Chapter V
Δ_2^1 and Beyond

Most of the analysis of the first level of the analytical hierarchy in Chapter IV rests on the representation of Π_1^1 sets in terms of well-orderings (Theorem IV.1.1), and for many years after these results were known there seemed to be no hope of extending any of the methods or results to higher levels. Since W is a Π_1^1 set it cannot be used directly to represent all Σ_2^1 or Π_2^1 relations, and no analogue of W at higher levels was apparent.

In § 1 we formulate the abstract pre-wellordering property and show that much of the structure of Π_1^1 and Δ_1^1 relations is due solely to the fact that Π_1^1 has this property. Furthermore, it is easily seen that Σ_2^1 also has the pre-wellordering property and this leads to the conclusion that a strong analogy exists between Π_1^1 and Σ_2^1. This correspondence will be reinforced in § VIII.3 where we discuss two generalizations of recursion theory for which Π_1^1 and Σ_2^1 are exactly the classes of "semi-recursive" relations.

The pre-wellordering property cannot be proved for other classes in the analytical hierarchy without further set-theoretical hypotheses beyond ZFC. In §§ 2 and 3 we discuss two such hypotheses — the hypothesis of constructibility ($V = L$) and the hypothesis of projective determinacy (PD). The principal results are (1) if $V = L$, then Σ_r^1 has the pre-wellordering property for all $r \geqslant 2$, whereas (2) if PD, then the classes which have the pre-wellordering property are $\Pi_1^1, \Sigma_2^1, \Pi_3^1, \Sigma_4^1, \Pi_5^1, \ldots$. These hypotheses also imply analogues of many of the results of §§ IV.5–7 for higher levels of the analytical and projective hierarchies.

We turn then to extensions of the results of §§ IV.3–4, which might be termed the study of Δ_1^1 and $\boldsymbol{\Delta}_1^1$ "from below". Here the results are mainly negative: no analogue of the Borel hierarchy suffices to exhaust any of the classes $\boldsymbol{\Delta}_r^1$ for $r \geqslant 2$, and similarly for the effective hierarchies and Δ_r^1. On the other hand, the classes of sets which comprise these analogues are themselves somewhat similar in structure to the class of (effective) Borel sets. The classical (boldface) versions lead to significant extensions of the reuslts of § IV.5, while the effective versions will be seen in §§ VI.5–6 to be closely connected with certain generalized recursion theories. Finally in § 6 we consider some facts peculiar to Δ_2^1 which lead to a hierarchy for the Δ_2^1 relations on numbers.

1. The Pre-Wellordering Property

We recall from I.1.6 that a pre-wellordering is a well-founded, transitive, reflexive, and connected relation — from being a well-ordering it lacks only antisymmetry. With any pre-wellordering $\preccurlyeq$ is associated a norm $|\ \ |$, a function from the field of $\preccurlyeq$ onto an ordinal such that

$$x \preccurlyeq y \leftrightarrow |x| \leqslant |y|.$$

The image of $|\ \ |$ is called the (pre-wellorder-) type of $\preccurlyeq$. Conversely, any function $|\ \ |$ from a set into the ordinals determines a pre-wellordering on this set by this equivalence. For example, the function $\|\ \ \|$ defined on ${}^{\omega}\omega$ by:

$$\|\gamma\| = \begin{cases} \text{the order-type of } \leqslant_\gamma, & \text{if } \gamma \in \mathsf{W}; \\ \aleph_1, & \text{otherwise}; \end{cases}$$

determines a pre-wellordering of type $\aleph_1 + 1$.

In what follows, we denote by X any one of the classes Σ^1_r or Π^1_r ($r \geqslant 1$) and by V the corresponding universal relation U^1_r or $\sim\mathsf{U}^1_r$. We sometimes think of V as the *set*, $\{\alpha : \mathsf{V}(\alpha(0), \alpha(1), \lambda p\,.\,\alpha(p+2))\}$. To improve readability, we shall write x for $(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)$ and y for $(b, \langle\mathbf{n}\rangle, \langle\boldsymbol{\beta}\rangle)$. Then also (x, γ) denotes $(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}, \gamma\rangle)$, etc., so for example,

$$\mathsf{U}^1_{r+1}(x) \leftrightarrow \exists\gamma \sim \mathsf{U}^1_r(x, \gamma).$$

1.1 Definition. X ($= \Sigma^1_r$ or Π^1_r) has the *pre-wellordering property* iff there exist relations $\preccurlyeq$, $\preccurlyeq_\Sigma$, and $\preccurlyeq_\Pi$ such that

(i) $\preccurlyeq$ is a pre-wellordering with field ${}^{2,1}\omega$ such that for all x and y,
 (a) $\sim\mathsf{V}(y) \to x \preccurlyeq y$, and
 (b) $\mathsf{V}(y) \wedge x \preccurlyeq y \to \mathsf{V}(x)$;

(ii) $\preccurlyeq_\Sigma$ is Σ^1_r and $\preccurlyeq_\Pi$ is Π^1_r;

(iii) for any x and y such that either $\mathsf{V}(x)$ or $\mathsf{V}(y)$,

$$(x \preccurlyeq_\Sigma y) \leftrightarrow (x \preccurlyeq y) \leftrightarrow (x \preccurlyeq_\Pi y).$$

(Cf. Exercise 1.20 for other characterizations).

Conditions (i) (a) and (b) mean that all $y \notin \mathsf{V}$ are $\preccurlyeq$-equivalent and strictly follow all $x \in \mathsf{V}$ in the pre-wellordering. Hence the pre-wellorder type of $\preccurlyeq$ is a successor ordinal. If X has the pre-wellordering property with notation as in the definition, then we set

$$x < y \leftrightarrow \neg(y \leqslant x), \qquad x <_\Sigma y \leftrightarrow \neg(y \leqslant_\Pi x),$$
$$\text{and} \quad x <_\Pi y \leftrightarrow \neg(y \leqslant_\Sigma x).$$

Then it is clear that $<_\Sigma$ is Σ^1_r, $<_\Pi$ is Π^1_r, and for any x and y such that either $\mathsf{V}(x)$ or $\mathsf{V}(y)$,

$$(x <_\Sigma y) \leftrightarrow (x < y) \leftrightarrow (x <_\Pi y).$$

Note also that if $\sim\mathsf{V}(y)$, then $[x < y \leftrightarrow \mathsf{V}(x)]$.

In the following, we shall write cX to denote $\{\mathsf{R}: \sim\mathsf{R} \in X\}$ and $\mathbf{X}$ to denote $\mathbf{\Sigma}^1_r$ when $X = \Sigma^1_r$, etc. In situations where X may refer to either Σ^1_r or Π^1_r we shall sometimes write $\leqslant_X$ to refer to $\leqslant_\Sigma$ in case X is Σ^1_r and to $\leqslant_\Pi$ in case X is Π^1_r. Similarly, $\leqslant_{cX}$ refers to $\leqslant_\Pi$ in case X is Σ^1_r and to $\leqslant_\Sigma$ in case X is Π^1_r.

To avoid confusion, we now write the relations of Definition IV.1.3 as $\leqslant^{\mathsf{W}}_\Sigma$, $<^{\mathsf{W}}_\Sigma$, etc.

1.2 Theorem. *Π^1_1 has the pre-wellordering property.*

Proof. Since $\sim\mathsf{U}^1_1$ is a Π^1_1 relation, there exists by Theorem IV.1.1 a recursive functional F such that for all $x, \sim\mathsf{U}^1_1(x) \leftrightarrow \mathsf{F}[x] \in \mathsf{W}$. Thus if we define

$$x \leqslant^1 y \leftrightarrow \|\mathsf{F}[x]\| \leqslant \|\mathsf{F}[y]\|;$$
$$x \leqslant^1_\Sigma y \leftrightarrow \mathsf{F}[x] \leqslant^{\mathsf{W}}_\Sigma \mathsf{F}[y];$$
$$x \leqslant^1_\Pi y \leftrightarrow \mathsf{F}[x] \leqslant^{\mathsf{W}}_\Pi \mathsf{F}[y];$$

it follows easily from Theorem IV.1.4 that the relations $\leqslant^1$, $\leqslant^1_\Sigma$, and $\leqslant^1_\Pi$ satisfy the conditions of Definition 1.1. $\square$

1.3 Theorem. *For any $r \geqslant 1$, if Π^1_r has the pre-wellordering property, then Σ^1_{r+1} also has the pre-wellordering property.*

Proof. Let $\leqslant^r$, $\leqslant^1_\Sigma$, and $\leqslant^r_\Pi$ be relations which establish the pre-wellordering property for Π^1_r, $|\ \ |^r$ the norm associated with $\leqslant^r$, and $\boldsymbol{\kappa}^r + 1$ the pre-wellorder type of $\leqslant^r$. Then for any x,

$$\mathsf{U}^1_{r+1}(x) \leftrightarrow \exists\gamma \sim \mathsf{U}^1_r(x, \gamma) \leftrightarrow \exists\gamma\, (|(x, \gamma)|^r < \boldsymbol{\kappa}^r).$$

Let

$$|x|^{r+1} = \inf\{|(x, \gamma)|^r : \gamma \in {}^\omega\omega\}.$$

Then

$$(*)\qquad \mathsf{U}^1_{r+1}(x) \leftrightarrow |x|^{r+1} < \boldsymbol{\kappa}^r.$$

We take $\preccurlyeq^{r+1}$ to be the pre-wellordering determined by $|\ \ |^{r+1}$:

$$x \preccurlyeq^{r+1} y \leftrightarrow |x|^{r+1} \leqslant |y|^{r+1},$$

and set

$$x \preccurlyeq^{r+1}_{\Sigma} y \leftrightarrow \exists\gamma\, \forall\delta\, [(x, \gamma) \preccurlyeq^r_{\Pi} (y, \delta)],$$

and

$$x \preccurlyeq^{r+1}_{\Pi} y \leftrightarrow \forall\delta\, \exists\gamma\, [(x, \gamma) \preccurlyeq^r_{\Sigma} (y, \delta)].$$

The provisions of clause (ii) of Definition 1.1 are clearly satisfied and those of (i) follow from (*). Towards (iii), we first observe that directly from the definition we have

$$(1)\qquad x \preccurlyeq^{r+1} y \leftrightarrow \exists\gamma\, \forall\delta\, [(x, \gamma) \preccurlyeq^r (y, \delta)].$$

From this, elementary logic, and the fact that $\preccurlyeq^r$ is well founded, we conclude also

$$(2)\qquad x \preccurlyeq^{r+1} y \leftrightarrow \forall\delta\, \exists\gamma\, [(x, \gamma) \preccurlyeq^r (y, \delta)].$$

We claim that if either $\mathsf{U}^1_{r+1}(x)$ or $\mathsf{U}^1_{r+1}(y)$, the following four equivalences hold:

$$(3)\qquad x \preccurlyeq^{r+1} y \leftrightarrow \exists\gamma\, \forall\delta\, [\sim\mathsf{U}^1_r(x, \gamma) \wedge (x, \gamma) \preccurlyeq^r (y, \delta)];$$

$$(4)\qquad x \preccurlyeq^{r+1} y \leftrightarrow \forall\delta\, \exists\gamma\, [\sim\mathsf{U}^1_r(x, \gamma) \wedge (x, \gamma) \preccurlyeq^r (y, \delta)];$$

$$(5)\qquad x \preccurlyeq^{r+1}_{\Sigma} y \leftrightarrow \exists\gamma\, \forall\delta\, [\sim\mathsf{U}^1_r(x, \gamma) \wedge (x, \gamma) \preccurlyeq^r_{\Pi} (y, \delta)];$$

$$(6)\qquad x \preccurlyeq^{r+1}_{\Pi} y \leftrightarrow \forall\delta\, \exists\gamma\, [\sim\mathsf{U}^1_r(x, \gamma) \wedge (x, \gamma) \preccurlyeq^r_{\Sigma} (y, \delta)].$$

All of the implications ($\leftarrow$) are immediate from (1), (2), and the definitions. For (3)($\rightarrow$) assume $x \preccurlyeq^{r+1} y$ and suppose first $\mathsf{U}^1_{r+1}(x)$. If γ_0 is such that $|(x, \gamma_0)|^r$ is as small as possible, then $\sim\mathsf{U}^1_r(x, \gamma_0)$ and for all δ, $(x, \gamma_0) \preccurlyeq^r (y, \delta)$. If, on the other hand, $\mathsf{U}^1_{r+1}(y)$, let δ_0 be such that $\sim\mathsf{U}^1_r(y, \delta_0)$. By (1), there is some γ_0 such that for all δ, $(x, \gamma_0) \preccurlyeq^r (y, \delta)$. In particular, $(x, \gamma_0) \preccurlyeq^r (y, \delta_0)$ so by (i)(b) applied to $\preccurlyeq^r$ we have $\sim\mathsf{U}^1_r(x, \gamma_0)$.

The proofs of the remaining implications ($\rightarrow$) are similar and are left to the reader. Condition (iii) of Definition 1.1 for $\preccurlyeq^{r+1}$, $\preccurlyeq^{r+1}_{\Sigma}$, and $\preccurlyeq^{r+1}_{\Pi}$ is immediate from (3)–(6) and (iii) for $\preccurlyeq^r$, $\preccurlyeq^r_{\Sigma}$, and $\preccurlyeq^r_{\Pi}$. $\square$

1.4 Corollary. Σ_2^1 *has the pre-wellordering property.*

Proof. Immediate from Theorems 1.2 and 1.3. $\square$

In the remainder of this section we shall derive a number of results which apply to any class X ($=\Sigma_r^1$ or Π_r^1) which has the pre-wellordering property. In the main, the proofs are translations of those of the corresponding results for Π_1^1 in §§ IV.1–2 into a more general setting. As corollaries we obtain many facts about Σ_2^1 and Δ_2^1, and in §§ 2 and 3, under additional set-theoretic assumptions, also about higher levels of the analytical hierarchy.

When X is assumed to have the pre-wellordering property we shall use the notation of Definition 1.1 and in addition write $|\ \ |$ for the norm associated with $\preccurlyeq$ and set $\kappa = \sup^+\{|x| : \mathsf{V}(x)\}$. On occasion we shall write $|a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle|$ instead of $|x|$ or $|(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)|$. We also set

$$V = \{(a, \langle\mathbf{m}\rangle) : \mathsf{V}(a, \langle\mathbf{m}\rangle, \langle\ \ \rangle)\}.$$

The pre-wellordering $\preccurlyeq$ restricted to sequences of the form $(a, \langle\mathbf{m}\rangle, \langle\ \ \rangle)$ induces a pre-wellordering of $\omega \times \omega$ in an obvious way and we write $|\ \ |_0$ for the norm associated with this pre-wellordering. Thus

$$|a, \langle\mathbf{m}\rangle|_0 \leqslant |b, \langle\mathbf{n}\rangle|_0 \leftrightarrow |a, \langle\mathbf{m}\rangle, \langle\ \ \rangle| \leqslant |b, \langle\mathbf{n}\rangle, \langle\ \ \rangle|,$$

and

$$|a, \langle\mathbf{m}\rangle|_0 = \sup{}^+\{|b, \langle\mathbf{n}\rangle|_0 : (b, \langle\mathbf{n}\rangle, \langle\ \ \rangle) \prec (a, \langle\mathbf{m}\rangle, \langle\ \ \rangle)\}.$$

In particular, $\kappa = \sup^+\{|a, \langle\mathbf{m}\rangle|_0 : V(a, \langle\mathbf{m}\rangle)\}$ is a countable ordinal. Note that $|a, \langle\mathbf{m}\rangle|_0 \leqslant |a, \langle\mathbf{m}\rangle, \langle\ \ \rangle|$ and the inequality may hold (Exercise 1.24).

1.5 Theorem. *If X ($=\Sigma_r^1$ or Π_r^1) has the pre-wellordering property, then*
(i) *X and $\mathbf{X}$ have the reduction property but not the separation property;*
(ii) *cX and $c\mathbf{X}$ have the separation property but not the reduction property.*

Proof. By Lemmas II.4.19 and II.4.21 it suffices to show that X and $\mathbf{X}$ have the reduction property. Let R and S be any two relations in X of the same rank. Since V is universal for X there exist indices a and b such that

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{V}(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle), \quad \text{and} \quad \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{V}(b, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle).$$

We set

$$\mathsf{R}^*(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \wedge (a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) \preccurlyeq_X (b, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle);$$

$$\mathsf{S}^*(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \wedge (b, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) \prec_X (a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle).$$

It is straightforward to verify that $(\mathsf{R}^*, \mathsf{S}^*)$ reduces (R, S).

For R and S belonging to $\mathbf{X}$, there exist a, b, β, and γ such that

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{V}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \beta \rangle), \quad \text{and} \quad \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{V}(b, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \gamma \rangle).$$

Then we take

$$\mathsf{R}^*(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \wedge (a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \beta \rangle) \leqslant_X (b, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \gamma \rangle);$$

$$\mathsf{S}^*(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \wedge (b, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \gamma \rangle) <_X (a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \beta \rangle). \quad \square$$

1.6 Boundedness Theorem. *If X $(=\Sigma_r^1$ or $\Pi_r^1)$ has the pre-wellordering property, then*

(i) *for any $R \in cX$, if $R \subseteq V$, then* $\sup^+\{|a, \langle \mathbf{m} \rangle|_0 : R(a, \langle \mathbf{m} \rangle)\} < \kappa$;

(ii) *if X is Π_r^1, then for any $\mathsf{R} \in c\mathbf{X}$ $(=\boldsymbol{\Sigma}_r^1)$, if $\mathsf{R} \subseteq \mathsf{V}$ $(= \sim\mathsf{U}_r^1)$ then* $\sup^+\{|x| : \mathsf{R}(x)\} < \boldsymbol{\kappa}$.

Proof. Suppose that for some $R \in cX$, $R \subseteq V$, the conclusion of (i) is false. Then for any b and $\mathbf{n}$,

$$V(b, \langle \mathbf{n} \rangle) \leftrightarrow \exists a\, \exists \mathbf{m} [R(a, \langle \mathbf{m} \rangle) \wedge |b, \langle \mathbf{n} \rangle|_0 \leqslant |a, \langle \mathbf{m} \rangle|_0]$$
$$\leftrightarrow \exists a\, \exists \mathbf{m} [R(a, \langle \mathbf{m} \rangle) \wedge (b, \langle \mathbf{n} \rangle, \langle\ \rangle) \leqslant_{cX} (a, \langle \mathbf{m} \rangle, \langle\ \rangle)].$$

Since cX is closed under $\exists^0$, this implies $V \in cX$, which is false as V is universal for relations on numbers in X.

