathbb{HYP}_{\mathfrak{M}}$ is projectible into $\mathfrak{M}$. $\square$

5.12 Theorem. *Let $\mathfrak{M}$ be a structure for* L *with an inductive pairing function and let α be an admissible ordinal. If the hyperelementary relations on $\mathfrak{M}$ consist of exactly the relations definable by α-finite formulas, then $\alpha = \kappa^{\mathfrak{M}}$.*

Proof. Lemma 5.11 shows us that if every hyperelementary relation is definable by an α-finite formula then $\kappa^{\mathfrak{M}} \leqslant \alpha$. We now show that if every relation definable by an α-finite formula is hyperelementary, then $\alpha \leqslant \kappa^{\mathfrak{M}}$. Suppose, to prove the contrapositive, that $\alpha > \kappa^{\mathfrak{M}}$ and let S be any inductive relation which is not hyperelementary. By 5.10, S is definable by a $\kappa^{\mathfrak{M}}$-recursive cotype. But then, S is definable by an α-finite formula since $\alpha > \kappa^{\mathfrak{M}}$, so not every relation definable by an α-finite formula is hyperelementary. ▯

It is interesting to compare the following corollary of 5.10 and 5.12 with a result in Moschovakis [1974].

5.13 Corollary. *Let $\mathfrak{M}$ be a structure with an inductive pairing function. The following conditions on $\mathfrak{M}$ are equivalent:*

(i) *$\mathfrak{M}$ is recursively saturated.*

(ii) *Every hyperelementary relation is first-order definable.*

(iii) $\kappa^{\mathfrak{M}} = \omega$.

Proof. Since $\kappa^{\mathfrak{M}} = o(\mathbb{H}\mathrm{YP}_{\mathfrak{M}})$, we proved (i) $\Longleftrightarrow$ (ii) back in § IV.5. We have the implication (iii) $\Rightarrow$ (ii) by 5.10 or by II.7.3. By 5.12 we have (ii) $\Rightarrow$ (iii). ▯

Moschovakis assumes that his structures are acceptable (see Exercise 4.15), a stronger condition than having an inductive pairing function. Corollary 5B.3 of Moschovakis [1974] asserts that if $\mathfrak{M}$ is acceptable then there is a hyperelementary relation that is not first order definable. Since an acceptable structure $\mathfrak{M}$ always has $\kappa^{\mathfrak{M}} > \omega$ (by 4.1), this follows from 5.13. But 5.13 also shows us that the restriction to acceptable structures rules out many of the most interesting structures, model theoretically interesting at any rate.

The general version of 5.13 reads as follows.

5.14 Corollary. *Let $\mathfrak{M}$ have an inductive pairing function and let α be an admissible ordinal. The following are equivalent:*

(i) *$\mathfrak{M}$ is α-recursively saturated and not β-recursively saturated for any admissible $\beta < \alpha$.*

(ii) *The hyperelementary relations are just those definable by α-finite formulas.*

(iii) $\kappa^{\mathfrak{M}} = \alpha$.

Proof. We have (ii) $\Longleftrightarrow$ (iii) by the theorems above and (i) $\Longleftrightarrow$ (iii) by Exercise IV.5.11 and the equality $\kappa^{\mathfrak{M}} = o(\mathbb{H}\mathrm{YP}_{\mathfrak{M}})$. ▯

Let $\mathfrak{M}$ have an inductive pairing function and let $\alpha = \kappa^{\mathfrak{M}}$. By 5.14 we see that the hyperelementary relations on $\mathfrak{M}$ are just the relations explicitly definable by α-finite formulas. One could imagine stronger notions of inductive and hyperelementary where one allowed an α-finite or even a $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}$-finite formula

$\varphi(x_1,\ldots,x_n,\mathsf{R}_+)$ to define an inductive operation Γ_φ. Refer to these notions, for the time being, as *α-inductive, α-hyperelementary, $\mathbb{HYP}_{\mathfrak{M}}$-inductive* and *$\mathbb{HYP}_{\mathfrak{M}}$-hyperelementary*. The next result shows that the notion of inductive on $\mathfrak{M}$ is "stable" in that it coincides with α-inductive and $\mathbb{HYP}_{\mathfrak{M}}$-inductive.

5.15 Theorem. *Let $\mathfrak{M}$ have an inductive pairing function and let $\alpha=\kappa^{\mathfrak{M}}$.*
(i) *The inductive, α-inductive and $\mathbb{HYP}_{\mathfrak{M}}$-inductive relations on $\mathfrak{M}$ all coincide.*
(ii) *Hence, the hyperelementary, α-hyperelementary and $\mathbb{HYP}_{\mathfrak{M}}$-hyperelementary relation on $\mathfrak{M}$ all coincide with the relations explicitly definable by α-finite formulas.*

Proof. It suffices to prove that if $\varphi(x_1,\ldots,x_n,\mathsf{R}_+)$ is a formula of $\mathsf{L}_{\mathbb{A}}$, where $\mathbb{A}=\mathbb{HYP}_{\mathfrak{M}}$, then I_φ is inductive on $\mathfrak{M}$. The proof uses the ideas from the two halves of 5.10 (ii). First note that I_φ^β is a $\mathbb{HYP}_{\mathfrak{M}}$-recursive function of β, for $\beta<\alpha$, since it is defined by Σ Recursion in $\mathbb{HYP}_{\mathfrak{M}}$. As before, the Σ Reflection theorem shows that $\|\Gamma_\varphi\|\leqslant\alpha$. Now define the formulas ψ_β as in the proof of 5.10:

$$\psi_0(x_1,\ldots,x_n)=\varphi(x_1,\ldots,x_n,\mathfrak{f}/\mathsf{R}),$$
$$\psi_\beta(x_1,\ldots,x_n)=\varphi(x_1,\ldots,x_n,\textstyle\bigvee_{\gamma<\beta}\psi\,(\ldots)/\mathsf{R})$$

so that $(x_1,\ldots,x_n)\in I_\varphi^\beta$ iff $\mathfrak{M}\models\psi_\beta[x_1,\ldots,x_n]$. Thus

$$(x_1,\ldots,x_n)\in I_\varphi \quad\text{iff}\quad \mathfrak{M}\models\textstyle\bigvee_{\beta<\alpha}\psi_\beta[x_1,\ldots,x_n].$$

But the set of $\mathbb{HYP}_{\mathfrak{M}}$-finite formulas $\{\psi_\beta\mid\beta<\alpha\}$ is α-r.e. (actually α-recursive) so I_φ is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ and hence inductive on $\mathfrak{M}$ by 5.1 (i). □

5.16—5.30 Exercises

5.16. Show that each of the following structures has a definable pairing function.
(i) $\mathcal{N}=\langle\omega,0,+,\cdot\rangle$.
(ii) Any model of Peano arithmetic.
(iii) Any model of ZF, KP or KPU.
(iv) $\mathrm{L}(a,\lambda)$ for any limit ordinal λ.
(v) $\mathscr{R}=\langle\omega^\omega\cup\omega,\omega,0,+,\cdot,\mathrm{App}\rangle$, where ω^ω is the set of all functions mapping ω into ω and

$$\mathrm{App}(f,n,m) \quad\text{iff}\quad f(m)=n.$$

5.17. Show that no nonstandard model of Peano arithmetic is acceptable. Show that some nonstandard models of Peano arithmetic are almost acceptable and that some are not. [Show that if $(\mathfrak{N},\mathscr{X})$ is a model of nonstandard analysis then $\mathfrak{N}$ is *not* almost acceptable.]

5.18 (Moschovakis [1974]). Let $\mathfrak{M}=\langle\alpha,<\rangle$ where α is any ordinal $\geqslant\omega$. Show that $\mathfrak{M}$ has an inductive pairing function. This is not easy. First assume $\alpha=\Sigma_{\beta<\alpha}(\beta\cdot 2+1)$.

5.19 (Moschovakis [1974]). Let $\mathfrak{M}$ be a structure with an inductive pairing function. Prove the following results using Theorem 5.1.

(i) $\kappa^{\mathfrak{M}} = \sup\{\rho(\prec) \mid \prec$ is a hyperelementary pre-wellordering of $\mathfrak{M}\}$.

(ii) If $\mathfrak{M}$ has a hyperelementary well-ordering then

$$\kappa^{\mathfrak{M}} = \sup\{\rho(\prec) \mid \prec \text{ is a hyperelementary well-ordering of } \mathfrak{M}\}.$$

(iii) (Reduction). Let B, C be inductive on $\mathfrak{M}$. Show that there are disjoint inductive sets $B_0 \subseteq B$, $C_0 \subseteq C$ such that $B_0 \cup C_0 = B \cup C$. [See V.4.10.]

(iv) (Separation). Let B, C be disjoint coinductive subsets of $\mathfrak{M}$. Show that there is a hyperelementary set D containing B which is disjoint from C. [Use (iii).]

(v) (Hyperelementary Selection Theorem). Let $S(x,y)$ be an inductive relation on $\mathfrak{M}$. Show that there are inductive relations S_0, S_1 such that

$$S_0 \subseteq S,$$

$$\mathrm{dom}(S_0) = \mathrm{dom}(S),$$

$$x \in \mathrm{dom}(S) \Rightarrow \forall y\,(S_0(x,y) \leftrightarrow \neg S_1(x,y)).$$

5.20. We give an application of the covering theorem; in fact, the original version of it due to Spector. We use the notation from Rogers [1967]. Let

$$W = \{e \mid \varphi^2 \text{ is the characteristic function of a well-ordering } <_e\}.$$

Let $\rho(e)$ = the order type of $<_e$, for $e \in W$.

(i) Show that W is Π^1_1 on $\mathcal{N}$.

(ii) Show that ρ is an inductive norm,

$$\rho : W \twoheadrightarrow \omega_1^c.$$

(iii) Let B be a Σ^1_1 set of natural numbers, $B \subseteq W$. Show that $\sup\{\rho(e) \mid e \in B\} < \omega_1^c$.

5.21. Show that 5.10(ii) remain true if "α-recursive type" is replaced by any of the following:

(i) α-r.e. type,

(ii) $\mathbb{HYP}_{\mathfrak{M}}$-recursive type,

(iii) $\mathbb{HYP}_{\mathfrak{M}}$-r.e. type.

5.22. Let $\mathfrak{M}$ be a structure with an inductive pairing function and let α be an admissible ordinal. Suppose that the inductive relations on $\mathfrak{M}$ are exactly the relations defined by an α-recursive cotype. Show that $\alpha = \kappa^{\mathfrak{M}}$.

5.23. Let $\mathfrak{M}$ have an inductive pairing function. Let S, T be inductive relations which are not hyperelementary.

(i) Show that $T \in \mathbb{HYP}_{(\mathfrak{M},S)}$, and hence that $\mathbb{HYP}_{(\mathfrak{M},S)}$ and $\mathbb{HYP}_{(\mathfrak{M},T)}$ have the same universe of sets. [Show that $o(\mathbb{HYP}_{(\mathfrak{M},S)}) > o(\mathbb{HYP}_{\mathfrak{M}})$ and then use 5.10(ii).]

(ii) (Moschovakis [1974]). Show that the two expanded structures $(\mathfrak{M}, S)$ and $(\mathfrak{M}, T)$ have the same inductive and hyperelementary relations.

5.24 (Moschovakis [1974]). Let $\mathfrak{M}$ be a structure with an inductive pairing function and let S be an inductive relation on $\mathfrak{M}$ which is not hyperelementary. Show that for any relation T on $\mathfrak{M}$,

$$S \text{ is hyperelementary on } (\mathfrak{M}, T) \quad \text{iff} \quad \kappa^{(\mathfrak{M}, T)} > \kappa^{\mathfrak{M}}.$$

5.25. Show that Theorem 5.15 is not true without the hypothesis that $\mathfrak{M}$ has an inductive pairing function. [Use the $\mathfrak{M}$ of Exercise 4.14.]

5.26. Our proof of the Abstract Kleene Theorem, Corollary 5.2, is a bit round about. Prove it directly from the $\mathfrak{M}$-completeness theorem and Proposition 3.13. (This proof, by the way, establishes the second order version given in Moschovakis [1974] without change.)

5.27. Let $\mathfrak{M}$ be a structure for L with an inductive pairing function.

(i) Show that $C_{\mathfrak{M}}(\mathrm{KPU}^+)$, in the notation of Proposition 3.13, is inductive but not hyperelementary.

(ii) Show that $\kappa^{\mathfrak{M}}$ = closure ordinal of the inductive definition of "provable from KPU^+ by the $\mathfrak{M}$-rule".

(iii) Show that $C_{\mathfrak{M}}(\mathrm{KPU}^+)$ can be used to parametrize the inductive relations on $\mathfrak{M}$. [Use the closure of the inductive relations under hyperelementary substitution and some hyperelementary coding of formulas.]