If $X = \Pi_r^1$ and for some $\mathsf{R} \in c\mathbf{X}$, $\mathsf{R} \subseteq \mathsf{V}$, the conclusion of (ii) is false, then for any b, $\mathbf{n}$, and $\boldsymbol{\beta}$

$$\mathsf{V}(b, \langle \mathbf{n} \rangle, \langle \boldsymbol{\beta} \rangle) \leftrightarrow \exists a\, \exists \mathbf{m} \exists \boldsymbol{\alpha} [\mathsf{R}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \wedge |b, \langle \mathbf{n} \rangle, \langle \boldsymbol{\beta} \rangle| \leqslant |a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle|]$$
$$\leftrightarrow \exists a\, \exists \mathbf{m} \exists \boldsymbol{\alpha} [\mathsf{R}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \wedge (b, \langle \mathbf{n} \rangle, \langle \boldsymbol{\beta} \rangle) \leqslant_\Sigma (a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle)].$$

Since Σ_r^1 is closed under both $\exists^0$ and $\exists^1$, this implies $\mathsf{V} \in \Sigma_r^1$, a contradiction. $\square$

Of course, the proof of (ii) does not work for $X = \Sigma_r^1$ because $\boldsymbol{\Pi}_r^1$ is not closed under $\exists^1$ — in fact, the result is false for $\boldsymbol{\Sigma}_2^1$ (cf. Exercise 1.25).

For each ordinal ρ, set

$$\mathsf{V}_\rho = \{(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) : |a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle| < \rho\}, \quad \text{and}$$
$$V_\rho = \{(a, \langle \mathbf{m} \rangle) : |a, \langle \mathbf{m} \rangle|_0 < \rho\}.$$

1.7 Theorem (Hierarchy). *If X $(=\Sigma_r^1$ or $\Pi_r^1)$ has the pre-wellordering property, then*

(i) *for all relations R on numbers,*

$$R \in \Delta^1_r \leftrightarrow R \preccurlyeq V_\rho \quad \textit{for some} \quad \rho < \kappa;$$

(ii) *if $X = \Pi^1_r$, then for all relations* R,

$$\mathsf{R} \in \boldsymbol{\Delta}^1_r \leftrightarrow \mathsf{R} \blacktriangleleft \mathsf{V}_\rho \quad \textit{for some} \quad \rho < \boldsymbol{\kappa}.$$

Proof. If $\rho < \kappa$, then there exists a $\langle b, \mathbf{n}\rangle \in V$ such that $\rho = |b, \langle\mathbf{n}\rangle|_0$ (we are treating V here as the *set* $\{\langle a, \mathbf{m}\rangle : V(a, \langle\mathbf{m}\rangle)\}$). Then

$$\begin{aligned}\langle a, \mathbf{m}\rangle \in V_\rho &\leftrightarrow (a, \langle\mathbf{m}\rangle, \langle\ \rangle) <_\Sigma (b, \langle\mathbf{n}\rangle, \langle\ \rangle)\\ &\leftrightarrow (\mathrm{a}, \langle\mathbf{m}\rangle, \langle\ \rangle) <_\Pi (b, \langle\mathbf{n}\rangle, \langle\ \rangle)\end{aligned}$$

which implies that $V_\rho \in \Delta^1_r$. It follows that if $R \preccurlyeq V_\rho$, then also $R \in \Delta^1_r$. Conversely, if $R \in \Delta^1_r$, let a be such that for all $\mathbf{m}$, $R(\mathbf{m}) \leftrightarrow \langle a, \mathbf{m}\rangle \in V$. Then $A = \{\langle a, \mathbf{m}\rangle : R(\mathbf{m})\}$ belongs to cX, so by the Boundedness Theorem $A \subseteq V_\rho$ for some $\rho < \kappa$. Thus $R \preccurlyeq V_\rho$.

For (ii), suppose $\mathsf{R} \in \boldsymbol{\Delta}^1_r$, say $\mathsf{R} \in \Delta^1_r[\beta]$. Then for some a, $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow V(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}, \beta\rangle)$. Then $\mathsf{S} = \{(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}, \beta\rangle) : \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha})\}$ is a $\Sigma^1_r[\beta]$ subrelation of V, so by the Boundeness Theorem, $\mathsf{S} \subseteq \mathsf{V}_\rho$ for some $\rho < \boldsymbol{\kappa}$. Then $R(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{V}_\rho(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}, \beta\rangle)$, so $\mathsf{R} \blacktriangleleft \mathsf{V}_\rho$. $\square$

Note that the proof of (i) also establishes:

1.8 Corollary. *If X $(= \Sigma^1_r$ or $\Pi^1_r)$ has the pre-wellordering property, then for all relations R on numbers, $R \in \Delta^1_r$ iff for some a and some $\rho < \kappa$,*

$$R(\mathbf{m}) \leftrightarrow V_\rho(a, \langle\mathbf{m}\rangle). \quad \square$$

1.9 Theorem (Upper Classification). *If X $(= \Sigma^1_r$ or $\Pi^1_r)$ has the pre-wellordering property, then $\{\alpha : \alpha \in \Delta^1_r\} \in X$.*

Proof. By Corollary 1.8, for any α,

$$\begin{aligned}\alpha \in \Delta^1_r &\leftrightarrow (\exists\rho < \kappa)\exists a\,\forall mn\,[\alpha(m) = n \leftrightarrow V_\rho(a, \langle m, n\rangle)]\\ &\leftrightarrow (\exists u \in V)\exists a\,\forall mn\,([\alpha(m) = n \rightarrow (a, \langle m, n\rangle) <_X u] \wedge\\ &\qquad \wedge [(a, \langle m, n\rangle) <_{cX} u \rightarrow \alpha(m) = n]). \quad \square\end{aligned}$$

1.10 Corollary. *If Π^1_r has the pre-wellordering property, then Δ^1_r is not a basis for Π^1_{r-1}. If Σ^1_r has the pre-wellordering property, then Δ^1_r is not a basis for Π^1_r.* $\square$

1.11 Selection Theorem. *If X $(=\Sigma_r^1$ or $\Pi_r^1)$ has the pre-wellordering property, then for any $\mathsf{R} \in X$, there exists a partial functional* $\mathrm{Sel}_{\mathsf{R}}$ *with graph in X such that for all* $\mathbf{m}$ *and* $\boldsymbol{\alpha}$,

$$\exists p\ \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathrm{Sel}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathrm{Sel}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}) \downarrow .$$

Proof. Suppose $\mathsf{R} \in X$ and a is an index such that $\mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{V}(a, \langle p, \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle)$. Then as in the proof of Lemma IV.2.5, it suffices to define

$$\begin{aligned}\mathrm{Sel}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq p \leftrightarrow \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \wedge \forall q\ &[(a, \langle p, \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle) \leqslant_X (a, \langle q, \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle)] \\ &\wedge (\forall q < p)[(a, \langle p, \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle) <_X (a, \langle q, \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle)]. \quad \square\end{aligned}$$

1.12 Lemma. *If X $(=\Sigma_r^1$ or $\Pi_r^1)$ has the pre-wellordering property, then for every $\rho < \kappa$, there exists a set $B \in \Delta_r^1$ such that $B \preccurlyeq V_\sigma$ for no $\sigma \leqslant \rho$.*

Proof. Similar to that of Lemma IV.2.4. $\square$

1.13 Theorem (Lower Classification). *If X $(=\Sigma_r^1$ or $\Pi_r^1)$ has the pre-wellordering property, then $\{\alpha : \alpha \in \Delta_r^1\} \notin \Delta_r^1$.*

Proof. If X is Π_r^1 we may proceed as in the second part of the proof of Theorem IV.2.6. Suppose now that X is Σ_r^1. Let

$$\mathsf{A} = \{\alpha : \alpha \text{ is the characteristic function of } V_\rho \text{ for some } \rho \leqslant \kappa\}.$$

First, $\mathsf{A} \in \Sigma_r^1$ since for all α,

$$\begin{aligned}\alpha \in \mathsf{A} \leftrightarrow \forall n\ [\alpha(n) \leqslant 1] \wedge \forall a\ \forall \mathbf{m}\ &[\alpha(\langle a, \mathbf{m}\rangle) = 0 \rightarrow \langle a, \mathbf{m}\rangle \in V] \\ \wedge \forall ab\ \forall \mathbf{mn}\ &[\alpha(\langle a, \mathbf{m}\rangle) = 0 \wedge (b, \langle \mathbf{n}\rangle, \langle\ \rangle) \leqslant_{\Pi} (a, \langle \mathbf{m}\rangle, \langle\ \rangle) \\ &\rightarrow \alpha(\langle b, \mathbf{n}\rangle) = 0].\end{aligned}$$

For $\rho < \kappa$, the characteristic function of V_ρ is Δ_r^1 by Theorem 1.6 whereas the characteristic function K_V of $V_\kappa = V$ is not Δ_r^1. Hence $\mathsf{A} \sim \{\alpha : \alpha \in \Delta_r^1\} = \{\mathsf{K}_V\}$. But if $\{\alpha : \alpha \in \Delta_r^1\} \in \Delta_r^1$, this set is Σ_r^1 which implies, by Corollary III.2.7 (vii), that K_V is Δ_r^1, a contradiction. $\square$

It would seem at first glance that the ordinals κ and $\boldsymbol{\kappa}$ might depend on the particular pre-wellordering used to establish the pre-wellordering property. It turns out, however, that in many cases these ordinals are uniquely determined.

1.14 Definition. For any $r \geqslant 1$,

(i) $\delta_r^1 = \sup^+\{\|R\| : R \in \Delta_r^1$ and R is a well-ordering on $\omega\}$;

(ii) $\boldsymbol{\delta}_r^1 = \sup^+\{\|\mathsf{R}\| : \mathsf{R} \in \boldsymbol{\Delta}_r^1$ and R is a pre-wellordering on ${}^\omega\omega\}$.

From Theorem IV.2.11 we have $\delta_1^1 = \omega_1$ and it easily follows from the techniques of that section that $\boldsymbol{\delta}_1^1 = \aleph_1$ (Exercise 1.26). Note that if R is a Δ_r^1 pre-wellordering on ω and

$$S(p,q) \leftrightarrow R(p,q) \wedge [\sim R(q,p) \vee (R(q,p) \wedge p \leqslant q)]$$

then S is a Δ_r^1 well-ordering and $\|S\| \geqslant \|R\|$. Hence δ_r^1 is also the supremum of the types of Δ_r^1 *pre*-wellorderings on ω.

1.15 Lemma. *If X $(=\Sigma_r^1$ or $\Pi_r^1)$ has the pre-wellordering property, then $\kappa \leqslant \delta_r^1$ and $\boldsymbol{\kappa} \leqslant \boldsymbol{\delta}_r^1$.*

Proof. For any ordinal $\rho < \kappa$, choose $w \in V$ such that $|w|_0 = \rho$. Then if $R_w(u,v) \leftrightarrow |u|_0 \leqslant |v|_0 < |w|_0$, R_w is a Δ_r^1 pre-wellordering of type ρ so by the preceding remark, $\rho < \delta_r^1$. Thus $\kappa \leqslant \delta_r^1$. The proof that $\boldsymbol{\kappa} \leqslant \boldsymbol{\delta}_r^1$ is similar. □

To prove the converse inequalities we shall need an effective version of the Boundeness Theorem. For the next two lemmas, let

$$R_b = \{(a, \langle \mathbf{m} \rangle) : \sim V(b, \langle a, \mathbf{m} \rangle)\}, \quad \text{and}$$

$$\mathsf{R}_b^{\beta,\delta} = \{(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) : \sim \mathsf{V}(b, \langle a, \mathbf{m} \rangle, \langle \boldsymbol{\alpha}, \beta, \delta \rangle)\}.$$

1.16 Lemma. *If X $(=\Sigma_r^1$ or $\Pi_r^1)$ has the pre-wellordering property, then there exist primitive recursive functions f and g such that for all b, β, and δ,*

(i) *if $R_b \subseteq V$, then $V(f(b), \langle f(b) \rangle)$ and*

$$\sup{}^+\{|a, \langle \mathbf{m} \rangle|_0 : R_b(a, \langle \mathbf{m} \rangle)\} \leqslant |f(a), \langle f(a) \rangle|_0 < \kappa;$$

(ii) *if X is Π_r^1 and $\mathsf{R}_b^{\beta,\delta} \subseteq \mathsf{V}(=\sim \mathsf{U}_r^1)$, then $\mathsf{V}(g(b), \langle g(b) \rangle, \langle \beta, \delta \rangle)$ and*

$$\sup{}^+\{|a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle| : \mathsf{R}_b^{\beta,\delta}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle)\} \leqslant |g(b), \langle g(b) \rangle, \langle \beta, \delta \rangle| < \boldsymbol{\kappa}.$$

Proof. Suppose first that $R_b \subseteq V$. Since V is universal, there exists a primitive recursive function f such that for all n,

$$\sim V(f(b), \langle n \rangle) \leftrightarrow \exists a\, \exists \mathbf{m} [R_b(a, \langle \mathbf{m} \rangle) \wedge (n, \langle n \rangle, \langle\ \rangle) \leqslant_{cX} (a, \langle \mathbf{m} \rangle, \langle\ \rangle)].$$

If $\sim V(f(b), \langle f(b) \rangle)$, then for some a and $\mathbf{m}$ such that $R_b(a, \langle \mathbf{m} \rangle)$, $(f(b), \langle f(b) \rangle, \langle\ \rangle) \leqslant_{cX} (a, \langle \mathbf{m} \rangle, \langle\ \rangle)$. Since $R_b \subseteq V$, this implies $V(f(b), \langle f(b) \rangle)$, a contradiction. Hence $V(f(b), \langle f(b) \rangle)$ and thus for all a and $\mathbf{m}$ such that $R_b(a, \langle \mathbf{m} \rangle)$, $(f(b), \langle f(b) \rangle, \langle\ \rangle) \not\leqslant_{cX} (a, \langle \mathbf{m} \rangle, \langle\ \rangle)$, that is, $|a, \langle \mathbf{m} \rangle|_0 < |f(b), \langle f(b) \rangle|_0$ as required.

If $\mathsf{R}_b^{\beta,\delta} \subseteq \mathsf{V}$ and $X = \Pi^1_r$, we take g to be a primitive recursive function such that for all n,

$$\sim\mathsf{V}(g(b), \langle n\rangle, \langle\beta, \delta\rangle) \leftrightarrow \exists a\, \exists \mathbf{m}\, \exists \boldsymbol{\alpha}\, [\mathsf{R}_b^{\beta,\delta}(a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) \wedge$$
$$(n, \langle n\rangle, \langle\beta, \delta\rangle) \preccurlyeq_\Sigma (a, \langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)]$$

and the argument proceeds similarly as above. □

1.17 Theorem. *If X ($=\Sigma^1_r$ or Π^1_r) has the pre-wellordering property, then*
(i) *for any $S \in \Delta^1_r$, if S is a well-ordering of ω, then $\|S\| < \kappa$;*
(ii) *if X is Π^1_r, then for any $\mathsf{S} \in \boldsymbol{\Delta}^1_r$, if S is a pre-wellordering of ${}^\omega\omega$, then $\|\mathsf{S}\| < \boldsymbol{\kappa}$.*

Proof. Let S be a Δ^1_r well-ordering. We shall construct a primitive recursive function h such that for all p and q,

$$(*) \qquad |p|_S < |q|_S \rightarrow |h(p), \langle h(p)\rangle|_0 < |h(q), \langle h(q)\rangle|_0.$$

It follows from $(*)$ that for all $p \in \mathrm{Fld}(S)$, $|p|_S \leqslant |h(p), \langle h(p)\rangle|_0$, so

$$\|S\| \leqslant \sup{}^+\{|h(p), \langle h(p)\rangle|_0 : p \in \mathrm{Fld}(S)\} < \kappa$$

by the Boundedness Theorem.

Since $S \in \Delta^1_r$, there exists a primitive recursive function h_0 such that for any $q \in \mathrm{Fld}(S)$ and any e,

$$R_{h_0(e,q)} = \{(\{e\}(p), \langle\{e\}(p)\rangle) : |p|_S < |q|_S\}$$
$$= \{(b, \langle b\rangle) : \exists p\, [S(p, q) \wedge p \neq q \wedge \{e\}(p) \simeq b]\}.$$

Let f be the function of the preceding lemma, by the Primitive Recursion Theorem choose $\bar{e}$ such that for all q, $\{\bar{e}\}(q) = f(h_0(\bar{e}, q))$, and set $h = \{\bar{e}\}$. We prove $(*)$ by induction on S. Suppose that $(*)$ holds for all p and q such that $|q|_S < |r|_S$. Then by Lemma 1.16

$$\sup{}^+\{|h(p), \langle h(p)\rangle|_0 : |p|_S < |r|_S\} \leqslant |h(r), \langle h(r)\rangle|_0$$

so that $(*)$ holds for all p and q such that $|q|_S \leqslant |r|_S$.

For (ii), let S be a $\boldsymbol{\Delta}^1_r$ pre-wellordering — say $\mathsf{S} \in \Delta^1_r[\delta]$. We shall construct a primitive recursive functional H such that for all α and β

$$(*) \qquad |\alpha|_{\mathsf{S}} < |\beta|_{\mathsf{S}} \rightarrow |\mathsf{H}(\alpha), \langle\mathsf{H}(\alpha)\rangle, \langle\alpha, \delta\rangle| < |\mathsf{H}(\beta), \langle\mathsf{H}(\beta)\rangle, \langle\beta, \delta\rangle|.$$

Since $\mathsf{S} \in \Delta^1_r[\delta]$, there exists a primitive recursive functional H_0 such that for any $\beta \in \mathrm{Fld}(\mathsf{S})$ and any e,

$$\begin{aligned} \mathsf{R}^{\beta,\delta}_{\mathsf{H}_0(e,\beta)} &= \{(\{e\}(\alpha), \langle\{e\}(\alpha)\rangle, \langle\alpha,\delta\rangle) : |\alpha|_{\mathsf{S}} < |\beta|_{\mathsf{S}}\} \\ &= \{(b, \langle b\rangle, \gamma) : \exists\alpha\, [\mathsf{S}(\alpha,\beta) \wedge \neg \mathsf{S}(\beta,\alpha) \wedge \gamma = \langle\alpha,\delta\rangle \wedge \{e\}(\alpha) \simeq b]\}. \end{aligned}$$

Let g be as in the preceding lemma, by the Primitive Recursion Theorem choose $\bar{e}$ such that $\{\bar{e}\}(\beta) = g(\mathsf{H}_0(\bar{e},\beta))$, and set $\mathsf{H} = \{\bar{e}\}$. The proof of $(*)$ is now similar to that of $(*)$ and the result is immediate from $(*)$. $\square$

1.18 Corollary. *If X $(= \Sigma^1_r$ or $\Pi^1_r)$ has the pre-wellordering property, then*

(i) $\kappa = \delta^1_r$;

(ii) *if X is Π^1_r, then $\boldsymbol{\kappa} = \boldsymbol{\delta}^1_r$.*

Proof. Immediate from 1.15 and 1.17. $\square$

Note that it is not necessarily true that when Σ^1_r has the pre-wellordering property, $\boldsymbol{\kappa} = \boldsymbol{\delta}^1_r$. Let κ^2 and $\boldsymbol{\kappa}^2$ be the ordinals associated with the pre-wellordering for Σ^1_2 obtained by the method of the proof of Theorem 1.3 from $\preccurlyeq^1$. Then clearly $\boldsymbol{\kappa}^2 = \boldsymbol{\delta}^1_1 = \aleph_1$, but the relation $\{(\gamma,\delta) : \gamma, \delta \in \mathsf{W} \wedge \|\gamma\| \leq \|\delta\|\}$ is a Π^1_1 pre-wellordering of type $\aleph_1$ so $\boldsymbol{\delta}^1_2 > \aleph_1$.