5.28. The following definition, due to Nyberg, will be useful in Exercise VIII.9.16 and in Theorem VIII.9.5. A structure $\mathfrak{M} = \langle M, R_1, \ldots, R_k \rangle$ is a *uniform Kleene structure* if for every Π_1^1 formula $\Phi(x, \mathsf{S}_+)$ in some extra relation symbols S there is a first order $\varphi(x, y, \mathsf{R}_+, \mathsf{S}_+)$ and a $y \in M$ such that for all x and all S

$$(\mathfrak{M}, S) \models \Phi(x, \mathsf{S}_+)$$

if and only if

$$(x, y) \in I_\varphi(\mathfrak{M}, S),$$

where the R in φ is used for the induction over the structure $(\mathfrak{M}, S)$. Prove that every countable structure with an inductive pairing function is a uniform Kleene structure. Let α be any ordinal of cofinality ω. Show that $\langle \mathrm{V}(\alpha), \in \rangle$ is a uniform Kleene structure. (This last is due to Chang-Moschovakis [1970].)

5.29 (Makkai and Schlipf, independently). Improve Theorem 5.10 as follows: Let $\mathfrak{M}$ be a structure for L and let $\alpha = O(\mathfrak{M})$. Let S be a relation on $\mathfrak{M}$. Show that:

(i) $S \in \mathbb{HYP}_{\mathfrak{M}}$ iff S is defined by an α-finite formula;

(ii) S is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ iff S is defined by an α-recursive cotype. [Hint: Use the fact that every $a \in \mathbb{HYP}_{\mathfrak{M}}$ is of the form $\mathscr{F}(p_1, \ldots, p_n, M, L(\lambda_1)_{\mathfrak{M}}, \ldots, L(\lambda_k)_{\mathfrak{M}})$ for some limit ordinals $\lambda_1, \ldots, \lambda_k$ and a substitutable function $\mathscr{F}$.]

5.30 (Moschovakis [1974]). Let $\mathfrak{M}$ *not* have an inductive pairing function. Prove that $\kappa^{\mathfrak{M}}$ is admissible or the limit of admissibles. It is an open problem to find an $\mathfrak{M}$ where $\kappa^{\mathfrak{M}}$ is not admissible.

5.31 Notes. Some of the results discussed above hold without the pairing function assumption. For example, all of 5.3 through 5.6 are proved directly in Moschovakis [1974]. On the other hand, some of the results are false without the pairing function (like 5.2, 5.8—5.12) and those that do hold are much harder to prove without the admissible set machinery. For structures without an inductive pairing function we are left with two distinct approaches, inductive definitions and $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}$ (equivalently, inductive* definitions). Only time will tell which is the most fruitful tool for definability theory.

6. Recursive Open Games

An *open game formula* is an infinitary expression $\mathscr{G}(\vec{x})$ of the form

$$\forall y_1 \, \exists z_1 \, \forall y_2 \, \exists y_2 \ldots \forall y_n \, \exists z_n \ldots \bigvee_{n<\omega} \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n)$$

where each φ_n is a formula of $\mathsf{L}_{\infty\omega}$. Note that $\mathscr{G}(\vec{x})$ itself is *not* a formula of $\mathsf{L}_{\infty\omega}$ due to the infinite string of quantifiers out front. If $\{\varphi_n \mid n<\omega\}$ is a recursive set of finitary formulas then $\mathscr{G}(\vec{x})$ is called a *recursive open game formula.*

For our study, the most important result on game formulas goes back to Svenonius [1965] where he proves that, for countable $\mathfrak{M}$, the Π_1^1 predicates are exactly those defined by recursive open game formulas (Theorem 6.8 below). This result went largely unnoticed until the formulas were rediscovered by Moschovakis [1971]. He established that for acceptable $\mathfrak{M}$ (of any cardinality), it is the inductive relations on $\mathfrak{M}$ which are definable by recursive open game formulas (Corollary 6.11 below). Thus, from our point of view, Moschovakis was proving the "absolute version" of the Svenonius theorem.

Before going into these results in detail, let's step back to examine the concept of "absolute version" with some detachment.

We have been using ZFC as a convenient informal metatheory and hence may construe all our results as statements about the universe $\mathbb{V}$ of sets. By a *class* C on $\mathbb{V}$ we mean a definable class,

$$x \in C \quad \text{iff} \quad \mathbb{V} \models \varphi[x]$$

for some formula $\varphi(v)$ of set theory. A predicate P on $\mathbb{V}$ is, by definition, given by

$$P(\vec{x}) \quad \text{iff} \quad \mathbb{V} \models \psi[\vec{x}]$$

for some formula $\psi(v_1, \ldots, v_n)$.

6.1 Definition. Let C be a class defined by a Σ_1 formula without parameters and let P be some predicate. A relation P^{abs} is an *absolute version of P on C* if the following conditions hold:

(i) P^{abs} is absolute on C (that is, there are Σ and Π formulas $\psi_1(v_1,\ldots,v_n)$, $\psi_2(v_1,\ldots,v_n)$ such that for all $\vec{x}\in C$

$$\begin{aligned} P^{\text{abs}}(\vec{x}) \quad &\text{iff} \quad \mathbb{V}\models\psi_1[\vec{x}] \\ &\text{iff} \quad \mathbb{V}\models\psi_2[\vec{x}]). \end{aligned}$$

(ii) P and P^{abs} agree on $C\cap H(\omega_1)$ (that is, for all $x_1,\ldots,x_n\in C\cap H(\omega_1)$,

$$P(\vec{x}) \quad \text{iff} \quad P^{\text{abs}}(\vec{x})).$$

While not every predicate has an absolute version, at least there can be at most one absolute version.

6.2 Metatheorem. *Let C be a Σ_1 definable class, let P be some predicate and let P_1, P_2 be absolute versions of P on C. Then for all $\vec{x}\in C$,*

$$P_1(\vec{x}) \quad \text{iff} \quad P_2(\vec{x}).$$

Proof. This is just a special case of the Lévy Absoluteness Principle, one we have used several times in special cases. The hypothesis can be written

$$\forall\vec{x}\in H(\omega_1)\,[\vec{x}\in C\rightarrow(P_1(\vec{x})\leftrightarrow P_2(\vec{x}))].$$

The part within brackets is equivalent to a Π formula so the conclusion follows from the Lévy Absoluteness Principle. ▯

6.3 Example. Let C be the class of pairs $(\mathfrak{M},S)$ where $\mathfrak{M}$ is a structure. Let $P(\mathfrak{M},S)$ assert that S is Π_1^1 on $\mathfrak{M}$. Let $P^{\text{abs}}(\mathfrak{M},S)$ assert that S is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. Then we have shown that P and P^{abs} agree on countable structures and that P^{abs} is absolute. For other examples, see Table 5 on page 254.