We conclude this section by listing the consequences of the pre-wellordering property for Σ^1_2.

1.19 Theorem.

(i) *Σ^1_2 and $\boldsymbol{\Sigma}^1_2$ have the reduction property but not the separation property*;

(ii) *Π^1_2 and $\boldsymbol{\Pi}^1_2$ have the separation property but not the reduction property*;

(iii) *for any $R \in \Pi^1_2$, if $R \subseteq U^1_2$, then* $\sup^+\{|a, \langle\mathbf{m}\rangle|_0 : R(a, \langle\mathbf{m}\rangle)\} < \delta^1_2$;

(iv) *for any R, $R \in \Delta^1_2 \leftrightarrow R \preccurlyeq U^1_{2,\rho}$ for some $\rho < \delta^1_2$*;

(v) $\{\alpha : \alpha \in \Delta^1_2\} \in \Sigma^1_2 \sim \Delta^1_2$;

(vi) *Δ^1_2 is not a basis for Π^1_2*;

(vii) *for any $\mathsf{R} \in \Sigma^1_2$, there exists a partial functional $\mathrm{Sel}_{\mathsf{R}}$ with Σ^1_2 graph such that for all $\mathbf{m}$ and $\boldsymbol{\alpha}$,*

$$\exists p\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow R(\mathrm{Sel}_{\mathsf{R}}(\mathbf{m},\boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathrm{Sel}_{\mathsf{R}}(\mathbf{m},\boldsymbol{\alpha})\downarrow\ ;$$

(viii) $\kappa^2 = \delta^1_2$ *and* $\boldsymbol{\kappa}^2 = \aleph_1 = \boldsymbol{\delta}^1_1$. $\square$

1.20–1.33 Exercises

1.20 (Moschovakis). Show that for $X = \Sigma^1_r$ or Π^1_r, each of the following is equivalent to the pre-wellordering property for X:

(i) for every $\mathsf{R} \in X$, there exist relation $\preccurlyeq$, $\preccurlyeq_\Sigma$, and $\preccurlyeq_\Pi$ which satisfy (i)–(iii) of Definition 3.1 with 'V' replaced by 'R';

(ii) for every $\mathsf{R} \in X$, there exists an ordinal λ and a function φ mapping ${}^{k,l}\omega$ onto $\lambda + 1$ such that $\mathsf{R}(x) \leftrightarrow \varphi(x) < \lambda$ and the two relations

$$x \leqslant_\varphi y \leftrightarrow \mathsf{R}(x) \wedge \varphi(x) \leqslant \varphi(y)$$

and

$$x <_\varphi y \leftrightarrow \mathsf{R}(x) \wedge \varphi(x) < \varphi(y)$$

both belong to X. (Such a φ is called an *X-norm for* R *of length* λ).

1.21. Give an alternative proof of Theorem 1.2 based on the results of Exercise III.3.33. For simplicity, consider in detail only relations on numbers.

1.22. Show that for all $r \geqslant 1$, Σ_r^0 has the pre-wellordering property (defined by making the obvious modification in Definition 1.1 (ii).)

1.23. Show that if X $(= \Sigma_r^1$ or $\Pi_r^1)$ has the pre-wellordering property, then so does $\mathbf{X}$ $(= \boldsymbol{\Sigma}_r^1$ or $\boldsymbol{\Pi}_r^1)$.

1.24 (Gandy). Show that it may happen that $|a, \langle \mathbf{m} \rangle|_0 < |a, \langle \mathbf{m} \rangle, \langle\ \ \rangle|$. (Take $X = \Sigma_2^1$ with the pre-wellordering defined by the proofs of Theorems 1.2 and 1.3. Apply the Basis Theorem to

$$\{\delta : \delta \in \mathsf{W} \wedge \exists \gamma\, (|a, \langle \mathbf{m} \rangle, \langle \gamma \rangle|^1 < \|\delta\| + 1) \rightarrow \exists \gamma\, (|a, \langle \mathbf{m} \rangle, \langle \gamma \rangle|^1 < \|\delta\|)\}$$

to show that not all Δ_2^1 ordinals are of the form

$$|a, \langle \mathbf{m} \rangle, \langle\ \ \rangle|^2 \quad \text{with} \quad \mathsf{U}_2^1(a, \langle \mathbf{m} \rangle, \langle\ \ \rangle).)$$

1.25. Show that the boldface boundedness property (1.6(ii)) fails for $X = \Sigma_2^1$.

1.26. Prove that $\boldsymbol{\delta}_1^1 = \aleph_1$. Show in fact that every $\boldsymbol{\Sigma}_1^1$ pre-wellordering of ${}^\omega\omega$ has type $< \aleph_1$. Note that there are $\boldsymbol{\Pi}_1^1$ pre-wellorderings of uncountable type.

1.27. (Cf. Exercise IV.2.25.) If X $(= \Sigma_r^1$ or $\Pi_r^1)$ has the pre-wellordering property, then the following two *effective choice principles* hold: for any $\mathsf{R} \in X$,

(i) if $\forall \mathbf{m} \forall \boldsymbol{\alpha}\, \exists p\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha})$, then there exists a Δ_r^1 functional F such that $\forall \mathbf{m} \forall \boldsymbol{\alpha}\, \mathsf{R}(\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha})$;

(ii) if $\forall \mathbf{m} \forall \boldsymbol{\alpha}\, (\exists \beta \in \Delta_r^1[\boldsymbol{\alpha}]) \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta)$, then there exists a Δ_r^1 functional G such that $\forall \mathbf{m} \forall \boldsymbol{\alpha}\, \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \lambda q\,.\, \mathsf{G}(q, \mathbf{m}, \boldsymbol{\alpha}))$.

1.28. Show that if Π_r^1 has the pre-wellordering property, then the image of a $\boldsymbol{\Delta}_r^1$

set B under a functional $\theta : {}^{\omega}\omega \to {}^{\omega}\omega$ with $\mathbf{\Delta}_r^1$ graph which is one–one on B is $\mathbf{\Delta}_r^1$ (cf. Theorem IV.6.9).

1.29. For any X ($=\Sigma_r^1$ or Π_r^1), let $\Sigma_1^{1,X}$ be the class of relations R such that for some $\mathsf{P} \in \Delta_r^1$,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (\exists \beta \in \Delta_r^1[\boldsymbol{\alpha}])\, \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta).$$

Show that if X has the pre-wellordering property, then $\Sigma_1^{1,X} \subseteq X$. (Cf. Theorem 3.8 below). Show that if $X = \Sigma_r^1$ and has the uniformization property, then the converse inclusion also holds.

1.30. Show that if X ($=\Sigma_r^1$ or Π_r^1) has the pre-wellordering property, then there exists a well-ordering of ω in X of order-type δ_r^1 but every well-ordering of ω in cX has order-type less than δ_r^1.

1.31 (Martin, Solovay). A natural conjecture based on Theorem III.4.9 is: if Π_r^1 has the pre-wellordering property, then $\{\alpha : \alpha$ is recursive in some $B \in \Sigma_r^1\}$ is a basis for Σ_r^1. Show that this conjecture is false for $r > 1$ (but cf. Theorem 3.7 below). (Show that for any $B \in \Pi_r^1$ and any $\gamma \in \mathsf{W}$ such that $\|\gamma\| \geq \delta_r^1$, $B \in \Delta_r^1[\gamma]$. Then consider $\{\alpha : \exists \gamma\, (\gamma \in \mathsf{W} \wedge \|\gamma\| \geq \delta_r^1 \wedge \alpha \notin \Delta_r^1[\gamma])\}$).

1.32 (Moschovakis). Let $W^2 = \{a : \exists \beta\,.\, a \in W[\beta]\}$ and $\|a\|^2 = \inf\{\|a\|_\beta : a \in W[\beta]\}$. Show that every Σ_2^1 relation on numbers is reducible to W^2 and that if A is a Π_2^1 subset of W^2, then $\sup^+\{\|a\|^2 : a \in A\} < \delta_2^1$.

1.33. Give another proof of the existence of a Σ_2^1 relation R of order-type δ_2^1 along the following lines. Using the Uniformization Theorem IV.7.8, obtain a relation $\mathsf{R} \in \Pi_1^1$, such that

$$\exists \beta\,.\, a \in W[\beta] \leftrightarrow \exists! \beta\, \mathsf{R}(a, \beta).$$

Use this to assign to each $a \in W^2$ a function $\gamma_a \in \mathsf{W}$ with $\|\gamma_a\| \geq \|a\|^2$ and obtain R by "piecing together" the orderings $\leq \gamma_a$.

1.34 Notes. The pre-wellordering property arose from an analysis of just what is used about Π_1^1 in proving the main structure theorems. Although we have for convenience formulated it here only for the classes Σ_r^1 and Π_r^1, essentially the same definition applies to any indexable or parametrizable class of relations, (cf. Definition VI.4.7 below) and by Exercise 1.20 even this restriction is unnecessary.

The observation that Σ_2^1 has the pre-wellordering property is due to Moschovakis; an early version appears in Rogers [1967, § 16.6], but the ideas go back to Addison [1959] and by attribution there to Novikov even earlier. The property was implicit in Moschovakis [1967] and [1969] and Addison–Moschovakis [1968], but the first abstract formulation seems to be Moschovakis [1970].

2. The Hypothesis of Constructibility

The Hypothesis of Constructibility (V = L) was formulated by Gödel in 1938 to prove the relative consistency with ZF (and other axiomatizations of set theory) of the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH). He showed Gödel [1939]

Theorem. (i) *If* ZF *is consistent, then* ZF + (V = L) *is also consistent*;
(ii) AC *and* GCH *are theorems of* ZF + (V = L).

Since then it has been recognized that many other mathematical assertions which are not provable in ZFC — or which at least have resisted proof to the present day — *are* provable in ZF + (V = L). It follows that any such assertion is at least relatively consistent with ZFC — that is, its negation is *not* a theorem of ZFC unless ZF itself is inconsistent. We shall consider here what can be proved about the analytical and projective hierarchies in ZF + (V = L).

The general theory of constructibility is beyond the scope of this book and we shall base our discussion on results (1)–(6) below of this theory, which we state without proof. They may be found in any treatment of constructibility, for example Devlin [1973] or Mostowski [1969].

The language of set theory $\mathscr{L}_{\mathrm{ZF}}$ has the symbols $\neg$, $\wedge$, $\exists$, $\doteq$, $\dot{\in}$, and variables $x_0, x_1, \ldots$. The only terms are the variables, the atomic formulas are the expressions $x \doteq y$ and $x \dot{\in} y$ for variables x and y, and the class of formulas is the smallest class containing the atomic formulas and such that if $\mathfrak{A}$ and $\mathfrak{B}$ are formulas, so are $\neg\mathfrak{A}$, $\mathfrak{A} \wedge \mathfrak{B}$, and $\exists x_i\, \mathfrak{A}$. A structure for this language is an ordered pair $\mathfrak{M} = (M, E)$ such that M is a set and E is a binary relation on M: $E \subseteq {}^2M$. If $E = \in\restriction M = \{(u, v) : u \in v \wedge u,\ v \in M\}$, we call $\mathfrak{M}$ an $\in$*-structure* and denote it simply by $(M, \in)$ (here and for the rest of this section we suspend the convention that u, v, and w denote natural numbers). We write $\mathfrak{M} \models \mathfrak{A}[u_0, \ldots, u_{k-1}]$ to mean that the elements $u_0, \ldots, u_{k-1}$ of M satisfy the formula $\mathfrak{A}$ in $\mathfrak{M}$ when the variable x_i is interpreted as u_i. A precise definition of this relation is similar to III.5.2. A set $X \subseteq M$ is called *definable over* (M, E) iff for some formula $\mathfrak{A}$ and some $u_0, \ldots, u_{k-1} \in M$,

$$X = \{v : (M, E) \models \mathfrak{A}[v, u_0, \ldots, u_{k-1}]\}.$$

The hierarchy of constructible sets is defined by (transfinite) recursion on ordinals:

2.1 Definition. For all ordinals ρ,

(i) $L_0 = \varnothing$;

(ii) $L_{\rho+1} = \{X : X \subseteq L_\rho$ and X is definable over $(L_\rho, \in)\}$;

(iii) $L_\rho = \bigcup\{L_\sigma : \sigma < \rho\}$, if ρ is a limit ordinal.

A set u is called *constructible* just in case $u \in L_\rho$ for some ρ. For constructible u, the *order* of u, $\mathrm{Od}(u)$, is the smallest ρ such that $u \in L_\rho$. It is easily verified from the definition that the following hold for all ordinals σ and ρ:

(1) $\sigma < \rho \to L_\sigma \cup \{L_\sigma, \sigma\} \subseteq L_\rho$;

(2) L_ρ is transitive — that is, $\forall u\, [u \in L_\rho \to u \subseteq L_\rho]$;

(3) $\mathrm{Card}(L_\rho) = \mathrm{Card}(\rho)$.

The property of being a constructible set, as with most properties of informal mathematics, may be expressed by a formula L of the language of set theory. If x denotes the unique free variable of L, then the formula $\forall x$ L, which asserts that all sets are constructible, is called the *Hypothesis of Constructibility*. In a set theory which admits proper classes as well as sets, L defines a class (also called L, the *class of constructible sets*) and the Hypothesis of Constructibility may be written '$V = L$', where V is the class of all sets. In accord with common practice we shall use '$V = L$' to denote this hypothesis even in ZF.

The proof that $V = L$ implies the Axiom of Choice proceeds by showing that if $V = L$, then the universe may be definably well ordered. The key fact in applications to the analytical hierarchy is that the restriction of this well-ordering to ${}^\omega\omega$ is Δ^1_2. To state this precisely, we need:

(4) there exists a relation $<_L$ which well orders $\{\alpha : \alpha$ is constructible$\}$ with order-type $\aleph_1$ and has the property that if $\mathrm{Od}(\alpha) < \mathrm{Od}(\beta)$, then $\alpha <_L \beta$. Furthermore, there is a formula $\circledS$ of the language of ZF such that for any ρ and any $\alpha, \beta \in L_\rho$,

$$\alpha <_L \beta \leftrightarrow (L_\rho, \in) \models \circledS[\alpha, \beta].$$

2.2 Theorem. *If* $V = L$, *then*

(i) $<_L$ *is* Δ^1_2;

(ii) *for any* $r \geqslant 2$ *and any* $\mathsf{R} \in \Sigma^1_r$ (Π^1_r), *if*

$$\mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \gamma) \leftrightarrow (\exists \beta <_L \gamma)\, \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta), \quad \textit{and}$$

$$\mathsf{Q}(\mathbf{m}, \boldsymbol{\alpha}, \gamma) \leftrightarrow (\forall \beta <_L \gamma)\, \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta),$$

then also P *and* Q *are* Σ^1_r (Π^1_r).

We shall prove the theorem after discussing some additional set-theoretic facts. The first of these is

(5) For all constructible $\alpha \in {}^\omega\omega$, $\mathrm{Od}(\alpha) < \aleph_1$.

From this we can already see the outline of the proof of Theorem 2.2. Under the assumption $V = L$, we have for any R,

$$
\begin{aligned}
&(\exists\beta <_L \gamma)\, R(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \\
(*)\quad &\leftrightarrow \exists\sigma\, \exists\beta\, [\sigma < \aleph_1 \wedge \beta, \gamma \in L_\sigma \wedge (L_\sigma, \in) \models \circledS[\beta, \gamma] \wedge R(\mathbf{m}, \boldsymbol{\alpha}, \beta)] \\
&\leftrightarrow \forall\sigma\, [\sigma < \aleph_1 \wedge \gamma \in L_\sigma \rightarrow \exists\beta\, (\beta \in L_\sigma \\
&\qquad\qquad \wedge (L_\sigma, \in) \models \circledS[\beta, \gamma] \wedge R(\mathbf{m}, \boldsymbol{\alpha}, \beta))].
\end{aligned}
$$

The reader familiar with the techniques of § III.3 (especially the discussion following III.3.8) will suggest that the first step in evaluating the complexity of these expressions is to replace quantification over countable ordinals by quantification over well-orderings. It turns out to be simpler to characterize directly the class of models L_σ ($\sigma < \aleph_1$). The other key point is that for $\sigma < \aleph_1$, L_σ is countable so that with suitable coding the quantifier '$\exists\beta \in L_\sigma$' may be replaced by a number quantifier.

A structure $\mathfrak{M}$ is called *well founded* iff the relation E is well founded — that is, there is no function $\phi : \omega \rightarrow M$ such that for all n, $E(\phi(n+1), \phi(n))$. If $\mathfrak{M}$ is well founded, then each $u \in M$ is assigned a unique ordinal number $\mathrm{hgt}(u)$ by the condition

$$\mathrm{hgt}(u) = \sup{}^{+}\{\mathrm{hgt}(v) : E(v, u)\}.$$

The least ordinal not assigned to any $u \in M$ is called the *height* of $\mathfrak{M}$. If $\mathfrak{M}$ is well founded and satisfies the axiom of extensionality, then there is a unique isomorphism ψ (the "collapsing map") of $\mathfrak{M}$ with an $\in$-structure $(M^\circ, \in)$. ψ is defined recursively by

$$\psi(u) = \{\psi(v) : E(v, u)\}.$$

If $\mathfrak{M}$ is a model of $V = L$ together with a certain finite collection of axioms of ZF, then in fact M° must be exactly L_ρ for ρ the height of $\mathfrak{M}$:

(6) there exists a theorem $\mathfrak{S}$ of $\mathrm{ZF} + (V = L)$ such that for any ρ, any well-founded model of $\mathfrak{S}$ of height ρ is isomorphic to $(L_\rho, \in)$. Furthermore, for any $\sigma < \aleph_1$, there exists a ρ such that $\sigma < \rho < \aleph_1$ and $(L_\rho, \in) \models \mathfrak{S}$.

The effect of (6) is to allow us in $(*)$ to replace quantification over the sets L_ρ ($\rho < \aleph_1$) by quantification over well-founded models of $\mathfrak{S}$.

For any $\varepsilon \in {}^\omega\omega$, let $\mathfrak{M}_\varepsilon = (\omega, \{(m, n) : \varepsilon(\langle m, n\rangle) = 0\})$. Any countable structure for the language of ZF is isomorphic to some $\mathfrak{M}_\varepsilon$, so we may restrict attention to these. Let $\ulcorner\ \urcorner$ denote a fixed Gödel numbering of this language.

2.3 Lemma. *There exist relations* $\mathsf{Wf} \in \Pi^1_1$ *and* $\mathsf{Mod} \in \Delta^1_1$ *such that for all* $\mathbf{m}$, ε, *and* $\mathfrak{A}$,

(i) $\mathsf{Wf}(\varepsilon) \leftrightarrow \mathfrak{M}_\varepsilon$ *is well founded*;

(ii) $\mathsf{Mod}(\ulcorner\mathfrak{A}\urcorner, \langle\mathbf{m}\rangle, \varepsilon) \leftrightarrow \mathfrak{M}_\varepsilon \models \mathfrak{A}[\mathbf{m}]$.