The distinction between P^{abs} and P is the distinction between Part B and Part C of this book.

In this section we apply these general considerations as follows. We first prove that for all countable $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$, a relation S on $\mathfrak{M}$ is Π_1^1 iff it is defined by a recursive open game formula. Next we show that the notion "S is definable on $\mathfrak{M}$ by a recursive open game formula" is absolute. It will then follow that *for any $\mathfrak{M}$*,

S is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ iff S is definable by a recursive open game formula

and hence, by Theorem 5.1, that *if $\mathfrak{M}$ has an inductive pairing function,*

S is inductive on $\mathfrak{M}$ iff S is definable by a recursive open game formula.

(For $\mathfrak{M}$ without an inductive pairing function, we must replace inductive by inductive*.)

The first question to settle is the very meaning of an infinite string of quantifiers. Given a relation $R(y_1, z_1, \ldots, y_n, z_n, \ldots)$ of infinite sequences from $\mathfrak{M}$, what is to be meant by

$$\mathfrak{M} \models \forall y_1 \, \exists z_1 \ldots \forall y_n \, \exists z_n \ldots R(y_1, z_1, \ldots)?$$

The sensible interpretation is by means of Skolem functions. The above is defined to mean

$$\exists F_1, F_2, \ldots [(\mathfrak{M}, F_1, \ldots, F_n, \ldots) \models \forall y_1 \, \forall y_2 \ldots R(y_1, F_1(y_1), y_2, F_2(y_1, y_2), \ldots)].$$

For ease in presenting informal proofs it is convenient to rephrase this in terms of an infinite two person game, one played by players $\forall$ and $\exists$. The players take turns choosing elements $a_1, b_1, a_2, b_2, \ldots$ from $\mathfrak{M}$. Player $\exists$ wins if $R(a_1, b_1, a_2, b_2, \ldots)$; otherwise $\forall$ wins. Then

$$\mathfrak{M} \models \forall y_1 \, \exists z_1 \ldots R(y_1, z_1, \ldots)$$

is equivalent to:

Player $\exists$ has a winning strategy in the above game.

Formally, of course, a strategy for $\exists$ simply consists of a set $\{F_1, F_2, \ldots\}$ of Skolem functions such that

$$(\mathfrak{M}, F_1, \ldots) \models \forall y_1 \, \forall y_2 \, R(y_1, F_1(y_1), y_2, F_2(y_1, y_2), \ldots).$$

For games which begin with a play by $\exists$,

$$\exists y_1 \, \forall z_1 \ldots R(y_1, z_1, \ldots),$$

we use the convention that a function of 0 arguments is simply an element of $\mathfrak{M}$.

We have already defined the notion of an open game formula $\mathscr{G}(\vec{x})$

$$\cdot \forall y_1 \, \exists z_1 \ldots \bigvee_n \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n).$$

The important part here is the infinite disjunction, not the fact that it begins with $\forall$ (we could always add a superfluous $\forall$ if it started with $\exists$) nor the fact the quantifiers exactly alternate one for one (again we could introduce superfluous quantifiers if necessary). The reason this is referred to as an "open" game formula is that in any given play

$$a_1, b_1, a_2, b_2, \ldots$$

of the game, if $\exists$ wins then he wins at some finite stage n and thus it wouldn't matter what he played after stage n. (That is, there is a whole neighborhood of winning plays for $\exists$ in the suitable product topology.)

The dual of an open game formula is a *closed game formula*, one of the form

$$\forall y_1 \, \exists z_1 \, \forall y_2 \, \exists z_2 \ldots \bigwedge_n \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n).$$

In a closed game, $\exists$ must remain eternally diligent if he is to win.

6.4 Examples. (i) The simplest example of an important recursive open game sentence is given by

$$\forall y_1 \, \forall y_2 \ldots \bigvee_{n<\omega} \neg(y_{n+1} E y_n).$$

This sentence holds in $\langle \mathfrak{M}, E \rangle$ iff E is well founded. This is a rather boring game for $\exists$ since he never gets to play. Once $\forall$ has played a sequence $a_1, a_2, \ldots, \exists$ wins if it is not a descending sequence. Hence, $\exists$ has a winning strategy iff there are no infinite descending sequences.

(ii) The Kleene normal form for Π_1^1 relations on $\mathcal{N} = \langle \omega, 0, +, \cdot \rangle$,

$$S(x) \quad \text{iff} \quad \forall f \, \exists n \, R(\bar{f}(n), x),$$

can be considered as a reduction of Π_1^1 relations to recursive open game formulas, namely $S(x)$ iff

$$\forall y_1 \, \forall y_2 \ldots \bigvee_n \exists s \, [s \text{ codes } \langle y_1, \ldots, y_n \rangle \wedge R(s, x)].$$

(iii) On arbitrary countable structures we must use game formulas in which both players get to play if we are to characterize Π_1^1 relations. Suppose M is countable and let $\mathfrak{M} = \langle M, R, S \rangle$ where R, S are binary. Then $\mathfrak{M}$ is a model of

$$\forall y_1 \, \exists z_1 \, \forall y_2 \, \exists z_2 \ldots \bigwedge_{n,m<\omega} R(y_n, y_m) \leftrightarrow S(z_n, z_m)$$

iff $\langle M, R \rangle \cong \langle M, S \rangle$. Here we have expressed a Σ_1^1 sentence by a recursive closed game sentence.

Given a game formula $\mathscr{G}(\vec{x})$ we write

$$\mathfrak{M} \models \neg \mathscr{G}(\vec{x})$$

as shorthand for

$$\text{not} \quad (\mathfrak{M} \models \mathscr{G}(\vec{x})).$$

In general one must resist certain impulses generated by experience with finite strings of quantifiers. There is no reason to suppose that

$$\mathfrak{M} \models \neg \forall y_1 \, \exists z_1 \ldots R(y_1, z_1, \ldots)$$

implies

$$\mathfrak{M} \models \exists y_1\, \forall z_1 \ldots \neg R(y_1, z_1, \ldots).$$

That is, just because $\exists$ has no winning strategy in the first game is no reason to suppose he does have a winning strategy in the second game. One can find R's for which this fails. For open and closed games, however, this tempting maneuver is perfectly acceptable, as Theorem 6.5 shows. We shall use the idea from this proof a couple of times later on.