Sketch of proof. For (i) *we have*

$$\mathfrak{M}_\varepsilon \text{ is well founded} \leftrightarrow \neg\exists\alpha\,\forall m\,[\varepsilon(\langle\alpha(m+1), \alpha(m)\rangle) = 0].$$

(ii) may be proved by constructing a Σ^0_1 family of inductive operators Γ_ε such that for all $\mathfrak{A}$, $\mathbf{m}$ and ε,

$$\mathfrak{M}_\varepsilon \models \mathfrak{A}[\mathbf{m}] \leftrightarrow \exists t.\langle t, \ulcorner\mathfrak{A}\urcorner, \langle\mathbf{m}\rangle\rangle \in \bar{\Gamma}_\varepsilon,$$

and applying Theorem III.3.17 (cf. Theorem III.3.6 and Exercise III.5.19). $\square$

Suppose now that $\mathfrak{M}_\varepsilon$ is a well-founded model of $\mathfrak{C}$ and define $\theta_\varepsilon : \omega \to \omega$ as follows:

$$\theta_\varepsilon(0) = \text{the unique } u \in \omega \text{ such that } \forall n.\,\varepsilon(\langle n, u\rangle) \neq 0;$$

$$\theta_\varepsilon(m+1) = \text{the unique } u \in \omega \text{ such that}$$

$$\forall n\,[\varepsilon(\langle n, u\rangle) = 0 \leftrightarrow \varepsilon(\langle n, \theta_\varepsilon(m)\rangle) = 0 \text{ or } n = \theta_\varepsilon(m)].$$

Then $\theta_\varepsilon(m)$ is the element of the model $\mathfrak{M}_\varepsilon$ which plays the role of the natural number m. In particular, if ψ_ε is the unique isomorphism of $\mathfrak{M}_\varepsilon$ with some $(\mathbf{L}_\rho, \in)$, then $\psi_\varepsilon(\theta_\varepsilon(m)) = m$. Similarly, we extend θ_ε by setting

$$\theta_\varepsilon(\beta) \simeq u \leftrightarrow \forall mn\,(\beta(m) = n \leftrightarrow \mathfrak{M}_\varepsilon \models (\langle x_1, x_2\rangle \in x_0)[u, \theta_\varepsilon(m), \theta_\varepsilon(n)]).$$

If $\theta_\varepsilon(\beta) \simeq u$, then u plays the role of β in $\mathfrak{M}_\varepsilon$ and $\psi_\varepsilon(\theta_\varepsilon(\beta)) = \beta$. It follows from Lemma 2.3 that the relation '$\theta_\varepsilon(\beta) \simeq u$' is Δ^1_1.

We can now conclude the proof of Theorem 2.2. By $(*)$ together with (6) we have

$$(\exists\beta <_{\mathrm{L}} \gamma)\,\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow$$

$$\leftrightarrow \exists\varepsilon\,\exists\beta\,\exists uv\,[\mathfrak{M}_\varepsilon \models \mathfrak{C} \wedge \mathfrak{M}_\varepsilon \text{ is well founded} \wedge \theta_\varepsilon(\beta) \simeq u \wedge \theta_\varepsilon(\gamma) \simeq v \wedge$$
$$\mathfrak{M}_\varepsilon \models \circledS[u, v] \wedge \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta)]$$

$$\leftrightarrow \forall\varepsilon\,\forall v\,[\mathfrak{M}_\varepsilon \models \mathfrak{C} \wedge \mathfrak{M}_\varepsilon \text{ is well founded} \wedge \theta_\varepsilon(\gamma) \simeq v \rightarrow$$
$$\exists u\,(\mathfrak{M}_\varepsilon \models \circledS[u, v] \wedge \exists\beta\,(\theta_\varepsilon(\beta) \simeq u) \wedge \forall\beta\,[\theta_\varepsilon(\beta) \simeq u \rightarrow \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta)])].$$

If $\mathsf{R} \in \Sigma^1_r$ $(r \geqslant 2)$, then the first equivalence shows that also $\mathsf{P} \in \Sigma^1_r$. Similarly, if $\mathsf{R} \in \Pi^1_r$, the second equivalence shows $\mathsf{P} \in \Pi^1_r$. The results for Q are immediate by complementation and (i) follows from the equivalence: $\alpha <_L \gamma \leftrightarrow (\exists \beta <_L \gamma)[\alpha = \beta]$. □

2.4 Theorem. *If* $V = L$, *then for all* $r \geqslant 2$, Σ^1_r *has the pre-wellordering property.*

Proof. Under the assumption $V = L$, $<_L$ well orders ${}^\omega\omega$ in type $\aleph_1$ and therefore assigns to each γ a countable ordinal $|\gamma|_L$. For each $x = (a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle)$ we define

$$|x|^r_L = \begin{cases} \inf\{|\gamma|_L : \sim \mathsf{U}^1_{r-1}(x, \gamma)\}, & \text{if } \mathsf{U}^1_r(x); \\ \aleph_1, & \text{otherwise.} \end{cases}$$

Then the relation $\leqslant^r_L$ defined by

$$x \leqslant^r_L y \leftrightarrow |x|^r_L \leqslant |y|^r_L$$

is a pre-wellordering on ${}^{2,1}\omega$. It is easy to check that the conditions of Definition 1.1 are satisfied if we set

$$x \leqslant^r_{L,\Sigma} y \leftrightarrow \exists \gamma [\sim \mathsf{U}^1_{r-1}(x, \gamma) \wedge (\forall \delta <_L \gamma) \mathsf{U}^1_{r-1}(y, \delta)],$$

and

$$x \leqslant^r_{L,\Pi} y \leftrightarrow \forall \delta [\sim \mathsf{U}^1_{r-1}(y, \delta) \rightarrow (\exists \gamma \leqslant_L \delta) \sim \mathsf{U}^1_{r-1}(x, \gamma)]. \quad \square$$

In accord with the notation of the preceding section, let $|\ \ |^r_{L,0}$ be the norm associated with the pre-wellordering induced on U^1_r by $\leqslant^r_L$,

$$\boldsymbol{\kappa}^r_L = \sup{}^+\{|x|^r_L : \mathsf{U}^1_r(x)\},$$

and

$$\kappa^r_L = \sup{}^+\{|a, \langle \mathbf{m} \rangle|^r_{L,0} : U^1_r(a, \langle \mathbf{m} \rangle)\}.$$

2.5 Theorem. *If* $V = L$, *then for all* $r \geqslant 2$,

(i) Σ^1_r *and* $\boldsymbol{\Sigma}^1_r$ *have the reduction property but not the separation property*;

(ii) Π^1_r *and* $\boldsymbol{\Pi}^1_r$ *have the separation property but not the reduction property*;

(iii) *for any* $R \in \Pi^1_r$, *if* $R \subseteq U^1_r$, *then*

$$\sup{}^+\{|a, \langle \mathbf{m} \rangle|^r_{L,0} : R(a, \langle \mathbf{m} \rangle)\} < \delta^1_r;$$

(iv) *for any* R, $R \in \Delta^1_r \leftrightarrow R \leqslant U^1_{r,\rho}$ *for some* $\rho < \delta^1_r$;

(v) $\{\alpha : \alpha \in \Delta_r^1\} \in \Sigma_r^1 \sim \Delta_r^1$;

(vi) Δ_r^1 *is not a basis for* Π_r^1;

(vii) *for any* $\mathsf{R} \in \Sigma_r^1$, *there exists a partial functional* $\mathrm{Sel}_{\mathsf{R}}$ *with* Σ_r^1 *graph such that for all* $\mathbf{m}$ *and* $\boldsymbol{\alpha}$,

$$\exists p\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathrm{Sel}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathrm{Sel}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}) \downarrow\ ;$$

(viii) $\kappa_{\mathrm{L}}^r = \delta_r^1$ *and* $\boldsymbol{\kappa}_{\mathrm{L}}^r = \aleph_1 = \boldsymbol{\delta}_1^1$.

Proof. Immediate from Theorem 2.4 and the results of § 1. □

The Hypothesis of Constructibility also has consequences concerning the properties of Σ_1^1 and Π_1^1 sets considered in § IV.5–7. As regards uniformization, we have

2.6 Theorem. *If* $\mathrm{V} = \mathrm{L}$, *then for all* $r \geq 2$, Σ_r^1 *and* $\boldsymbol{\Sigma}_r^1$ *have the uniformization property.*

Proof. Suppose $\mathrm{V} = \mathrm{L}$ and

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \exists \gamma\, \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \gamma)$$

with $\mathsf{P} \in \Pi_{r-1}^1$. Let

$$\mathsf{Q}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \gamma) \leftrightarrow \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \gamma) \wedge (\forall \delta <_{\mathrm{L}} \langle \beta, \gamma \rangle) \sim \mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, (\delta)_0, (\delta)_1)$$

and

$$\theta(\mathbf{m}, \boldsymbol{\alpha}) \simeq \beta \leftrightarrow \exists \gamma\, \mathsf{Q}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \gamma).$$

Then θ uniformizes R and $\mathsf{Q} \in \Delta_r^1$ so $\mathrm{Gr}_\theta \in \Sigma_r^1$. □

2.7 Corollary. *If* $\mathrm{V} = \mathrm{L}$, *then for all* $r \geq 2$, Δ_r^1 *is a basis for* Σ_r^1. □

The main conclusion we draw concerning the results of § IV.5 is that they cannot be extended to higher levels of the projective hierarchy without assumptions which contradict $\mathrm{V} = \mathrm{L}$. Consider first the results on cardinality. Of course, if $\mathrm{V} = \mathrm{L}$, then the Continuum Hypothesis holds and *every* uncountable subset of ${}^\omega\omega$ has power $2^{\aleph_0}$. We are interested, however, in the method used to prove Theorem IV.5.12. The construction shows that any uncountable $\boldsymbol{\Sigma}_1^1$ set has a perfect subset (which therefore has power $2^{\aleph_0}$).

2.8 Theorem. *If* $\mathrm{V} = \mathrm{L}$, *then there exists an uncountable* Π_1^1 *subset of* ${}^\omega\omega$ *which has no perfect subset.*

Proof. Let $W^* = \{\gamma : \gamma \in W \wedge \neg(\exists\delta <_L \gamma)(|\gamma| = |\delta|)\}$. For each countable ordinal ρ, W^* contains exactly one function γ such that $\|\gamma\| = \rho$ and thus W^* is uncountable. It follows from Theorem 2.2 that $W^* \in \Sigma^1_2$, so let R be a Π^1_1 relation such that $\gamma \in W^* \leftrightarrow \exists\beta\, R(\gamma, \beta)$. By the Uniformization Theorem there exists a Π^1_1 relation $S \subseteq R$ such that $\gamma \in W^* \leftrightarrow \exists\beta\, S(\gamma,\beta) \leftrightarrow \exists!\beta\, S(\gamma,\beta)$. Let $B = \{\langle\gamma,\beta\rangle : S(\gamma,\beta)\}$. B is Π^1_1, and as the projection function $\langle\gamma,\beta\rangle \mapsto \gamma$ is one–one from B onto W^*, B is uncountable. Suppose B had a perfect subset P, and let $C \subseteq W^*$ be the projection of P. On the one hand, C is the one–one image of the uncountable set P so is uncountable. But P is closed so by (1) of § IV.6, $C \in \mathbf{\Sigma}^1_1$ (indeed by Theorem IV.6.9, C is Borel). Hence, by the Boundedness Theorem, $C \subseteq W_\rho$ for some $\rho < \aleph_1$. Since $W^* \cap W_\rho$ is countable, this is a contradiction. □

Concerning measurability and the Baire property, we need the following two standard results. The proof of Fubini's Theorem may be found in almost any text on Measure Theory, while that of the Kuratowski–Ulam Theorem is in Oxtoby [1971] and Kuratowski [1966]. For any $R \subseteq {}^{0,2}\omega$, let

$$R^\alpha = \{\beta : R(\alpha,\beta)\} \quad \text{and} \quad R_\beta = \{\alpha : R(\alpha,\beta)\}.$$

We denote the usual Lebesgue measure in the plane also by mes. The phrase "for almost all α (measure) (category)" means "for all α except those in some set (of measure 0) (which is meager)".

Fubini's Theorem. *For any measurable relation* $R \subseteq {}^{0,2}\omega$, *the following are equivalent*:

(i) $\text{mes}(R) = 0$;
(ii) $\text{mes}(R^\alpha) = 0$ *for almost all* α (*measure*);
(iii) $\text{mes}(R_\beta) = 0$ *for almost all* β (*measure*).

Kuratowski–Ulam Theorem. *For any relation* $R \subseteq {}^{0,2}\omega$ *which has the Baire Property, the following are equivalent*:

(i) R *is meager*;
(ii) R^α *is meager for almost all* α (*category*);
(iii) R_β *is meager for almost all* β (*category*).

2.9 Theorem. *If* V = L, *then there exists a* Δ^1_2 *relation which is neither measurable nor has the Baire Property.*

Proof. The relation is $<_L$. Since the order-type of $<_L$ is $\aleph_1$, $<_{L,\beta}$ is countable for each β and thus is meager and of measure 0. Similarly, for each α, $<_L^\alpha$ is the complement of a countable set, hence is comeager and of measure 1, hence is not meager (by the Baire Category Theorem) and is not of measure 0 (by additivity). These facts contradict the preceding theorems if $<_L$ is either measurable or has the Baire Property.

2.10–2.13 Exercises

2.10. Without the hypothesis $V = L$, $\{\alpha : \alpha \text{ is constructible}\}$ may be a proper subset of ${}^{\omega}\omega$. Show that it is Σ_2^1.

2.11. Show that if $V = L$, then for all $r \geqslant 2$ and all β,

$$\beta \in \Delta_r^1 \leftrightarrow \beta \in L_{\delta_r^1} \leftrightarrow |\beta|_L < \delta_r^1.$$

(Use the Basis Theorem (2.7) and the fact that the functions belonging to any L_σ form an initial segment in the $<_L$ ordering).

2.12. Show that if $V = L$, then for all r, Δ_r^1 is a model of the Δ_r^1-Comprehension schema.

2.13 (Spector [1958]–Addison [1959a]). Show that there exist α and β such that neither $\alpha \in \Delta_1^1[\beta]$ nor $\beta \in \Delta_1^1[\alpha]$ (α and β have incomparable hyperdegrees), but that if $V = L$, then any two functions are Δ_2^1-comparable. (For the first part use Fubini's Theorem).

2.14 Notes. The history of the consequences of $V = L$ for the analytical and projective hierarchies is rather complex. A good summary of it appears in Addison [1959a]. That Theorem 2.2 leads to the pre-wellordering property for all Σ_r^1 ($r \geqslant 2$) was obvious as soon as this property was formulated. Much of the material of this section may be found also in Devlin [1973] and Mostowski [1969].

In most of the literature the assertion $V = L$ is called the *Axiom* of Constructibility. We have used the term "hypothesis" to reflect more accurately the light in which this assertion is regarded by most logicians.

3. The Hypothesis of Projective Determinacy

Although the Hypothesis of Constructibility leads to a reasonably pleasant and elegant world of sets, it has not been accepted by many as a true statement about the intuitive world of sets. The case for the intuitive truth of any assertion is supported by the "correctness" of its consequences, but there seems to be little support to be gained from the structure of the analytical hierarchy described in the preceding section. In this section we discuss an alternative hypothesis (PD) which leads to a quite different picture of the analytical hierarchy. The reader may judge for himself whether or not these results are arguments in favor of the intuitive truth of PD.

Determinacy is an assertion concerning the existence of strategies for a certain class of infinite two-person games. With each set $\mathsf{A} \subseteq {}^\omega\omega$ we associate a game as follows. Players I and II choose alternately the values $\varepsilon(0), \varepsilon(1), \varepsilon(2), \ldots$ of a function ε. If, after an ω-sequence of moves, the completed function ε belongs to A, then I is the winner, if $\varepsilon \notin \mathsf{A}$, then II wins.

Let $\varepsilon_{\mathrm{I}}(m) = \varepsilon(2m)$ and $\varepsilon_{\mathrm{II}}(m) = \varepsilon(2m+1)$. Player I plays the number $\varepsilon_{\mathrm{I}}(m)$ at his m-th turn and player II plays the number $\varepsilon_{\mathrm{II}}(m)$ at his. We say that I plays according to the *strategy* γ iff for all m, $\varepsilon_{\mathrm{I}}(m) = \gamma(\bar{\varepsilon}(2m))$. Similarly, II plays according to the strategy δ iff for all m, $\varepsilon_{\mathrm{II}}(m) = \delta(\bar{\varepsilon}(2m+1))$. If each player plays according to his respective strategy γ or δ, the unique function generated is denoted by $\gamma \# \delta$ — that is, for all m,

$$(\gamma \# \delta)(2m) = \gamma((\overline{\gamma \# \delta})(2m)) \quad \text{and} \quad (\gamma \# \delta)(2m+1) = \delta((\overline{\gamma \# \delta})(2m+1)).$$

A strategy γ is called *winning for* I iff $\forall\delta\,[\gamma \# \delta \in \mathsf{A}]$. δ is *winning for* II iff $\forall\gamma\,[\gamma \# \delta \notin \mathsf{A}]$. A is *determined* iff either I or II has a winning strategy. $\mathrm{Det}(X)$ means that all $\mathsf{A} \in X$ are determined. The *Hypothesis of Projective Determinacy* (PD) is the assertion $\mathrm{Det}(\boldsymbol{\Delta}^1_{(\omega)})$.

It would be inappropriate here to enter into a full-scale study of all the consequences of PD or other forms of determinacy. Our main aim here is to show that under the assumption PD, the classes $\Pi^1_3, \Sigma^1_4, \Pi^1_5, \ldots$ have the pre-wellordering property. However, to orient the reader who is completely unfamiliar with determinacy, we shall mention a few of the simpler general facts about it. Others are treated in the exercises.

Using the Axiom of Choice, one can easily construct a non-determined set — given a function from a cardinal λ onto ${}^\omega\omega$, $\langle \gamma_\sigma : \sigma < \lambda \rangle$, one constructs A in λ stages to ensure at stage σ that γ_σ is not a winning strategy for either I or II (Exercise 3.13). Thus $\neg\mathrm{Det}(\mathbf{P}({}^\omega\omega))$ is a theorem of ZFC. Without the Axiom of Choice, however, there seems to be no way to construct a non-determined set, and it may well be that $\mathrm{Det}(\mathbf{P}({}^\omega\omega))$ is relatively consistent with ZF. It is known, however, that even the consistency of $\mathrm{ZF} + \mathrm{Det}(\Delta^1_2)$ cannot be proved in the theory ZF + (ZF is consistent) (Friedman [1971]).

In another direction, $\mathrm{Det}(\boldsymbol{\Delta}^1_1)$ is a theorem of ZFC (Martin [1975]). $\mathrm{Det}(\boldsymbol{\Sigma}^1_1 \cup \boldsymbol{\Pi}^1_1)$ is provable from the existence of a measurable cardinal (Martin [1970]), but $\neg\mathrm{Det}(\Sigma^1_1 \cup \Pi^1_1)$ holds in ZF + (V = L) and is therefore relatively consistent with ZFC (Corollary 3.11).