6.5 Gale-Stewart Theorem. *For all $\mathfrak{M}$ and $\vec{x}$,*

$$\mathfrak{M} \models \neg \forall y_1\, \exists z_1 \ldots \bigvee_n \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n)$$

iff

$$\mathfrak{M} \models \exists y_1\, \forall z_1 \ldots \bigwedge_n \neg \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n).$$

Proof. Let game I be the game given by

$$\mathfrak{M} \models \forall y_1\, \exists z_1 \ldots \bigvee_n \varphi_n(y_1, z_1, \ldots, y_n, z_n)$$

(we are suppressing the $\vec{x}$ since they play no role) and let game II be given by

$$\mathfrak{M} \models \exists y_1\, \forall z_1 \ldots \bigwedge_n \neg \varphi_n(y_1, z_1, \ldots, y_n, z_n).$$

It is clear that $\exists$ cannot have a winning strategy in both games, for then $\forall$ could use $\exists$'s strategy from game II to defeat him in game I. Thus we have the ($\Leftarrow$) half of the theorem. (This part does not use the openness hypothesis.) Now suppose $\exists$ has no strategy in game I. We show that $\forall$ has a winning strategy in I which of course amounts to a winning strategy for $\exists$ in II. Now since $\exists$ has no strategy in I there must be a fixed a_1 such that $\exists$ still has no strategy in the game

$$\mathfrak{M} \models \exists z_1\, \forall y_2\, \exists z_2 \ldots \bigvee_n \varphi_n(a_1, z_1, \ldots, y_n, z_n).$$

Why? Because if each a_1 gave rise to a strategy $s(a_1)$ for $\exists$ then he would have had a winning strategy at the start; namely

answer $\forall$'s play of a_1 by using $s(a_1)$.

Thus $\forall$'s first play is to play an a_1 such that

$$\mathfrak{M} \models \neg \exists z_1\, \forall y_2\, \exists z_2 \ldots \bigvee_n \varphi_n(a_1, z_1, \ldots, y_n, z_n).$$

Now after $\exists$ makes some play $z_1 = b_1$, $\forall$ again plays an a_2 so that $\exists$ still has no winning strategy; i.e.

$$\mathfrak{M} \models \neg \exists z_2\, \forall y_3\, \exists y_3 \ldots \bigvee_n \varphi_n(a_1, b_1, a_2, z_2, \ldots, y_n, z_n).$$

The same reasoning as above shows that such an a_2 exists. Now $\forall$ keeps on playing at the m^{th} play some a_m so that

$$\mathfrak{M} \models \neg \exists z_m \forall y_{m+1} \exists z_{m+1} \ldots \bigvee_n \varphi_n(a_1, b_1, \ldots, a_m, z_m, \ldots, z_n),$$

and, in particular

$$\mathfrak{M} \models \bigwedge_{k<m} \neg \varphi_k(a_1, b_1, \ldots, a_k, b_k).$$

Then, at the conclusion of play we have

$$\mathfrak{M} \models \bigwedge_{k<\omega} \neg \varphi_k(a_1, b_1, \ldots, a_k, b_k),$$

a win for $\forall$ in game I. We have thus defined a winning strategy for $\forall$ in game I. □

6.6 Corollary. *For all* $\mathfrak{M}, \vec{x}$,

$$\mathfrak{M} \models \neg \forall y_1 \exists z_1 \ldots \bigwedge_n \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n)$$

iff

$$\mathfrak{M} \models \exists y_1 \forall z_1 \ldots \bigvee_n \neg \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n).$$

Proof. The following are equivalent:

$$\mathfrak{M} \models \exists y_1 \forall z_1 \ldots \bigvee_n \neg \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n)$$

$$\text{not } [\mathfrak{M} \models \neg \exists y_1 \forall z_1 \ldots \bigvee_n \neg \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n)] \qquad \text{by 6.5}$$

$$\text{not } [\mathfrak{M} \models \forall y_1 \exists z_1 \ldots \bigwedge_n \neg\neg \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n)]$$

$$\mathfrak{M} \models \neg \forall y_1 \exists z_1 \ldots \bigwedge \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n). \quad \square$$

A simple application of the Gale-Stewart Theorem is to show that recursive open game formulas define Π_1^1 sets. We'll improve this later by improving the Gale-Stewart Theorem.

6.7 Corollary. *Let $\mathcal{G}(\vec{x})$ be a recursive open game formula of* L. *There is a Π_1^1 formula $\Theta(\vec{x})$ such that for all infinite* L*-structures $\mathfrak{M}$ and all $x_1, \ldots, x_k \in \mathfrak{M}$,*

$$\mathfrak{M} \models \mathcal{G}(\vec{x}) \quad \textit{iff} \quad \mathfrak{M} \models \Theta(\vec{x}).$$

Proof. Let $\mathcal{G}(\vec{x})$ be

$$\forall y_1 \exists z_1 \ldots \bigvee_n \varphi_n(\vec{x}, y_1, \ldots, z_n).$$

To prove the corollary it suffices, by the Gale-Stewart Theorem, to find a Σ_1^1 formula equivalent to

$$\exists y_1 \forall z_1 \ldots \bigwedge_n \neg \varphi_n(\vec{x}, y_1, \ldots, z_n).$$

This expression is equivalent to

$$\exists F\ [F \text{ is a function with } \mathrm{dom}(F) = \text{all finite sequences from } M \wedge \text{ for all } n \text{ and all } y_1,\ldots,y_n \in M,\ \neg\varphi_n(\vec{x}, y_1, F(\langle y_1\rangle),\ldots,y_n, F(\langle y_1,\ldots,y_n\rangle))].$$

This is co-extended Σ_1^1 by Proposition IV.2.11 and hence is Σ_1^1 by Proposition IV.2.8. To see that the same Σ_1^1 formula works in all structures one simply notices that the proofs in § IV.2 were uniform. □

We now come to the theorem of Svenonius referred to above, a partial converse to 6.7.

6.8 Svenonius' Theorem. *For every Π_1^1 formula $\Theta(\vec{x})$ of L there is an recursive open game formula $\mathscr{G}(\vec{x})$ of L such that for all countable structures $\mathfrak{M}$ and all $x_1,\ldots,x_k \in \mathfrak{M}$,*

$$\mathfrak{M} \models \mathscr{G}(\vec{x}) \quad \textit{iff} \quad \mathfrak{M} \models \Theta(\vec{x}).$$

Proof. It suffices, by the addition of constant symbols for the variables $x_1,\ldots,x_n$, to prove the theorem for Π_1^1 sentences. We actually prove the dual, that every Σ_1^1 sentence is defined by some recursive closed game sentence in all countable structures. By the Skolem Lemma of V.8.7, any Σ_1^1 sentence is equivalent to one of the form

$$\exists \mathsf{S}_1,\ldots,\mathsf{S}_m\, \forall y_1,\ldots,y_l\, \exists z_1,\ldots,z_k\, \varphi(\vec{y},\vec{z},\vec{\mathsf{S}})$$

where φ is quantifier free with no function symbols. We prove the special case

$$\exists \mathsf{S}\, \forall y_1\, y_2\, \exists z_1\, z_2\, \varphi(y_1, y_2, z_1, z_2, \mathsf{S}),$$

the general case being only notationally more complicated. We need the following fact.