We shall continue to work in ZFC even when we assume PD. The preceding remarks do not imply that the theory ZFC + PD is inconsistent because the non-determined set constructed above is not projective (or definable in any way). It is worth noting, however, that the results of this section depend only on two special forms of the Axiom of Choice. The first is used in the general development of the analytical and projective hierarchies to prove that Σ^1_r and Π^1_r are closed under number quantification and in the last part of the proof of Theorem 3.1:

$$\forall p\, \exists \beta\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}, \beta) \leftrightarrow \exists \beta\, \forall p\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}, (\beta)^p).$$

This equivalence for projective R in fact follows from PD (Exercise 3.14). The other use of the Axiom of Choice is in the proof of Theorem 3.1 below when we use the alternative condition (4′) of I.6 for well-foundedness. To prove the equivalence of (4′) with (4) requires not the full Axiom of Choice, but only the weaker axiom of Dependent Choice (DC). Thus the results we obtain are all theorems of ZF + DC + PD.

3.1 Theorem. *If* PD, *then for any* $r > 0$, *if* Σ_r^1 *has the pre-wellordering property, then* Π_{r+1}^1 *also has the pre-wellordering property.*

Proof. As in the preceding sections we shall write x for $(a, \langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle)$, (x, γ) for $(a, \langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}, \gamma\rangle)$, etc. Let $\leqslant^r$, $\leqslant_\Sigma^r$ and $\leqslant_\Pi^r$ be relations which establish the pre-wellordering property for Σ_r^1 and $|\quad|^r$ the norm associated with $\leqslant^r$. We aim to define relations $\leqslant^{r+1}$, $\leqslant_\Sigma^{r+1}$, and $\leqslant_\Pi^{r+1}$ which establish the pre-wellordering property for Π_{r+1}^1.

To motivate the definitions, we first consider a construction which does *not* work. Since $\sim \mathsf{U}_{r+1}^1(x) \leftrightarrow \forall \gamma\, \mathsf{U}_r^1(x, \gamma)$, the method used in the proof of Theorem 1.3 suggests that we define

$$|x\dot{|}^{r+1} = \sup{}^+\{|x, \gamma|^r : \gamma \in {}^\omega\omega\}$$

and define $\dot{\leqslant}^{r+1}$ so that $|\quad\dot{|}^{r+1}$ is the associated norm (the dot signifies that these are *not* our eventual definitions). Corresponding to formula (2) in that proof we have

$$(\dot{2}) \qquad x \dot{\leqslant}^{r+1} y \leftrightarrow \forall \gamma\, \exists \delta\, [(x, \gamma) \leqslant^r (y, \delta)]$$

so that if we set

$$x \dot{\leqslant}_\Pi^{r+1} y \leftrightarrow \forall \gamma\, \exists \delta\, [(x, \gamma) \leqslant_\Sigma^r (y, \delta)]$$

we can prove as before that $\dot{\leqslant}^{r+1}$ and $\dot{\leqslant}_\Pi^{r+1}$ coincide when one of the arguments lies in $\sim \mathsf{U}_{r+1}^1$. However, the equivalence corresponding to (1) is false:

$$(\dot{1}) \qquad x \dot{\leqslant}^{r+1} y \not\leftrightarrow \exists \delta\, \forall \gamma\, [(x, \gamma) \leqslant^r (y, \delta)].$$

The implication ($\rightarrow$) fails because it is not the case that every non-empty set of ordinals has a largest element. Thus there is no good candidate for $\dot{\leqslant}_\Sigma^{r+1}$.

The contribution of determinacy is essentially to provide a new sort of "quantifier" which avoids this problem. We set

$$x \leqslant^{r+1} y \leftrightarrow \forall\gamma\, \exists\delta\, [(x, (\gamma \# \delta)_{\mathrm{I}}) \leqslant^r (y, (\gamma \# \delta)_{\mathrm{II}})];$$
$$x \leqslant^{r+1}_{\Sigma} y \leftrightarrow \exists\delta\, \forall\gamma\, [(x, (\gamma \# \delta)_{\mathrm{I}}) \leqslant^r_{\Pi} (y, (\gamma \# \delta)_{\mathrm{II}})];$$
$$x \leqslant^{r+1}_{\Pi} y \leftrightarrow \forall\gamma\, \exists\delta\, [(x, (\gamma \# \delta)_{\mathrm{I}}) \leqslant^r_{\Sigma} (y, (\gamma \# \delta)_{\mathrm{II}})].$$

It is clear that $\leqslant^{r+1}_{\Sigma}$ is Σ^1_{r+1} and $\leqslant^{r+1}_{\Pi}$ is Π^1_{r+1}. For any x and y, let

$$\mathsf{A}_{xy} = \{\varepsilon : \neg[(x, \varepsilon_{\mathrm{I}}) \leqslant^r (y, \varepsilon_{\mathrm{II}})]\}.$$

Thus

(1) $\quad x \leqslant^{r+1} y \leftrightarrow$ Player I does not have a winning strategy in A_{xy}.

Suppose that one of $\sim\mathsf{U}^1_{r+1}(x)$ or $\sim\mathsf{U}^1_{r+1}(y)$. Then for all ε, $\mathsf{U}^1_r(x, \varepsilon_{\mathrm{I}})$ or $\mathsf{U}^1_r(y, \varepsilon_{\mathrm{II}})$, so that for all ε

(2) $\quad (x, \varepsilon_{\mathrm{I}}) \leqslant^r_{\Sigma} (y, \varepsilon_{\mathrm{II}}) \leftrightarrow (x, \varepsilon_{\mathrm{I}}) \leqslant^r (y, \varepsilon_{\mathrm{II}}) \leftrightarrow (x, \varepsilon_{\mathrm{I}}) \leqslant^r_{\Pi} (y, \varepsilon_{\mathrm{II}})$

by clause (iii) of Definition 1.1. It follows that A_{xy} is $\boldsymbol{\Delta}^1_r$ and thus, by the assumption PD, is determined. Hence, whenever one of $\sim\mathsf{U}^1_{r+1}(x)$ or $\sim\mathsf{U}^1_{r+1}(y)$,

(3) $\quad x \leqslant^{r+1} y \leftrightarrow$ player II has a winning strategy in A_{xy}

$$\leftrightarrow \exists\delta\, \forall\gamma\, [(x, (\gamma \# \delta)_{\mathrm{I}}) \leqslant^r (y, (\gamma \# \delta)_{\mathrm{II}})].$$

It follows immediately from (2), (3) and the definitions that whenever one of $\sim\mathsf{U}^1_{r+1}(x)$ or $\sim\mathsf{U}^1_{r+1}(y)$,

(4) $\quad x \leqslant^{r+1}_{\Sigma} y \leftrightarrow x \leqslant^{r+1} y \leftrightarrow x \leqslant^{r+1}_{\Pi} y,$

that is, condition (iii) of Definition 1.1 is satisfied.

It remains to check condition (i). Since V here is $\sim\mathsf{U}^1_{r+1}$, (i)(a) and (b) become

(a) $\mathsf{U}^1_{r+1}(y) \rightarrow x \leqslant^{r+1} y$;

(b) $\sim\mathsf{U}^1_{r+1}(y) \wedge x \leqslant^{r+1} y \rightarrow \sim\mathsf{U}^1_{r+1}(x)$.

For (a), suppose $\mathsf{U}^1_{r+1}(y)$, so for some δ, $\sim\mathsf{U}^1_r(y, \delta)$. If player II plays in the game A_{xy} so that $\varepsilon_{\mathrm{II}} = \delta$, then by the corresponding property for $\leqslant^r$, $(x, \varepsilon_{\mathrm{I}}) \leqslant^r (y, \varepsilon_{\mathrm{II}})$ and thus II wins. Hence II has a winning strategy in A_{xy}, hence I does not so $x \leqslant^{r+1} y$.

For (b), suppose $\sim\mathsf{U}^1_{r+1}(y)$, so for all δ, $\sim\mathsf{U}^1_r(y, \delta)$, and $x \leqslant^{r+1} y$. If ε is any play of A_{xy} in which II follows his winning strategy, then $(x, \varepsilon_{\mathrm{I}}) \leqslant^r (y, \varepsilon_{\mathrm{II}})$. Since $\mathsf{U}^1_r(y, \varepsilon_{\mathrm{II}})$, the corresponding property for $\leqslant^r$ implies also $\mathsf{U}^1_r(x, \varepsilon_{\mathrm{I}})$. Clearly player I may realize any function as ε_{I} and thus for all γ, $\mathsf{U}^1_r(x, \gamma)$ — that is, $\sim\mathsf{U}^1_{r+1}(x)$.

Finally, we prove that $\leqslant^{r+1}$ is a pre-wellordering. First let δ be any function such that for all s and n, $\delta(s * \langle n \rangle) = n$. Then for any γ, if $\varepsilon = \gamma \# \delta$, then

$$\varepsilon_{\mathrm{II}}(m) = \delta(\bar{\varepsilon}(2m+1)) = \varepsilon(2m) = \varepsilon_{\mathrm{I}}(m),$$

and thus by the reflexivity of $\preccurlyeq^r$, $(x, \varepsilon_{\mathrm{I}}) \preccurlyeq^r (x, \varepsilon_{\mathrm{II}})$. In other words, δ is a winning strategy for II in the game A_{xx}, so I has no winning strategy and thus $x \preccurlyeq^{r+1} x$ — that is, $\preccurlyeq^{r+1}$ is reflexive.

We next establish that $\preccurlyeq^{r+1}$ is connected. Suppose $y \not\preccurlyeq^{r+1} x$, so I has a winning strategy, say γ^0, for A_{yx}. Hence

$$\forall\delta\, [(y, (\gamma^0 \# \delta)_{\mathrm{I}}) \not\preccurlyeq^r (x, (\gamma^0 \# \delta)_{\mathrm{II}})].$$

Then because $\preccurlyeq^r$ is connected,

$$\forall\delta\, [(x, (\gamma^0 \# \delta)_{\mathrm{II}}) \preccurlyeq^r (x, (\gamma^0 \# \delta)_{\mathrm{I}})].$$

We aim to show that player I does not have a winning strategy in A_{xy} and thus that $x \preccurlyeq^{r+1} y$. For this it will suffice to show

(5) there exists δ such that for any γ there exists δ_0 such that if $\varepsilon = (\gamma \# \delta)$ and $\varepsilon^0 = (\gamma^0 \# \delta^0)$, then $\varepsilon_{\mathrm{I}} = \varepsilon^0_{\mathrm{II}}$ and $\varepsilon^0_{\mathrm{I}} = \varepsilon_{\mathrm{II}}$.

Given (5), we have for any γ,

$$(x, \varepsilon_{\mathrm{I}}) = (x, \varepsilon^0_{\mathrm{II}}) \preccurlyeq^r (y, \varepsilon^0_{\mathrm{I}}) = (y, \varepsilon_{\mathrm{II}})$$

so that δ is a winning strategy for II.

To prove (5) we define δ and δ^0 by the recursive conditions:

$$\begin{aligned} &\delta^0(\bar{\varepsilon}^0(2m+1)) = \gamma(\bar{\varepsilon}(2m)); \quad \text{and} \\ &\delta(\bar{\varepsilon}(2m+1)) = \gamma^0(\bar{\varepsilon}^0(2m)). \end{aligned} \tag{6}$$

The solution to these equations may best be visualized by considering the following diagram:

$$\begin{array}{lllllll}
 & \mathrm{I} & m_0 = \gamma(\langle\ \rangle) & & m_2 = \gamma(\langle m_0, n_0\rangle) & \ldots & \varepsilon_{\mathrm{I}} \\
\mathsf{A}_{xy} & & & & & & \\
 & \mathrm{II} & & n_0 & \quad n_2 & \ldots & \varepsilon_{\mathrm{II}} \\
\hline
 & \mathrm{I} & n_0 = \gamma^0(\langle\ \rangle) & & n_2 = \gamma^0(\langle n_0, m_0\rangle) & \ldots & \varepsilon^0_{\mathrm{I}} \\
\mathsf{A}_{yx} & & & & & & \\
 & \mathrm{II} & & m_0 & \quad m_2 & \ldots & \varepsilon^0_{\mathrm{II}}\ .
\end{array}$$

As player I plays in A_{xy} according to his strategy γ, player II uses these moves to construct δ^0 which he then plays against γ^0 in A_{yx} to determine his moves in A_{xy}. δ describes this "strategy".

We show next that $\leqslant^{r+1}$ is transitive. Suppose $x \leqslant^{r+1} y$ and $y \leqslant^{r+1} z$. If $\mathsf{U}^1_{r+1}(z)$, then by (a) above, $x \leqslant^{r+1} z$ as desired, so we assume $\sim\mathsf{U}^1_{r+1}(z)$. By (b) above, we have then $\sim\mathsf{U}^1_{r+1}(y)$ and $\sim\mathsf{U}^1_{r+1}(x)$, so condition (3) holds for both pairs (x, y) and (y, z). Let δ^0 and δ^1 be winning strategies for II in A_{xy} and A_{yz}, respectively — that is,

$$\forall\gamma\,[(x,(\gamma \# \delta^0)_{\mathrm{I}}) \leqslant^r (y,(\gamma \# \delta^0)_{\mathrm{II}})], \quad \text{and}$$
(7)
$$\forall\gamma\,[(y,(\gamma \# \delta^1)_{\mathrm{I}}) \leqslant^r (z,(\gamma \# \delta^1)_{\mathrm{II}})].$$

We aim to show that player I does not have a winning strategy in A_{xz}. For this it will suffice to show:

there exists δ such that for any γ there exist γ^0 and γ^1 such that if

(8) $\varepsilon = (\gamma \# \delta)$ and $\varepsilon^i = (\gamma^i \# \delta^i)$ $(i = 0, 1)$, then

$\varepsilon_{\mathrm{I}} = \varepsilon^0_{\mathrm{I}}, \quad \varepsilon^0_{\mathrm{II}} = \varepsilon^1_{\mathrm{I}}, \quad \text{and} \quad \varepsilon^1_{\mathrm{II}} = \varepsilon_{\mathrm{II}},$

as if (8) holds, then for any γ,

$$(x, \varepsilon_{\mathrm{I}}) = (x, \varepsilon^0_{\mathrm{I}}) \leqslant^r (y, \varepsilon^0_{\mathrm{II}}) = (y, \varepsilon^1_{\mathrm{I}}) \leqslant^r (z, \varepsilon^1_{\mathrm{II}}) = (z, \varepsilon_{\mathrm{II}})$$

so that by the transitivity of $\leqslant^r$, $(x, \varepsilon_{\mathrm{I}}) \leqslant^r (z, \varepsilon_{\mathrm{II}})$ and thus δ is a winning strategy for II. Again, the solution to (8) may best by visualized by means of a diagram:

$$\begin{array}{llllll}
 & \mathrm{I} & m_0 = \gamma(\langle\ \rangle) & & m_2 = \gamma(\langle m_0, p_0\rangle) & \ldots\ \varepsilon_{\mathrm{I}} \\
\mathsf{A}_{xz} & & & & & \\
 & \mathrm{II} & & p_0 & \qquad\qquad p_2 & \ldots\ \varepsilon_{\mathrm{II}} \\
\hline
 & \mathrm{I} & m_0 & & m_2 & \ldots\ \varepsilon^0_{\mathrm{I}} \\
\mathsf{A}_{xy} & & & & & \\
 & \mathrm{II} & n_0 = \delta^0(\langle m_0\rangle) & & n_2 = \delta^0(\langle m_0, n_0, m_2\rangle) & \ldots\ \varepsilon^0_{\mathrm{II}} \\
 & \mathrm{I} & n_0 & & n_2 & \ldots\ \varepsilon^1_{\mathrm{I}} \\
\mathsf{A}_{yz} & & & & & \\
 & \mathrm{II} & p_0 = \delta^1(\langle n_0\rangle) & & p_2 = \delta^1(\langle n_0, p_0, n_2\rangle) & \ldots\ \varepsilon^1_{\mathrm{II}}\ .
\end{array}$$

As player I plays according to γ in A_{xz}, player II constructs γ^0 and γ^1 as indicated, plays them against δ^0 and δ^1 in A_{xy} and A_{yz}, respectively, and uses the resulting moves in A_{yz} as his moves in A_{xz}.

Finally, suppose that $\leqslant^{r+1}$ is not well founded so there exist x_i such that for all $i \in \omega$, $x_{i+1} \leqslant^{r+1} x_i$ but $x_i \not\leqslant^{r+1} x_{i+1}$. Then in each game $\mathsf{A}_{x_i, x_{i+1}}$, player I has a winning strategy, say γ^i. We aim to show:

$$\text{(9)} \qquad \begin{array}{l} \text{there exist strategies } \delta^i \ (i \in \omega) \text{ such that for all } i, \\ \text{if } \varepsilon^i = (\gamma^i \# \delta^i), \text{ then } \varepsilon_{\mathrm{I}}^{i+1} = \varepsilon_{\mathrm{II}}^i. \end{array}$$

From (9), it follows that for all i,

$$(x_i, \varepsilon_{\mathrm{I}}^i) \not\leqslant^r (x_{i+1}, \varepsilon_{\mathrm{II}}^i) = (x_{i+1}, \varepsilon_{\mathrm{I}}^{i+1})$$

which contradicts the well-foundedness of $\leqslant^r$. Strategies δ^i satisfying (9) may be constructed as in the following diagram:

$$\begin{array}{llllll}
\mathsf{A}_{x_0 x_1} & \mathrm{I} \ \ m_0 = \gamma^0(\langle\ \rangle) & & m_2 = \gamma^0(\langle m_0, n_0\rangle) & & \ldots \ \varepsilon_{\mathrm{I}}^0 \\
 & \mathrm{II} & n_0 & & n_2 & \ldots \ \varepsilon_{\mathrm{II}}^0 \\
\mathsf{A}_{x_1 x_2} & \mathrm{I} \ \ n_0 = \gamma^1(\langle\ \rangle) & & n_2 = \gamma^1(\langle n_0, p_0\rangle) & & \ldots \ \varepsilon_{\mathrm{I}}^1 \\
 & \mathrm{II} & p_0 & & p_2 & \ldots \ \varepsilon_{\mathrm{II}}^1 \\
\mathsf{A}_{x_2 x_3} & \mathrm{I} \ \ p_0 = \gamma^2(\langle\ \rangle) & & p_2 = \gamma^2(\langle p_0, q_0\rangle) & & \ldots \ \varepsilon_{\mathrm{I}}^2 \\
 & \mathrm{II} & q_0 & & q_2 & \ldots \ \varepsilon_{\mathrm{II}}^2 . \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & &
\end{array}$$

□

3.2 Theorem. *If* PD, *then for all odd r,* Π_r^1 *and* Σ_{r+1}^1 *have the pre-wellordering property.*

Proof. By induction using Theorems 1.2, 1.3, and 3.1. □

To express concisely the properties of the analytical and projective hierarchies which now follow from the general results of § 1, let $\leqslant_{\mathrm{D}}^1$ be $\leqslant^1$ as defined in the proof of Theorem 1.2 and for all $r > 1$, let $\leqslant_{\mathrm{D}}^r$ be the pre-wellorderings which arise by application of the proofs of Theorems 1.3 and 3.1. Let $|\ \ |_{\mathrm{D}}^r$ be the norm associated with $\leqslant_{\mathrm{D}}^r$, and $\boldsymbol{\kappa}_{\mathrm{D}}^r = \sup^+\{|x|_{\mathrm{D}}^r : \mathsf{V}_r(x)\}$, where V_r is $\sim\mathsf{U}_r^1$, if r is odd, and U_r^1, if r is even. Similarly, $|\ \ |_{\mathrm{D},0}^r$ is the norm associated with the restriction of $\leqslant_{\mathrm{D}}^r$ to sequences of the form $(a, \langle \mathbf{m}\rangle, \langle\ \ \rangle)$ and $\kappa_{\mathrm{D}}^r = \sup^+\{|a, \langle \mathbf{m}\rangle|_{\mathrm{D},0}^r : V_r(a, \langle \mathbf{m}\rangle)\}$, where V_r is $\sim U_r^1$, if r is odd, and U_r^1, if r is even. Note that there is no reason to believe that V_r is an initial segment of V_r with respect to $\leqslant^r_{\mathrm{D}}$ so that we may have $|a, \langle \mathbf{m} \rangle|^r_{\mathrm{D},0} < |a, \langle \mathbf{m} \rangle, \langle \quad \rangle|^r_{\mathrm{D}}$.