(1) *For each quantifier free formula $\theta(\vec{v},\mathsf{S})$ there is another quantifier free formula $\theta^0(\vec{v})$ such that*

$$\theta^0(\vec{v}) \leftrightarrow \exists \mathsf{S}\, \theta(\vec{v},\mathsf{S})$$

is valid. Moreover, one can find θ^0 effectively from θ.

To prove (1), first write $\theta(\vec{v},\mathsf{S})$ as a disjunction

$$\theta_1(\vec{v},\mathsf{S}) \vee \cdots \vee \theta_p(\vec{v},\mathsf{S})$$

where each θ_i is a conjunction of atomic and negated atomic formulas. Since $\exists$ commutes with $\bigvee$ it suffices to prove (1) for formulas which are conjunctions of atomic and negated atomic formulas. So suppose we have to get rid of the $\exists \mathsf{S}$ from $\exists \mathsf{S}\, \theta(\vec{v},\mathsf{S})$ where $\theta(\vec{v},S)$ is

$$[\psi_1(\vec{v},\mathsf{S}) \wedge \cdots \wedge \psi_q(\vec{v},\mathsf{S})]$$

and each ψ_i is atomic or negated atomic. This just amounts to propositional logic. First *remove all equalities* like $(x=y)$ and make up for them by replacing x by y and y by x everywhere they occur (see examples below). Next we simply inspect the new list of formulas to see if it is consistent in propositional logic. If it is, θ^0 consists of the conjunction of all the formulas in the original list that don't mention S; if it isn't consistent, θ^0 consists of some false formula like $(x \neq x)$. We give three examples.

Example 1. Suppose $\theta(\vec{v}, \mathsf{S})$ consists of

$$\mathsf{R}(x,z), \quad \mathsf{S}(x), \quad (x=y), \quad \neg\mathsf{S}(y).$$

The new list consists of

$$\mathsf{R}(x,z), \quad \mathsf{R}(y,z), \quad \mathsf{S}(x), \quad \mathsf{S}(y), \quad \neg\mathsf{S}(y), \quad \neg\mathsf{S}(x).$$

This is not consistent so there can be no such S.

Example 2. Suppose $\theta(\vec{v}, \mathsf{S})$ consists of

$$\mathsf{R}(x,z), \quad \mathsf{S}(x), \quad (x \neq y), \quad (y=z).$$

The new list consists of

$$\mathsf{R}(x,z), \quad \mathsf{R}(x,y), \quad \mathsf{S}(x), \quad (x \neq y), \quad (x \neq z).$$

This is consistent so there will be such an S iff

$$\mathsf{R}(x,z) \wedge (x \neq y) \wedge (y=z).$$

Example 3. Suppose $\varphi(v, \mathsf{S})$ consists of

$$\mathsf{S}(x), \quad (x=y), \quad (y=z), \quad (x \neq z).$$

The new list will contain $(y \neq y)$ which is not consistent; there is no such S.

These examples should convince the student that the procedure decribed above actually works. It is obviously effective. This proves (1).

Now, using (1), let $\psi_n(y_{11}, y_{12}, z_{11}, z_{12}, y_{21}, y_{22}, z_{21}, z_{22}, \ldots, y_{n1}, y_{n2}, z_{n1}, z_{n2})$ be a quantifier free formula equivalent to

$$\exists \mathsf{S} \bigwedge\nolimits_{1 \leqslant m \leqslant n} \varphi(y_{m1}, y_{m2}, z_{m1}, z_{m2}, \mathsf{S})$$

and let the closed sentence $\mathscr{G}$ be

$$\forall y_{11}, y_{12}\, \exists z_{11}, z_{12}\, \forall y_{21}, y_{22}\, \exists z_{21}, z_{22} \ldots \bigwedge\nolimits_n \psi_n(y_{11}, \ldots, z_{m2}).$$

First we prove that:

(2) *For any model* $\mathfrak{M}$, *if* $\mathfrak{M} \models \exists \mathrm{S} \forall y_1, y_2 \exists z_1, z_2 \varphi$, *then* $\mathfrak{M} \models \mathscr{G}$.

For suppose $(\mathfrak{M}, S) \models \forall y_1 \forall y_2 \exists z_1 \exists z_2 \varphi(y_1, y_2, z_1, z_2, \mathrm{S})$. Let $\exists$ play with the strategy:

if $\forall$ plays a_1, a_2 at stage n, then choose b_1, b_2 so that $(\mathfrak{M}, S) \models \varphi(a_1, a_2, b_1, b_2, \mathrm{S})$. This clearly presents $\exists$ with a win.

To conclude the proof we need only prove

(3) *If* $\mathfrak{M}$ *is countable and* $\mathfrak{M} \models \mathscr{G}$ *then there is a relation* S *on* $\mathfrak{M}$ *so that*

$$(\mathfrak{M}, S) \models \forall y_1, y_2 \exists z_1, z_2 \varphi(y_1, y_2, z_1, z_2, \mathrm{S}).$$

Suppose $\mathfrak{M} \models \mathscr{G}$ so that player $\exists$ has a winning strategy. Since $\mathfrak{M}$ is countable, so is M^2, so enumerate M^2, $M^2 = \{\langle a_{n1}, a_{n2} \rangle \mid n < \omega\}$. Let $\forall$ play $y_{ni} = a_{ni}$ and let $\exists$ play $z_{ni} = b_{ni} \in M$ using his winning strategy. Thus, we end up with

$$\mathfrak{M} \models \exists \mathrm{S} \bigwedge_{1 \leqslant m \leqslant n} \varphi(a_{m1}, a_{m2}, b_{m1}, b_{m2}, \mathrm{S})$$

for each $n < \omega$. Then, by the ordinary Compactness Theorem for propositional logic

$$\mathrm{Diagram}(\mathfrak{M}) \cup \{\varphi(a_{m1}, a_{m2}, b_{m1}, b_{m2}, \mathrm{S}) \mid m < \omega\}$$

is consistent. Thus there really is an S such that

$$(\mathfrak{M}, S) \models \varphi(a_{m1}, a_{m2}, b_{m1}, b_{m2}, \mathrm{S})$$

for each m, since φ is quantifier free. Thus, since every pair is $\langle a_{m1}, a_{m2} \rangle$ for some m,

$$(\mathfrak{M}, S) \models \forall y_1, y_2 \exists z_1, z_2 \varphi(y_1, y_2, z_1, z_2, \mathrm{S}).$$

This proves (3).