3.3 Theorem. *If* PD, *then for all odd* r,

(i) Π^1_r *and* $\boldsymbol{\Pi}^1_r$ *have the reduction property but not the separation property*;

(ii) Σ^1_r *and* $\boldsymbol{\Sigma}^1_r$ *have the separation property but not the reduction property*;

(iii) (a) *for any* $R \in \Sigma^1_r$, *if* $R \subseteq \sim U^1_r$, *then*

$$\sup{}^+\{|a, \langle \mathbf{m} \rangle|^r_{\mathrm{D},0} : R(a, \langle \mathbf{m} \rangle)\} < \delta^1_r,$$

(b) *for any* $\mathsf{R} \in \boldsymbol{\Sigma}^1_r$, *if* $\mathsf{R} \subseteq \sim \mathsf{U}^1_r$, *then*

$$\sup{}^+\{|x|^r_{\mathrm{D}} : \mathsf{R}(x)\} < \boldsymbol{\delta}^1_r;$$

(iv) (a) *for all* R, $R \in \Delta^1_r \leftrightarrow R \leqslant \sim U^1_{r,\rho}$ *for some* $\rho < \delta^1_r$;
(b) *for all* R, $\mathsf{R} \in \boldsymbol{\Delta}^1_r \leftrightarrow \mathsf{R} \blacktriangleleft \sim \mathsf{U}^1_{r,\rho}$ *for some* $\rho < \boldsymbol{\delta}^1_r$;

(v) $\{\alpha : \alpha \in \Delta^1_r\} \in \Pi^1_r \sim \Delta^1_r$;

(vi) Δ^1_r *is not a basis for* Π^1_{r-1};

(vii) *for any* $\mathsf{R} \in \Pi^1_r$, *there exists a partial functional* $\mathrm{Sel}_{\mathsf{R}}$ *with* Π^1_r *graph such that for all* $\mathbf{m}$ *and* $\boldsymbol{\alpha}$,

$$\exists p\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathrm{Sel}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathrm{Sel}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}) \downarrow;$$

(viii) $\kappa^r_{\mathrm{D}} = \delta^1_r$ *and* $\boldsymbol{\kappa}^r_{\mathrm{D}} = \boldsymbol{\delta}^1_r$. □

3.4 Theorem. *If* PD, *then for all even* $r \geqslant 2$,

(i) Σ^1_r *and* $\boldsymbol{\Sigma}^1_r$ *have the reduction property but not the separation property*;

(ii) Π^1_r *and* $\boldsymbol{\Pi}^1_r$ *have the separation property but not the reduction property*;

(iii) *for any* $R \in \Pi^1_r$, *if* $R \subseteq U^1_r$, *then*

$$\sup{}^+\{|a, \langle \mathbf{m} \rangle|^r_{\mathrm{D},0} : R(a, \langle \mathbf{m} \rangle)\} < \delta^1_r;$$

(iv) *for all* R, $R \in \Delta^1_r \leftrightarrow R \leqslant U^1_{r,\rho}$ *for some* $\rho < \delta^1_r$;

(v) $\{\alpha : \alpha \in \Delta^1_r\} \in \Sigma^1_r \sim \Delta^1_r$;

(vi) Δ^1_r *is not a basis for* Π^1_r;

(vii) *for any* $\mathsf{R} \in \Sigma^1_r$, *there exists a partial functional* $\mathrm{Sel}_{\mathsf{R}}$ *wtih* Σ^1_r *graph such that for all* $\mathbf{m}$ *and* $\boldsymbol{\alpha}$,

$$\exists p\, \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathrm{Sel}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathrm{Sel}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}) \downarrow;$$

(viii) $\kappa^r_{\mathrm{D}} = \delta^1_r$ *and* $\boldsymbol{\kappa}^r_{\mathrm{D}} = \boldsymbol{\kappa}^{r-1}_{\mathrm{D}} = \boldsymbol{\delta}^1_{r-1}$. □

In the remainder of this section we shall give a brief survey of some of the other consequences of PD and other forms of determinacy for the analytical and projective hierarchies. The theory is very rich; in fact, nearly all of the theory of Π_1^1 and Σ_2^1 applies to Π_r^1 and Σ_{r+1}^1 for all odd r under the hypothesis PD. We put in parentheses following each result the number of the result(s) which it generalizes.

3.5 Theorem (Moschovakis [1971a]). *If* PD, *then for all odd r*, Π_r^1, $\boldsymbol{\Pi}_r^1$, Σ_{r+1}^1 *and* $\boldsymbol{\Sigma}_{r+1}^1$ *have the uniformization property* (IV.7.8 *and* IV.7.13). □

3.6 Corollary. *If* PD, *then for all even* $r \geqslant 2$, Δ_r^1 *is a basis for* Σ_r^1 (IV.7.9). □

3.7 Theorem. *If* PD, *then for all odd r, and any* $\beta \notin \Delta_r^1$ *which is implicitly* Π_r^1, $\{\alpha : \alpha$ *is* Δ_r^1 *in* $\beta\}$ *is a basis for* Σ_r^1 (III.4.7 — *cf. Exercise* 1.31). □

3.8 Theorem (Moschovakis). *If* PD, *then for all odd r and all* $\mathsf{R} \subseteq {}^{k,l}\omega$, $\mathsf{R} \in \Pi_r^1$ *iff for some* Π_{r-1}^1 *relation* P,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (\exists \beta \in \Delta_r^1[\boldsymbol{\alpha}])\mathsf{P}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \qquad \text{(IV.2.9).} \ \square$$

3.9 Theorem (Davis [1964], Mycielski–Swierczkowski [1964]). *If* PD, *then every uncountable projective set has a perfect subset* (IV.5.12) *and every projective set is Lebesgue measurable* (IV.5.3) *and has the Baire property* (IV.5.10). □

3.10 Theorem (Moschovakis). *If* PD, *then for all odd r and all* $\mathsf{A} \subseteq {}^{\omega}\omega$, *the following are equivalent*:

(i) $\mathsf{A} \in \boldsymbol{\Delta}_r^1$;

(ii) A *is the image of a* $\boldsymbol{\Delta}_r^1$ *set* B *under a continuous functional which is one–one on* B;

(iii) A *is the image of a* $\boldsymbol{\Pi}_{r-1}^1$ *set* C *under a continuous functional which is one–one on* C. (IV.6.9 — *cf. Exercise* 1.28). □

In each of the preceding theorems where the hypothesis PD is assumed, the proof establishes a somewhat sharper theorem. In the proof of Theorem 3.1, to define the pre-wellordering $\leqslant^{r+1}$ from $\leqslant^r$ we needed only the determinacy of certain $\boldsymbol{\Delta}_r^1$ sets. Hence, for example, that Π_3^1 has the pre-wellordering property requires only Det($\boldsymbol{\Delta}_2^1$). Similarly, Theorem 3.9 can be refined to show that if every $\boldsymbol{\Pi}_r^1$ set is determined, then every uncountable $\boldsymbol{\Pi}_r^1$ set has a perfect subset. Thus with Theorem 2.8 we have a clear measure of the incompatibility of V = L and PD:

3.11 Corollary. *If* V = L, *then* $\neg$Det($\boldsymbol{\Pi}_1^1$). □

A parallel result we simply mention is

3.12 Theorem (Solovay). *If* $\mathrm{Det}(\boldsymbol{\Delta}_2^1)$, *then there exists a* Δ_3^1 *non-constructible subset of* ω (*Friedman* [1971]). □

The conclusion of this Theorem is optimal in that all Σ_2^1 and Π_2^1 subsets of ω are provably constructible (Shoenfield [1962] and Theorem VIII.3.7).

The ordinals δ_r^1 are all countable, but the ordinals $\boldsymbol{\delta}_r^1$ may be quite large. It turns out that their size relative to the sequence $\aleph_0, \aleph_1, \ldots$ depends rather delicately on just what set-theoretical assumptions are used. Of course, if the Axiom of Choice is assumed, then $2^{\aleph_0} = \aleph_\sigma$ for some σ and any pre-wellordering of ${}^\omega\omega$ has length less than $\aleph_{\sigma+1}$. In any case $\boldsymbol{\delta}_1^1 = \aleph_1$ and $\boldsymbol{\delta}_2^1 \leqslant \aleph_2$ Martin [1977]. Under the assumption of PD together with the Axiom of Choice, one has also $\boldsymbol{\delta}_3^1 \leqslant \aleph_3$ and $\boldsymbol{\delta}_4^1 \leqslant \aleph_4$, for all r, $\boldsymbol{\delta}_r^1 < \boldsymbol{\delta}_{r+1}^1$, and if r is odd, $\boldsymbol{\delta}_{r+1}^1$ is not larger than the next cardinal greater than $\boldsymbol{\delta}_r^1$. It is conjectured that in this theory $\boldsymbol{\delta}_r^1 \leqslant \aleph_r$ for all r.

If one assumes $\mathrm{Det}(\mathbf{P}({}^\omega\omega))$, the determinacy of all subsets of ${}^\omega\omega$, then it is no longer consistent to use the Axiom of Choice. It seems, however, to be consistent to assume the Axiom of Dependent Choice. In this theory it can be proved that all $\boldsymbol{\delta}_r^1$ are regular cardinals and for odd r, $\boldsymbol{\delta}_{r+1}^1$ is the least cardinal greater than $\boldsymbol{\delta}_r^1$. Thus $\boldsymbol{\delta}_2^1 = \aleph_2$ here. Curiously, however, all $\aleph_n$ with $2 < n \leqslant \omega$ are singular in this theory and it turns out that $\boldsymbol{\delta}_3^1 = \aleph_{\omega+1}$ and $\boldsymbol{\delta}_4^1 = \aleph_{\omega+2}$. In fact, for all odd r, $\boldsymbol{\delta}_r^1$ is the least cardinal greater than some cardinal with cofinality ω. It would then be expected that $\boldsymbol{\delta}_5^1 = \aleph_{\omega\cdot 2+1}$, but it is known that $\boldsymbol{\delta}_5^1$ is still larger.

We mentioned earlier that it is not possible to prove the consistency of ZFC + PD without some strong hypotheses. Hence the results of this section do not provide us with any proofs of consistency or independence. Harrington [1978] has shown that it is consistent with ZFC that neither of the classes Σ_r^1 or Π_r^1 $(r \geqslant 3)$ has either the reduction or separation properties. It follows that reduction for Σ_r^1 $(r \geqslant 3)$ and separation for Π_r^1 $(r \geqslant 3)$ are independent of ZFC. For each α and n, let $\mathfrak{A}_{r,n}$ be an assertion in the language of set theory as follows:

$\mathfrak{A}_{r,0}$: Σ_r^1 has the reduction property;

$\mathfrak{A}_{r,1}$: Π_r^1 has the reduction property;

$\mathfrak{A}_{r,n+2}$: neither Σ_r^1 nor Π_r^1 has the reduction property.

It is an open question in general for which α, the assertion $\forall r\, \mathfrak{A}_{r,\alpha(r)}$ is consistent with ZFC. The same question with "reduction" replaced by "separation" or "pre-wellordering" is also open.

3.13–3.32 Exercises

3.13. Complete the proof sketched above that if the Axiom of Choice holds, then there is a non-determined set. (*Hint*: construct two sequences α_σ and β_σ

and take $A = \{\alpha_\sigma : \sigma < \lambda\}$. The role of α_σ (β_σ) is to ensure that γ_σ is not a winning strategy for player II (I).)

3.14 (Mycielski [1964]). Let A_n ($n \in \omega$) be a countable family of subsets of ${}^\omega\omega$ such that the relation "$\beta \in A_n$" is projective. Show that if PD, then there exists a choice function θ such that $\forall n.\ \theta(n) \in A_n$. Derive from this fact that Σ^1_r and Π^1_r are closed under number quantification.

3.15 (Gale–Stewart [1953]). Prove that every open game is determined. (If I has no winning strategy, then player II may win by ensuring that at each stage of the game he still has a chance.)

3.16 (Martin [1968]). The following is an outline of an alternative proof for Theorem 3.1. We write "$\gamma \leqslant \delta$" for "γ is recursive in δ" and "$\gamma \equiv \delta$" for "$\gamma \leqslant \delta$ and $\delta \leqslant \gamma$".

(i) Let A be any determined subset of ${}^\omega\omega$ which is closed under $\equiv$ (A may be thought of as a set of degrees.) Show that there is a function γ such that either

$$(\forall \delta \geqslant \gamma)\ \delta \in A \quad \text{or} \quad (\forall \delta \geqslant \gamma)\ \delta \notin A.$$

(ii) We say $\overline{\mathrm{mes}}(A) = 1$ if the first alternative holds, $\overline{\mathrm{mes}}(A) = 0$ if the second holds. Show that if PD, then $\overline{\mathrm{mes}}$ is a countably additive measure on the projective sets of degrees.

(iii) Let $\leqslant^r$, $\leqslant^r_\Sigma$, and $\leqslant^r_\Pi$ be relations which establish the pre-wellordering property for Σ^1_r and $|\ \ |^r$ the norm associated with $\leqslant^r$. For any ε, let

$$|x|^r = \sup{}^+\{|x, \gamma|^r : \gamma \leqslant \varepsilon\}$$

and set

$$x \leqslant^{r+1} y \leftrightarrow \overline{\mathrm{mes}}(\{\varepsilon : |x|^r_\varepsilon \leqslant |y|^r_\varepsilon\}) = 1.$$

Show that if PD then there exist relations $\leqslant^{r+1}_\Sigma$ and $\leqslant^{r+1}_\Pi$ which together with $\leqslant^{r+1}$ establish the pre-wellordering property for Π^1_{r+1}.

3.17 (Blackwell [1967]). Complete the following game-theoretic proof that Π^1_1 has the reduction property. Suppose

$$R(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall \beta\ \exists p\ P(\bar{\beta}(p), \mathbf{m}, \boldsymbol{\alpha})$$

and

$$S(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \forall \gamma\ \exists q\ Q(\bar{\gamma}(q), \mathbf{m}, \boldsymbol{\alpha})$$

with P and Q closed-open. For each $(\mathbf{m}, \boldsymbol{\alpha})$, set

$$\mathsf{A}_{\mathbf{m},\boldsymbol{\alpha}} = \{\varepsilon : \exists p\, [\mathsf{P}(\bar{\varepsilon}_{\mathrm{II}}(p), \mathbf{m}, \boldsymbol{\alpha}) \wedge (\forall q \leqslant p) \neg \mathsf{Q}(\bar{\varepsilon}_{\mathrm{I}}(q), \mathbf{m}, \boldsymbol{\alpha})]\}.$$

Let

$$\mathsf{R}^*(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \text{ and player I has a winning strategy for } \mathsf{A}_{\mathbf{m},\boldsymbol{\alpha}};$$

$$\mathsf{S}^*(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \text{ and player II has a winning strategy for } \mathsf{A}_{\mathbf{m},\boldsymbol{\alpha}}.$$

Show that $(\mathsf{R}^*, \mathsf{S}^*)$ reduces (R, S). (*Hint*: $\mathsf{A}_{\mathbf{m},\boldsymbol{\alpha}}$ is an open, hence determined game by Exercise 3.15.)

The following series of exercises leads to a proof of Theorem 3.5. We say that X $(= \Sigma_r^1$ or $\Pi_r^1)$ has the *scale property* iff for all $n \in \omega$ there exists relations $\leqslant_n$, $\leqslant_{\Sigma,n}$, and $\leqslant_{\Pi,n}$ which satisfy $(\mathrm{i})_n$ and $(\mathrm{iii})_n$ as in Definition 1.1,

(ii)′ the relations $\mathsf{R}_\Sigma(n, x, y) \leftrightarrow x \leqslant_{\Sigma,n} y$ and $\mathsf{R}_\Pi(n, x, y) \leftrightarrow x \leqslant_{\Pi,n} y$ are Σ_r^1 and Π_r^1, respectively, and if $|\ \ |_n$ is the norm associated with $\leqslant_n$,

(iv) for any x and any ordinals λ_n, if $\langle x_i : i \in \omega\rangle$ is a sequence of elements of V which converges to x and $\forall n\, \exists i_n\, (\forall i \geqslant i_n)\, |x_i|_n = \lambda_n$, then $\mathsf{V}(x)$ and $\forall n . |x|_n \leqslant \lambda_n$.

We say X has the *nice scale property* iff there exist $\leqslant_n$, etc. which in addition satisfy the following (when $x = \langle a, \langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle\rangle$, we write $x(n)$ for $\langle a, \mathbf{m}, \alpha_0(n), \ldots, \alpha_{l-1}(n)\rangle\rangle$):

(v) for any x and y, if $|x|_n \leqslant |y|_n$, then also $|x|_m \leqslant |y|_m$ for all $m < n$; if $|x|_n = |y|_n$, then $x(m) = y(m)$ for all $m \leqslant n$.

Note that (iv) and (v) together imply

(vi) if $\langle x_i : i \in \omega\rangle$ is a sequence such that $\forall i\, \mathsf{V}(x_i)$ and there exist $n_0 < n_1 < \cdots$ and ordinals λ_{n_k} such that $\forall k\, \exists j_k\, (\forall i \geqslant j_k) . |x_i|_{n_k} = \lambda_{n_k}$, then $\langle x_i : i \in \omega\rangle$ converges to some $x \in \mathsf{V}$ and for *all* n there are ordinals λ_n such that $\exists i_n\, (\forall i \geqslant i_n) . |x_i|_n = \lambda_n$, and $|x|_n \leqslant \lambda_n$.

3.18. Show that if X has the scale property, then X has the nice scale property. (If $\leqslant_n$, etc., establish the scale property, set

$$x^{(n)} = (|x|_0, x(0), |x|_1, x(1), \ldots, |x|_n, x(n))$$

and

$$x \leqslant_n^* y \leftrightarrow x^{(n)} \text{ precedes } y^{(n)} \text{ lexicographically or } x, y \notin \mathsf{V}.)$$

3.19. Show that if Π_r^1 has the nice scale property, then Π_r^1 has the uniformization property. (Suppose $\mathsf{R}(x, \gamma) \leftrightarrow \mathsf{V}(a, x, \gamma)$ is a Π_r^1 relation. Let $\lambda_{x,n} = \inf\{|a, x, \gamma|_n : \mathsf{R}(x, \gamma)\}$ and for each n choose γ_n such that $|a, x, \gamma_n|_n = \lambda_{x,n}$.