The proof of the theorem is complete except that $\mathscr{G}$ is not quite in the form demanded of a recursive closed game formula. But trivial modifications with superfluous quantifiers, renaming variables and renaming the subformulas obviously puts it in the desired form. □

We have carried out half our task by showing Π_1^1 is the same as "defined by a recursive open game formula" for countable structures. It remains to show that it is absolute. We prove more than this in the next two results.

The next theorem can be viewed as an effective version of the main theorem of Keisler [1965]. The proof is rather different.

Given a recursive open game formula $\mathscr{G}(\vec{x})$, say,

$$\forall y_1 \exists z_1 \forall y_2 \exists z_2 \ldots \bigvee_n \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n)$$

we define its *finite approximations* $\delta_m(\vec{x})$ by:

$$\delta_m \text{ is } \quad \forall y_1 \exists z_1 \ldots \forall y_m \exists z_m \bigvee_{n \leqslant m} \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n).$$

It is obvious, from a gamesmanship point of view, that

$$\forall \vec{x} [\delta_m(\vec{x}) \rightarrow \mathcal{G}(\vec{x})]$$

is true in all structures.

6.9 Theorem. *Let $\mathfrak{M}$ be recursively saturated. Then, using the notation of the previous paragraph,*

$$\mathfrak{M} \models \forall \vec{x} [\mathcal{G}(\vec{x}) \leftrightarrow \bigvee_{m<\omega} \delta_m(\vec{x})].$$

Proof. We already have the trivial implication ($\leftarrow$). To prove the contrapositive of the other direction we imitate the proof of the Gale-Stewart theorem. We assume

$$\mathfrak{M} \models \bigwedge_{m<\omega} \neg \delta_m(\vec{x})$$

and exhibit a winning strategy for $\forall$ in the game

$$\forall y_1 \exists z_1 \forall y_2 \exists z_2 \ldots \bigvee_n \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n).$$

We claim that there is an a_1 such that, for each $m < \omega$,

$$\mathfrak{M} \models \neg [\exists z_1 \forall y_2 \exists z_2 \ldots \forall y_m \exists z_m \bigvee_{n \leqslant m} \varphi_n(\vec{x}, a_1, z_1, y_2, z_2, \ldots, y_n, z_n)].$$

Why? Suppose that for every $a_1 \in M$ there is an m such that

$$\mathfrak{M} \models \exists z_1 \ldots \bigvee_{n \leqslant m} \varphi_n(\vec{x}, a_1, \ldots).$$

Now this all holds in $\mathbb{HYP}_{\mathfrak{M}}$, which has ordinal ω, so, by Σ Reflection there is a $k < \omega$ such that m can always be chosen less than k. (Here we are using the fact that φ_n is a recursive function of n, so is Σ_1 in $\mathbb{HYP}_{\mathfrak{M}}$.) But then

$$\mathfrak{M} \models \delta_k(\vec{x}),$$

contrary to assumption. Thus there is such an a_1 and we let $\forall$ play it. Let $\exists$ play $z_1 = b_1$. We claim that there is an a_2 such that, for all $m < \omega$,

$$\mathfrak{M} \models \neg [\exists z_2 \forall y_3 \exists z_3 \ldots \forall y_m \exists z_m \bigvee_{n \leqslant m} \varphi_n(\vec{x}, a_1, b_1, a_2, z_2, \ldots, y_n, z_n)].$$

The reasoning is just as for a_1. If $\forall$ continues in this way, do what $\exists$ will, a sequence $a_1 b_1 a_2 b_2 \ldots$ will be generated which satisfies

$$\mathfrak{M} \models \neg \varphi_n(a_1, b_1, \ldots, a_n, b_n)$$

for each n. Hence we have described a winning strategy for $\forall$. $\square$

Now, if $\mathfrak{M}$ is a structure with $\alpha = o(\mathbb{HYP}_{\mathfrak{M}})$ one would hope to show that, on $\mathfrak{M}$, $\mathscr{G}(x)$ is equivalent to the disjunction of its α-finite approximations:

$$\mathfrak{M} \models \forall \vec{x}[\mathscr{G}(\vec{x}) \leftrightarrow \bigvee_{\beta < \alpha} \delta_\beta(\vec{x})].$$

This turns out to be true once one has the correct definition of the δ_β's.

Let $\mathscr{G}(\vec{x})$ be a recursive open game formula, say

$$\forall y_1 \exists z_1 \forall y_2 \exists z_2 \ldots \bigvee_n \varphi_n(\vec{x}, y_1, z_1, \ldots, y_n, z_n).$$

Define formulas $\delta_\beta^n(x, y_1, z_1, \ldots, y_n z_n)$

$$\delta_0^n(\vec{x}, y_1, \ldots, z_n) \text{ is } \bigvee_{m \leq n} \varphi_m(\vec{x}, y_1, z_1, \ldots, y_m z_m),$$
$$\delta_{\beta+1}^n(\vec{x}, y_1, \ldots, z_n) \text{ is } \forall y_{n+1} \exists z_{n+1} \delta_\beta^{n+1}(\vec{x}, y_1, z_1, \ldots, y_{n+1}, z_{n+1}),$$
$$\delta_\lambda^n(\vec{x}, y_1, \ldots, z_n) \text{ is } \bigvee_{\beta < \lambda} \delta_\beta^n \text{ if } \lambda \text{ is a limit ordinal}.$$

Let $\delta_\beta(\vec{x})$ be $\delta_\beta^0(\vec{x})$. Note that δ_n, for $n < \omega$, has the same meaning as it did in Theorem 6.9. Also note that δ_β is an α-recursive function of $\beta < \alpha$, whenever α is an admissible ordinal.

6.10 Theorem. *Let $\alpha = o(\mathbb{HYP}_{\mathfrak{M}})$. Then, using the notation of the previous paragraph,*

$$\mathfrak{M} \models \forall \vec{x}[\mathscr{G}(\vec{x}) \leftrightarrow \bigvee_{\beta < \alpha} \delta_\beta(\vec{x})].$$

Proof. To prove the easy half ($\leftarrow$) one first proves by a straightforward induction on β that

$$\delta_\beta^n(\vec{x}, y_1, z_1, \ldots, y_n, z_n) \rightarrow \forall y_{n+1} \exists z_{n+1} \ldots \bigvee_m \varphi_m(\vec{x}, y_1, z_1, \ldots, y_m, z_m)$$

for all n. For $n = 0$ this gives the desired result. The proof of the other half is so similar to the proof of Theorem 6.9 (a special case of 6.10) that we leave it to the student. $\square$

6.11 Corollary. *For any structure $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ and any relation S on $\mathfrak{M}$, the following are equivalent:*

(i) *S is definable by a recursive open game formula on $\mathfrak{M}$.*

(ii) *S is inductive* on $\mathfrak{M}$.*

(iii) *S is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$.*

If $\mathfrak{M}$ has an inductive pairing function, these are also equivalent to

(iv) *S is inductive on $\mathfrak{M}$.*

Proof. It follows from 6.10 that

"S is definable on $\mathfrak{M}$ by a recursive open game formula"

is absolute so the theorem follows from Theorem 6.2. We present a slightly more direct proof which shows a bit more uniformity.

We see immediately that (i)⇒(iii), from Theorem 6.9, since

$$\exists\beta<\alpha[\mathfrak{M}\models\delta_\beta(\vec{x})]$$

is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. It thus suffices to prove (ii)⇒(i). Let $\varphi(\vec{x}, \mathsf{R}_+)$ be any extended first order formula. Write $I_\varphi(\mathfrak{M})$ for the fixed point defined on $\mathfrak{M}$ by Γ_φ. We prove that there is a fixed recursive open game formula $\mathscr{G}(\vec{x})$ such that

(4) for all $\mathfrak{M}$, $\vec{x}\in I_\varphi(\mathfrak{M})$ iff $\mathfrak{M}\models\mathscr{G}(\vec{x})$

Now $I_\varphi(\mathfrak{M})$ is extended Π_1^1 on $\mathfrak{M}$, hence Π_1^1 on $\mathfrak{M}$ by Proposition IV.2.8, and the same Π_1^1 formula $\Phi(\vec{x})$ defines $I_\varphi(\mathfrak{M})$ for all $\mathfrak{M}$;

(5) for all $\mathfrak{M}$, $\vec{x}\in I_\varphi(\mathfrak{M})$ iff $\mathfrak{M}\models\Phi(\vec{x})$.

Now use Theorem 6.8 to choose $\mathscr{G}(\vec{x})$ such that

(6) for all countable $\mathfrak{M}$, $\mathfrak{M}\models\Phi(\vec{x})$ iff $\mathfrak{M}\models\mathscr{G}(\vec{x})$.

Now, combining lines (5) and (6) we have

for all countable $\mathfrak{M}[\vec{x}\in I_\varphi(\mathfrak{M})$ iff $\mathfrak{M}\models\mathscr{G}(\vec{x})]$

and the part in brackets is absolute. Hence, by Lévy Absoluteness, we have (4). ☐

6.12 Exercise. The Interpolation Theorem for $\mathsf{L}_{\omega\omega}$ can be stated as follows. Let $\Phi(x_1,\ldots,x_n)$ be a finitary Σ_1^1 formula of $\mathsf{L}_{\omega\omega}$ and let $\Psi(x_1,\ldots,x_n)$ be a finitary Π_1^1 formula of $\mathsf{L}_{\omega\omega}$. If every L-structure $\mathfrak{M}$ is a model of

(*) $\quad\forall x_1,\ldots,x_n[\Phi(\vec{x})\to\Psi(\vec{x})]$

then there is a first order formula $\theta(\vec{x})$ such that every L-structure $\mathfrak{M}$ is a model of

(**) $\quad\forall x_1,\ldots,x_n[[\Phi(\vec{x})\to\theta(\vec{x})]\wedge[\theta(\vec{x})\to\Psi(\vec{x})]]$.

We can turn this into a local result as follows.

(i) Let $\mathfrak{M}$ be a recursively saturated countable model of (*). Show that there is a $\theta(\vec{x})$ such that $\mathfrak{M}$ is a model of (**). [This is easy from Exercise V.4.8. A more direct proof goes via Svenonius' Theorem and the Approximation Theorem 6.9. Of course one could also cheat and apply the Interpolation Theorem for $\mathsf{L}_{\mathbb{A}}$ with $\mathbb{A}=\mathbb{HYP}_{\mathfrak{M}}$.]

(ii) Prove the interpolation theorem for $\mathsf{L}_{\omega\omega}$ directly from (i).

6.13 Notes. The student would profit from a comparison of our treatment with that in Moschovakis [1971], [1974]. His proof [1971] makes it clear where the approximations δ_β originate. The model theoretic interest of the Moschovakis-Svenonius results was brought out by the important paper Vaught [1973]. The student is urged to read this and Makkai [1973] in the same volume. This section (VI.6) of the book is included partly to make these papers more accessible.

Table 5. Absolute versions of some nonabsolute notions

Primitive notion P	*Absolute version P^{abs}*	*Relevant class C of objects*
1. S is Π_1^1 on $\mathfrak{M}$	S is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$	all structures $\mathfrak{M}=\langle M, R_1, \ldots, R_l\rangle$ and relations S on $\mathfrak{M}$
2. S is Π_1^1 on $\mathfrak{M}$	S is inductive* on $\mathfrak{M}$	same as (1)
3. S is Π_1^1 on $\mathfrak{M}$	S is inductive on $\mathfrak{M}$	$\mathfrak{M}, S$ as in (1) when $\mathfrak{M}$ has an inductive pairing function
4. S is Π_1^1 on $\mathfrak{M}$	S is defined by an open recursive game	same as (1)
5. $\models \varphi$	$\vdash \varphi$	all sentences of $\mathsf{L}_{\infty\omega}$
6. $\mathfrak{M} \cong \mathfrak{N}$	$\mathfrak{M} \cong_p \mathfrak{N}$ (cf. § VII.5)	all structures $\mathfrak{M}, \mathfrak{N}$
7. $\mathfrak{M} \cong \mathfrak{N}$	$\mathfrak{M} \equiv \mathfrak{N}(L_{\infty\omega})$ (cf. § VII.5)	same as (6)
8. S is strict Π_1^1 on $\mathbb{A}$	S is Σ_1 on $\mathbb{A}$ (cf. § VIII.3)	all admissible sets $\mathbb{A}$ and relations S on $\mathbb{A}$
9. $\mathfrak{M}$ is rigid (cf. § VII.7)	every element of $\mathfrak{M}$ is definable by a formula of $\mathsf{L}_{\infty\omega} \cap \mathbb{HYP}_{\mathfrak{M}}$ without parameters	all L-structures $\mathfrak{M}=\langle M, R_1, \ldots, R_l\rangle$