Show that the (a, x, γ_n) converge to some (a, x, γ_x) such that $\mathsf{R}(x, \gamma_x)$ and that the function θ such that $\theta(x) = \gamma_x$ has Π^1_r graph.)

3.20. Show that Π^1_1 has the scale property. (To each x assign a linear ordering as usual such that $\sim\mathsf{U}^1_1(x)$ iff $\leqslant_x$ is a well-ordering. Set $p \leqslant_{x,n} q$ iff $p \leqslant_x q \wedge q <_x n$, and take

$$\begin{aligned} x \leqslant_n y \quad \text{iff} \quad & \|\leqslant_x\| < \|\leqslant_y\| \\ & \text{or} \quad (\|\leqslant_x\| = \|\leqslant_y\| < \aleph_1 \wedge \|\leqslant_{x,n}\| \leqslant \|\leqslant_{y,n}\|) \\ & \text{or} \quad (\|\leqslant_x\| = \|\leqslant_y\| = \aleph_1)). \end{aligned}$$

Towards (iv), show that if $\langle x_i : i \in \omega\rangle$ converges to x and for $i \geqslant i_n$, $\|\leqslant_{x_i}\| = \bar{\lambda}$ and $\|\leqslant_{x_i,n}\| = \bar{\lambda}_n$, then the function $n \mapsto \bar{\lambda}_n$ is order-preserving on the field of $\leqslant_x$.)

3.21. Show that if Π^1_r has the scale property, then also Σ^1_{r+1} has the scale property. (Let $\leqslant^r_n$, etc., be a nice scale for Π^1_r and let θ be constructed as in Exercise 3.19 to uniformize $\sim\mathsf{U}^1_r$. Thus θ has Π^1_r graph and $\mathsf{U}^1_{r+1} = \mathrm{Dm}\,\theta$. Set

$$|x|^{r+1}_n = \begin{cases} |x, \theta(x)|^r_n, & \text{if} \quad \mathsf{U}^1_{r+1}(x); \\ \kappa^r_n, & \text{otherwise}, \end{cases}$$

let $\leqslant^{r+1}_n$ be the associated pre-wellordering,

$$x \leqslant^{r+1}_{\Sigma,n} y \leftrightarrow \exists\gamma\, \exists\delta\, [\theta(x) \simeq \gamma \wedge \theta(y) \simeq \delta \wedge (x, \gamma) \leqslant^r_{\Pi,n} (y, \delta)],$$

and

$$x \leqslant^{r+1}_{\Pi,n} y \leftrightarrow \forall\delta\, \exists\gamma\, [\theta(y) \simeq \delta \to (x, \gamma) \leqslant^r_{\Sigma,n} (y, \delta)].)$$

3.22. Show that if PD, then for any $r \geqslant 1$, if Σ^1_r has the scale property, then also Π^1_{r+1} has the scale property. (Let $\leqslant^r_n$, etc. be a nice scale for Σ^1_r and let $s_0, s_1, \ldots$ be a recursive one–one enumeration of codes for finite sequences of natural numbers such that $s_0 = \langle\ \rangle$ and if s_i is a proper initial segment of s_j, then $i < j$. Set

$$\mathsf{A}^n_{x,y} = \{\varepsilon : \neg[(x, s_n * \varepsilon_{\mathrm{I}}) \leqslant^r_n (y, s_n * \varepsilon_{\mathrm{II}})]\};$$

$$x \lessdot^{r+1}_n y \leftrightarrow \text{player II has a winning strategy for } \mathsf{A}^n_{xy};$$

$$\begin{aligned} x \leqslant^{r+1}_n y \leftrightarrow x \mathrel{\dot{<}}^{r+1}_0 y \vee (\sim\mathsf{U}^1_{r+1}(x) \wedge x \lessdot^{r+1}_0 y \wedge \\ y \lessdot^{r+1}_0 x \wedge x \lessdot^{r+1}_n y) \vee \mathsf{U}^1_{r+1}(y). \end{aligned}$$

Define $\leqslant^{r+1}_{\Sigma,n}$ and $\leqslant^{r+1}_{\Pi,n}$ and prove (i), (ii)′, and (iii) analogously to the proof of Theorem 3.1. For (iv), suppose $\langle x_i : i \in \omega\rangle$ converges to x and

$\forall n\,(\forall i \geqslant n).\,|x_i|_n^{r+1} = \lambda_n$. For any fixed β, let $s_{n_i} = \bar{\beta}(i)$, and choose δ^i to be winning strategies for player II in $A^{n_i}_{x_{n_{i+1}}, x_{n_i}}$. Construct strategies γ^i such that if $\varepsilon^i = \gamma^i \# \delta^i$, then $\varepsilon^i_{\mathrm{I}} = \langle \beta(i) \rangle * \varepsilon^{i+1}_{\mathrm{II}}$ (hence $s_{n_i} * \varepsilon^i_{\mathrm{I}} = s_{n_{i+1}} * \varepsilon^{i+1}_{\mathrm{II}}$). For each m, the ordinals $|(x_{n_{i+1}}, s_{n_i} * \varepsilon^i_{\mathrm{I}})|^r_{n_j}$ decrease with i, for $i \geqslant j$, and are thus eventually constant. Then $(x_{n_{i+1}}, s_{n_i} * \varepsilon^i_{\mathrm{I}})$ converges to (x, β) and $\mathsf{U}^1_r(x, \beta)$. To see that $|x|^{r+1}_n \leqslant \lambda_n$, it suffices to show that player II has a winning strategy in A^n_{x, x_n}. Fix n and a strategy γ for player I. For each k and all $m \geqslant k$, player II wins $\mathsf{A}^k_{x_m, x_k}$, say by strategy $\delta^{m,k}$. Show that there exist $m_0 = n < m_1 < m_2 < \ldots$ and strategies γ_i and δ such that if $\varepsilon = (\gamma \# \delta)$ and for all i, $\varepsilon^i = (\gamma^i \# \delta^{m_{i+1}, m_i})$, then $s_{m_i} = s_n * \bar{\varepsilon}_{\mathrm{I}}(i)$, $\varepsilon^i_{\mathrm{I}} = \langle \varepsilon_{\mathrm{I}}(i) \rangle * \varepsilon^{i+1}_{\mathrm{II}}$ (hence $s_{m_i} * \varepsilon^i_{\mathrm{I}} = s_{m_{i+1}} * \varepsilon^{i+1}_{\mathrm{II}}$), and $\varepsilon_{\mathrm{II}} = \varepsilon^0_{\mathrm{II}}$. It follows that the ordinals $|(x_{m_{i+1}}, s_{m_i} * \varepsilon^i_{\mathrm{I}})|^r_{m_j}$ decrease with i, for $i \geqslant j$, hence are eventually constant, say at λ'_j. Then $(x_{m_{i+1}}, s_{m_i} * \varepsilon^i_{\mathrm{I}})$ converges to $(x, s_n * \varepsilon_{\mathrm{I}})$ and

$$|(x, s_n * \varepsilon_{\mathrm{I}})|^r_n \leqslant \lambda'_0 \leqslant |(x_{m_0}, s_{m_0} * \varepsilon^0_{\mathrm{II}})|^r_{m_0} = |(x_n, s_n * \varepsilon_{\mathrm{II}})|^r_n.)$$

3.23 (Davis [1964]). Fill in the following sketch of a proof of the first clause of Theorem 3.9. Given $\mathsf{A} \subseteq {}^\omega 2$, consider the game played as before except that player I at each of his turns may play any finite sequence of natural numbers. Show first that if player I has a winning strategy for this game, then A has a perfect subset. If player II has a winning strategy δ, for each (code for a) finite sequence s of 0's and 1's, let β_s be the unique function in ${}^\omega 2$ such that

$$\text{for all } i < \lg(s),\ \beta_s(i) = (s)_i, \quad \text{and}$$
$$\text{for all } i \geqslant \lg(s),\ \beta_s(i) \neq \delta(\bar{\beta}_s(i)).$$

Show that in this case $\mathsf{A} \subseteq \{\beta_s : s \in \omega\}$ and thus A is countable.

There now remain two steps in the proof:

(i) if every uncountable projective subset of ${}^\omega 2$ has a perfect subset, then the same is true for every uncountable projective subset of ${}^\omega \omega$;

(ii) if PD, then for every projective set $\mathsf{A} \subseteq {}^\omega 2$, the game described above is determined.

Towards (ii), to each $\varepsilon \in {}^\omega 2$ assign a function $\varepsilon^* \in {}^\omega 2$ by interpreting the even values of ε as codes for finite segments of ε^* as follows: if $\lg(\varepsilon(0)) = n_0$, then $\bar{\varepsilon}^*(n_0) = \varepsilon(0)$ and $\varepsilon^*(n_0) = \varepsilon(1)$; if $\lg(\varepsilon(2)) = n_1$, then $\bar{\varepsilon}^*(n_0 + 1 + n_1) = \varepsilon(0) * \langle \varepsilon(1) \rangle * \varepsilon(1)$ and $\varepsilon^*(n_0 + 1 + n_1) = \varepsilon(3)$; etc. Given a projective set A, let $\mathsf{B} = \{\varepsilon : \varepsilon^* \in \mathsf{A}\}$. Then B is also projective and a winning strategy for either player in the ordinary game associated with B can be converted into one for the same player in the new game associated with A. Finally, verify that this proof gives also the refined version of Theorem 3.9 needed for Corollary 3.11.

3.24 (Solovay). Show that if all sets are determined, then the following choice principle holds: for any function $\varphi : \aleph_1 \to P({}^\omega \omega)$ such that if $\sigma < \tau < \aleph_1$, then

$\varnothing \neq \varphi(\tau) \subseteq \varphi(\sigma)$, there exists a continuous functional $\theta : {}^\omega\omega \to {}^\omega\omega$ such that for all $\gamma \in \mathsf{W}$, $\theta(\gamma) \in \varphi(\|\gamma\|)$.

3.25. Use the preceding exercise to show that if all sets are determined, then every union of an $\aleph_1$-sequence of Borel sets is Σ_2^1 (cf. Corollary IV.5.2).

3.26 (Mycielski [1964]). (For readers versed in set theory). Show that if all sets are determined, then $\aleph_1$ is a strongly inaccessible cardinal in the constructible universe L. Conclude from this that the relative consistency of the statement "all sets are determined" with ZF cannot be proved in ZF. (It is obvious that $\aleph_1$ is still regular in L. If $\aleph_1$ were a successor cardinal $\aleph_{\sigma+1}^{\mathrm{L}}$ in L, then show that there exists a set of relations $\mathsf{R}_\rho \subseteq \aleph_\sigma^{\mathrm{L}} \times \aleph_\sigma^{\mathrm{L}}$ of power $\aleph_1$. Since $\aleph_\sigma^{\mathrm{L}}$ is countable, this leads to a subset of ${}^\omega\omega$ of power $\aleph_1$. Since under the hypothesis ${}^\omega\omega$ is not well-orderable, this yields $\aleph_1 < 2^{\aleph_0}$, which contradicts an extension of Exercise 3.23. The proof is completed by use of Gödel's second incompleteness theorem.)

3.27. Show that if PD, then for all r there exist α and β which are Δ_r^1-incomparable. (Cf. Exercise 2.13.)

3.28. Show that if PD, then Δ_3^1 is a model of the Δ_3^1-Comprehension schema.

3.29. Strengthen Corollary 3.11 to: if Det $(\mathbf{\Pi}_1^1)$, then $\{\alpha : \alpha$ is constructible$\}$ is countable. (If Det $(\mathbf{\Pi}_1^1)$ but $\{\alpha : \alpha$ is constructible$\}$ is uncountable, then using Exercises 2.10 and 3.23 it contains a perfect subset. Use Lemma IV.6.3 and a modification of Theorem 2.2 to construct a $\mathbf{\Delta}_2^1$ well-ordering of ${}^\omega\omega$ and reach a contradiction via Theorem 2.8.)

3.30 (Wadge, Martin). For sets $\mathsf{A}, \mathsf{B} \subseteq {}^\omega\omega$, let

$$[\mathsf{A}, \mathsf{B}] = \{\varepsilon : \varepsilon_{\mathrm{I}} \in \mathsf{A} \leftrightarrow \varepsilon_{\mathrm{II}} \notin \mathsf{B}\}.$$

(These are known as *Wadge Games.*) Set

$\mathsf{A} \boxed{\leqslant} \mathsf{B}$ iff player II has a winning strategy in either

$$[\mathsf{A}, \mathsf{B}] \quad \text{or} \quad [\mathsf{A}, \sim\mathsf{B}]$$

and

$$\mathsf{A} \boxed{<} \mathsf{B} \quad \text{iff} \quad \mathsf{A} \boxed{\leqslant} \mathsf{B} \quad \text{but } \textit{not} \quad \mathsf{B} \boxed{\leqslant} \mathsf{A}.$$

Show that if PD, then $\boxed{\leqslant}$ restricted to projective sets is a pre-wellordering. (For well-foundedness, suppose $\forall i.\mathsf{A}_{i+1} \boxed{<} \mathsf{A}_i$. Then I has a winning strategy in both

$[\mathsf{A}_i, \mathsf{A}_{i+1}]$ and $[\mathsf{A}_i, \sim\mathsf{A}_{i+1}]$, say (by Exercise 3.14) γ_i^0 and γ_i^1, respectively. For any $\delta \in {}^{\omega}2$, imitate the proof of well-foundedness in Theorem 3.1 using the strategies $\gamma_i^{\delta(i)}$ to construct functions $\varepsilon^{i,\delta}$. Show that for any finite sequence s of 0's and 1's,

$$\text{mes}(\{\delta : \varepsilon_{\text{I}}^{i,\delta} \in \mathsf{A}_i\} \cap [s]) = \tfrac{1}{2} \cdot \text{mes}([s]).$$

Derive a contradiction from the 0–1 Law, Exercise I.2.10.)

3.31. Show that the closed-open subsets of ${}^{\omega}\omega$ are pre-wellordered by $\boxed{\leq}$ in type $\aleph_1$.

3.32 (Martin). Show that if *every* Wadge game is determined, then for any $X \subseteq \aleph_1$, X is $\mathbf{\Pi}^1_1$ *in the codes* — that is, Code $X = \{\gamma : \|\gamma\| \in X\}$ is $\mathbf{\Pi}^1_1$. (Consider the game [Code X, W].)

3.33 Notes. The question of determinacy of infinite games was first discussed in Gale–Stewart [1953], but the first suggestion of its relevance for set theory came in Mycielski–Steinhaus [1962]. Since closed-open games are essentially finite, the Gale–Stewart proof that open games are determinate was an extension of Von Neumann's proof of determinacy for finite two-person games. Determinacy for $\mathbf{\Sigma}^0_2$ was proved by P. Wolfe in 1956, for $\mathbf{\Sigma}^0_3$ by Morton Davis [1964], and for $\mathbf{\Sigma}^0_4$ by J. Paris [1972]. Friedman [1971] shows that $\mathbf{\Sigma}^0_5$ determinacy is not provable in Zermelo set theory (ZF without the replacement schema).

The inspiration for applications of determinacy to the projective hierarchy was Blackwell's [1967] game-theoretic proof of the reduction property for $\mathbf{\Pi}^1_1$ (Exercise 3.17). Theorem 3.1 appeared almost simultaneously in Martin [1968] and Addison–Moschovakis [1968]. There followed an avalanche of results which is still rolling. As a guide for further reading, we suggest beginning with the survey articles, Fenstad [1971a] and Moschovakis [1973]. A comprehensive treatment will appear in Moschovakis [1979?] and Martin [1979?].

4. *Classical Hierarchies in* $\mathbf{\Delta}^1_r$

We turn now to analogues of the Borel and effective Borel hierarchies. The main results are negative: for $r \geq 2$ there is no way of building up $\mathbf{\Delta}^1_r$ or Δ^1_r from below as the Borel and effective Borel hierarchies do for $\mathbf{\Delta}^1_1$ and Δ^1_1, respectively (Theorems IV.3.3 and IV.4.12).

The operations $\bigcup$ and $\bigcap$ which generate the class of Borel relations may be thought of as applied either to a countable *set* of relations or to a countable *family* — that is, a function $\langle \mathsf{P}_p : p \in \omega \rangle$:

$$\bigcup\langle \mathsf{P}_p : p \in \omega\rangle = \{(\mathbf{m}, \boldsymbol{\alpha}) : \exists p \,.\, \mathsf{P}_p(\mathbf{m}, \boldsymbol{\alpha})\}.$$

We shall consider classes of relations which are constructed by use of other operations on families. One such operation is $\mathscr{A}$ discussed in Exercise III.2.19:

$$\mathscr{A}\langle \mathsf{P}_p : p \in \omega\rangle = \{(\mathbf{m}, \boldsymbol{\alpha}) : \exists \beta \, \forall p \,.\, \mathsf{P}_{\bar{\beta}(p)}(\mathbf{m}, \boldsymbol{\alpha})\}.$$

4.1 Definition. An *operation* is a function Φ which for any k and l, to any family $\langle \mathsf{P}_p : p \in \omega\rangle$ of relations of rank (k, l) assigns a relation $\Phi\langle \mathsf{P}_p : p \in \omega\rangle$ of rank (k, l). An operation Φ is *positive analytic* iff

(i) for any constant family $\langle \mathsf{P} : p \in \omega\rangle$, $\Phi\langle \mathsf{P} : p \in \omega\rangle = \mathsf{P}$;

(ii) for any families $\langle \mathsf{P}_p : p \in \omega\rangle$ and $\langle \mathsf{Q}_p : p \in \omega\rangle$, if for all p, $\mathsf{P}_p \subseteq \mathsf{Q}_p$, then $\Phi\langle \mathsf{P}_p : p \in \omega\rangle \subseteq \Phi\langle \mathsf{Q}_p : p \in \omega\rangle$;

(iii) for any family $\langle \mathsf{P}_p : p \in \omega\rangle$ of relations of rank (k, l) and any $\psi : {}^{k', l'}\omega \to {}^{k, l}\omega$,

$$\psi^{-1}(\Phi\langle \mathsf{P}_p : p \in \omega\rangle) = \Phi\langle \psi^{-1}(\mathsf{P}_p) : p \in \omega\rangle.$$

It is obvious that $\bigcup$, $\bigcap$, and $\mathscr{A}$ are positive analytic operations. For any $\mathsf{B} \subseteq \mathbf{P}(\omega)$, let Θ_{B} be the operation defined by

$$\Theta_{\mathsf{B}}\langle \mathsf{P}_p : p \in \omega\rangle = \{(\mathbf{m}, \boldsymbol{\alpha}) : (\exists A \in \mathsf{B})(\forall p \in A)\, \mathsf{P}_p(\mathbf{m}, \boldsymbol{\alpha})\}.$$

B is called a *base* of the operation Θ_{B}. It is similarly easy to check that for any B, if $\mathsf{B} \neq \varnothing$ and $\varnothing \notin \mathsf{B}$, then Θ_{B} is a positive analytic operation. Conversely, if Φ is positive analytic, let

$$\mathsf{B}(\Phi) = \Phi\langle \{A : p \in A\} : p \in \omega\rangle.$$

4.2 Theorem. *For every positive analytic operation Φ, $\mathsf{B}(\Phi)$ is a base of Φ — that is, $\Phi = \Theta_{\mathsf{B}(\Phi)}$.*

Proof. Let $\langle \mathsf{P}_p : p \in \omega\rangle$ be a fixed family of relations of rank (k, l). For each $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$, let $\psi_{\mathbf{m}, \boldsymbol{\alpha}}$ be the constant function with domain $\mathbf{P}(\omega)$ and value $(\mathbf{m}, \boldsymbol{\alpha})$. Note that for any R,

$$(1) \qquad \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \psi_{\mathbf{m}, \boldsymbol{\alpha}}^{-1}(\mathsf{R}) \neq \varnothing \leftrightarrow \psi_{\mathbf{m}, \boldsymbol{\alpha}}^{-1}(\mathsf{R}) = \mathbf{P}(\omega).$$

Similarly, if for any $B \subseteq \omega$, ψ_B is the constant function with domain $\mathbf{P}(\omega)$ and value B, we have for all p

$$(2) \qquad p \in B \leftrightarrow \psi_B^{-1}(\{A : p \in A\}) \neq \varnothing \leftrightarrow \psi_B^{-1}(\{A : p \in A\}) = \mathbf{P}(\omega).$$

For a fixed $(\mathbf{m}, \boldsymbol{\alpha})$, let $C = \{p : \mathsf{P}_p(\mathbf{m}, \boldsymbol{\alpha})\}$. Then by (1) and (2), for all p

(3) $$\psi^{-1}_{\mathbf{m},\boldsymbol{\alpha}}(\mathsf{P}_p) = \psi^{-1}_C(\{A : p \in A\}).$$

Hence we have

$$\begin{aligned}
(\mathbf{m}, \boldsymbol{\alpha}) \in \Phi\langle \mathsf{P}_p \rangle &\leftrightarrow \psi^{-1}_{\mathbf{m},\boldsymbol{\alpha}}(\Phi\langle \mathsf{P}_p \rangle) \neq \varnothing && \text{by (1)}\\
&\leftrightarrow \Phi\langle \psi^{-1}_{\mathbf{m},\boldsymbol{\alpha}}(\mathsf{P}_p)\rangle \neq \varnothing && \text{by (iii) of 4.1}\\
&\leftrightarrow \Phi\langle \psi^{-1}_C(\{A : p \in A\})\rangle \neq \varnothing && \text{by (3) and (ii) of 4.1}\\
&\leftrightarrow \psi^{-1}_C(\mathsf{B}(\Phi)) \neq \varnothing && \text{by (iii) of 4.1}\\
&\leftrightarrow C \in \mathsf{B}(\Phi)\\
&\rightarrow (\mathbf{m}, \boldsymbol{\alpha}) \in \Theta_{\mathsf{B}(\Phi)}\langle \mathsf{P}_p \rangle.
\end{aligned}$$

On the other hand, if $(\mathbf{m}, \boldsymbol{\alpha}) \in \Theta_{\mathsf{B}(\Phi)}\langle \mathsf{P}_p \rangle$ and C is any set in $\mathsf{B}(\Phi)$ such that $(\forall p \in C)\mathsf{P}_p(\mathbf{m}, \boldsymbol{\alpha})$, then $\psi^{-1}_C(\{A : p \in A\}) \subseteq \psi^{-1}_{\mathbf{m},\boldsymbol{\alpha}}(\mathsf{P}_p)$ and the implications $(\leftarrow)$ above all hold, so we conclude $(\mathbf{m}, \boldsymbol{\alpha}) \in \Phi\langle \mathsf{P}_p \rangle$. Hence $\Phi\langle \mathsf{P}_p \rangle = \Theta_{\mathsf{B}(\Phi)}\langle \mathsf{P}_p \rangle$ as required. □

In particular, the class of positive analytic operations coincides with the class of operations Θ_{B} with $\mathsf{B} \neq \varnothing$ and $\varnothing \notin \mathsf{B}$. $\mathsf{B}(\Phi)$ is called the *canonical base* for Φ and is easily seen to be the unique base B for Φ which satisfies the condition: for all A and B, if $A \in \mathsf{B}$ and $A \subseteq B$, then $B \in \mathsf{B}$.

Positive analytic operations may also be thought of as quantifiers or operators on relations:

(4) $$\begin{aligned}
(\Phi\mathsf{R})(\mathbf{m}, \boldsymbol{\alpha}) &\leftrightarrow (\exists A \in \mathsf{B}(\Phi))(\forall p \in A)\,\mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha})\\
&\leftrightarrow \{p : \mathsf{R}(p, \mathbf{m}, \boldsymbol{\alpha})\} \in \mathsf{B}(\Phi).
\end{aligned}$$

In this notation, $\bigcup$ coincides with $\exists^0$ and $\bigcap$ with $\forall^0$.

4.3 Definition. For any operation Φ, the *dual* Φ° of Φ is defined by:

$$\Phi^\circ\langle \mathsf{P}_p : p \in \omega \rangle = {\sim}\Phi\langle {\sim}\mathsf{P}_p : p \in \omega \rangle.$$

It is trivial that if Φ is positive analytic, so is Φ° — in fact, by a direct computation we see that

(5) $$\mathsf{B}(\Phi^\circ) = \{B : (\forall A \in \mathsf{B}(\Phi))\, A \cap B \neq \varnothing\}.$$

As an immediate consequence of (5) we have for any family $\langle \mathsf{P}_p : p \in \omega \rangle$:

$$(6)\qquad \begin{aligned}&(\exists B \in \mathsf{B}(\Phi^\circ))(\forall p \in B)\,\mathsf{P}_p(\mathbf{m},\boldsymbol{\alpha}) \leftrightarrow (\forall A \in \mathsf{B}(\Phi))(\exists p \in A)\,\mathsf{P}_p(\mathbf{m},\boldsymbol{\alpha});\\ &(\forall B \in \mathsf{B}(\Phi^\circ))(\exists p \in B)\,\mathsf{P}_p(\mathbf{m},\boldsymbol{\alpha}) \leftrightarrow (\exists A \in \mathsf{B}(\Phi))(\forall p \in A)\,\mathsf{P}_p(\mathbf{m},\boldsymbol{\alpha}).\end{aligned}$$

Another useful characterization is given by:

4.4 Lemma. *For any positive analytic operation Φ,*

$$\mathsf{B}(\Phi^\circ) = \{B : \sim B \notin \mathsf{B}(\Phi)\}.$$

Proof. If $B \in \mathsf{B}(\Phi^\circ)$, then $B \cap A \neq \varnothing$ for all $A \in \mathsf{B}(\Phi)$ so clearly $\sim B$ cannot be among such A. Conversely, if $\sim B \notin \mathsf{B}(\Phi)$, since $\mathsf{B}(\Phi)$ is closed under superset, for no $A \in \mathsf{B}(\Phi)$ is $A \subseteq \sim B$. Hence for every $A \in \mathsf{B}(\Phi)$, $A \cap B \neq \varnothing$ — that is, $B \in \mathsf{B}(\Phi^\circ)$. □

4.5 Definition. For any operation Φ, $\nabla(\Phi)$ is the smallest class of relations containing the closed-open relations and closed under Φ and Φ°.

Thus $\nabla(\bigcup) = \nabla(\bigcap) =$ the class of Borel relations and Theorem IV.3.3 asserts that $\nabla(\bigcup) = \Delta_1^1$. The main result of this section is that if $r \geq 2$, then $\Delta_r^1 \neq \nabla(\Phi)$ for *any* positive analytic operation Φ. We call Φ a Δ_r^1 *operation* iff Δ_r^1 is closed under Φ. Then we have:

4.6 Theorem. *For any positive analytic operation Φ, and any $r \geq 1$, Φ is a Δ_r^1 operation iff $\mathsf{B}(\Phi) \in \Delta_r^1$.*

Proof. Since $\mathsf{B}(\Phi)$ results from applying Φ to a family of open relations, if Φ is a Δ_r^1 operation, then $\mathsf{B}(\Phi) \in \Delta_r^1$. Suppose now that $\mathsf{B}(\Phi) \in \Delta_r^1$ and let $\langle \mathsf{P}_p : p \in \omega\rangle$ be any family of Δ_r^1 relations. Let $\mathsf{P}(p,\mathbf{m},\boldsymbol{\alpha}) \leftrightarrow \mathsf{P}_p(\mathbf{m},\boldsymbol{\alpha})$. We essentially showed in the proof of Theorem III.1.16 that also $\mathsf{P} \in \Delta_r^1$. Then by Theorem 4.2 we have

$$(\mathbf{m},\boldsymbol{\alpha}) \in \Phi\langle \mathsf{P}_p : p \in \omega\rangle \leftrightarrow (\exists A \in \mathsf{B}(\Phi))(\forall p \in A)\,\mathsf{P}(p,\mathbf{m},\boldsymbol{\alpha})$$

which yields immediately that $\Phi\langle \mathsf{P}_p : p \in \omega\rangle \in \Sigma_r^1$. On the other hand, it follows from Lemma 4.4 that also $\mathsf{B}(\Phi^\circ) \in \Delta_r^1$ and by (6) we have

$$(\mathbf{m},\boldsymbol{\alpha}) \in \Phi\langle \mathsf{P}_p : p \in \omega\rangle \leftrightarrow (\forall B \in \mathsf{B}(\Phi^\circ))(\exists p \in B)\,\mathsf{P}(p,\mathbf{m},\boldsymbol{\alpha})$$

which implies that $\Phi\langle \mathsf{P}_p : p \in \omega\rangle \in \Pi_r^1$. □

We aim next to show that for any Δ_r^1 operation Φ, $\nabla(\Phi) \subsetneq \Delta_r^1$ for all $r \geq 2$. To this end we introduce the operator $*$ on operations and prove that if Φ is Δ_r^1, so is Φ^* and that $\nabla(\Phi) \subsetneq \nabla(\Phi^*) \subseteq \Delta_r^1$. The operator $*$ turns out to be closely related to certain sorts of inductive definability.

The idea behind $*$ is an attempt to generalize the relationship between $\bigcup$ and $\mathscr{A}$. For any operation Φ, let Σ_1^{Φ} be the class of relations of the form $\Phi\langle \mathsf{P}_p : p \in \omega\rangle$ with all P_p closed-open, Π_1^{Φ} the class of complements of such relations, and $\Delta_1^{\Phi} = \Sigma_1^{\Phi} \cap \Pi_1^{\Phi}$. Then $\Sigma_1^{\cup} = \Sigma_1^0$ and $\Sigma_1^{\mathscr{A}} = \Sigma_1^1$ and Theorem IV.3.3 may be stated in the form: $\nabla(\bigcup) = \Delta_1^{\mathscr{A}}$. We shall define $*$ in such a way that $\Delta_1^{\cup^*} = \Delta_1^{\mathscr{A}}$ (Exercise 4.15) and for all positive analytic Φ, $\nabla(\Phi) \subseteq \Delta_1^{\Phi^*}$, where in general the inclusion may be proper.

To motivate the definition of $*$, consider the operation $\mathscr{A}$ in the form

$$\mathscr{A}\langle \mathsf{P}_p : p \in \omega\rangle(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p_0 \exists p_1 \cdots \forall n\, \mathsf{P}_{\langle p_0, \ldots, p_{n-1}\rangle}(\mathbf{m}, \boldsymbol{\alpha}).$$

In this form it is obvious that $\Sigma_1^{\mathscr{A}}$ is closed under $\exists^0$ and $\bigcup$, but less obvious that it is closed under $\forall^0$ and $\bigcap$. In Exercise III.3.22 we showed that $\Sigma_1^{\mathscr{A}}$ also coincides with the class of relations R expressible in the form

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p_0 \forall p_1 \exists p_2 \forall p_3 \cdots \forall n\, \mathsf{P}_{\langle p_0, \ldots, p_{n-1}\rangle}(\mathbf{m}, \boldsymbol{\alpha})$$

with all P_s closed-open. This expression is interpreted to mean that player I has a winning strategy in the game determined by $\{\varepsilon : \forall n\, \mathsf{P}_{\bar\varepsilon(n)}(\mathbf{m}, \boldsymbol{\alpha})\}$. This leads to:

4.7 Definition. For any positive analytic operation Φ, Φ^* is the operation such that for any family $\langle \mathsf{P}_s : s \in \omega\rangle$,

$$\begin{aligned}\Phi^*\langle \mathsf{P}_s : s \in \omega\rangle(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (\exists A_0 \in \mathsf{B}(\Phi))(\forall p_0 \in A_0)(\forall A_1 \in \mathsf{B}(\Phi))\\ (\exists p_1 \in A_1)(\exists A_2 \in \mathsf{B}(\Phi)) \cdots \forall n\, \mathsf{P}_{\langle p_0, \ldots, p_{n-1}\rangle}(\mathbf{m}, \boldsymbol{\alpha}),\end{aligned}$$

where the right-hand expression is true just in case player I has a winning strategy in the game $\mathscr{G}_{\mathbf{m},\boldsymbol{\alpha}}$ played as follows: I chooses $A_0 \in \mathsf{B}(\Phi)$, then II chooses both $p_0 \in A_0$ and $A_1 \in \mathsf{B}(\Phi)$, then I chooses both $p_1 \in A_1$ and $A_2 \in \mathsf{B}(\Phi)$, etc; I wins iff $\forall n\, \mathsf{P}_{\langle p_0, \ldots, p_{n-1}\rangle}(\mathbf{m}, \boldsymbol{\alpha})$.

There is no difficulty in verifying that if Φ is positive analytic, so is Φ^*. The canonical base $\mathsf{B}(\Phi^*)$ may be described as follows. Call a set C a *Φ-fan* iff C is a set of sequence numbers closed under subsequence, $\langle\ \rangle \in C$, and for every $s \in C$,

$$\begin{aligned}&\text{if } \lg(s) \text{ is even, then } (\exists A \in \mathsf{B}(\Phi))(\forall p \in A)[s * \langle p\rangle \in C];\\ (7)\quad &\\ &\text{if } \lg(s) \text{ is odd, then } (\forall A \in \mathsf{B}(\Phi))(\exists p \in A)[s * \langle p\rangle \in C].\end{aligned}$$

Then

$$\mathsf{B}(\Phi^*) = \{B : \exists C\,[C \subseteq B \wedge C \text{ is a } \Phi\text{-fan}]\}.$$

Indeed, a Φ-fan C such that $(\forall s \in C)\,\mathsf{P}_s(\mathbf{m}, \boldsymbol{\alpha})$ is a natural way of encoding a winning strategy for player I in the game $\mathscr{G}_{\mathbf{m},\boldsymbol{\alpha}}$.

4.8 Theorem. *For any positive analytic operation Φ, $\mathbf{\Delta}_1^{\Phi^*}$ is closed under both Φ and Φ°. Hence $\nabla(\Phi) \subseteq \mathbf{\Delta}_1^{\Phi^*}$.*

Proof. It will suffice to show that $\mathbf{\Sigma}_1^{\Phi^*}$ is closed under both Φ and Φ°, as then by complementation so is $\mathbf{\Pi}_1^{\Phi^*}$. By condition (i) of Definition 4.1, $\mathbf{\Delta}_1^{\Phi^*}$ contains all closed-open relations, so the second assertion follows immediately from the first.

Let $\langle \mathsf{P}_p : p \in \omega\rangle$ be a family of $\mathbf{\Sigma}_1^{\Phi^*}$ relations and $\langle \mathsf{P}_{p,s} : s \in \omega\rangle$ corresponding families of closed-open relations such that $\mathsf{P}_p = \Phi^*\langle \mathsf{P}_{p,s} : s \in \omega\rangle$. Let $\langle \mathsf{Q}_t : t \in \omega\rangle$ be a family of closed-open relations such that for all p, q, and s,

$$\mathsf{Q}_{\langle p\rangle} = {}^{k,l}\omega \quad \text{and} \quad \mathsf{Q}_{\langle p,q\rangle * s} = \mathsf{P}_{p,s}.$$

Then

$$\begin{aligned}
(\mathbf{m}, \boldsymbol{\alpha}) \in \Phi\langle \mathsf{P}_p : p \in \omega\rangle &\leftrightarrow (\exists A \in \mathsf{B}(\Phi))(\forall p \in A)[(\mathbf{m}, \boldsymbol{\alpha}) \in \Phi^*\langle \mathsf{P}_{p,s} : s \in \omega\rangle]\\
&\leftrightarrow (\exists A \in \mathsf{B}(\Phi))(\forall p \in A)(\forall B \in \mathsf{B}(\Phi))(\exists q \in B)\,[(\mathbf{m}, \boldsymbol{\alpha}) \in \Phi^*\langle \mathsf{Q}_{\langle p,q\rangle * s}\rangle]\\
&\leftrightarrow (\mathbf{m}, \boldsymbol{\alpha}) \in \Phi^*\langle \mathsf{Q}_t : t \in \omega\rangle.
\end{aligned}$$

The last equivalence may be seen by reflecting on the notion of strategy. The proof that $\mathbf{\Sigma}_1^{\Phi^*}$ is closed under Φ° is similar except that we take

$$\mathsf{Q}_{\langle q\rangle} = {}^{k,l}\omega \quad \text{and} \quad \mathsf{Q}_{\langle q,p\rangle * s} = \mathsf{P}_{p,s}. \quad \square$$

The dual operation $\Phi^{*\circ}$ may also be expressed in terms of the existence of a winning strategy for a game. Observe first that the game $\mathscr{G}_{\mathbf{m},\boldsymbol{\alpha}}$ is open from the point of view of player II: to win he must force the choice of a finite sequnce $\langle p_0, \ldots, p_{n-1}\rangle$ such that $\sim\mathsf{P}_{\langle p_0,\ldots,p_{n-1}\rangle}(\mathbf{m}, \boldsymbol{\alpha})$ and from that point on the moves are irrelevant. The argument sketched for Exercise 3.15 shows that $\mathscr{G}_{\mathbf{m},\boldsymbol{\alpha}}$ is determined. Let $\langle \mathsf{P}_s : s \in \omega\rangle$ be a fixed family, $\mathscr{G}_{\mathbf{m},\boldsymbol{\alpha}}$ the associated games which define Φ^*, and $\mathscr{G}^{\sim}_{\mathbf{m},\boldsymbol{\alpha}}$ the games associated with the family $\langle \sim \mathsf{P}_s : s \in \omega\rangle$. Then

$$\begin{aligned}
&(\mathbf{m}, \boldsymbol{\alpha}) \in \Phi^{*\circ}\langle \mathsf{P}_s : s \in \omega\rangle\\
&\quad \leftrightarrow \text{I does not have a winning strategy in } \mathscr{G}^{\sim}_{\mathbf{m},\boldsymbol{\alpha}}\\
(8)\qquad &\quad \leftrightarrow \text{II does have a winning strategy in } \mathscr{G}^{\sim}_{\mathbf{m},\boldsymbol{\alpha}}\\
&\quad \leftrightarrow (\forall A_0 \in \mathsf{B}(\Phi))(\exists p_0 \in A_0)(\exists A_1 \in \mathsf{B}(\Phi))\\
&\qquad\qquad (\forall p_1 \in A_1)\cdots \exists n\, \mathsf{P}_{\langle p_0,\ldots,p_{n-1}\rangle}(\mathbf{m}, \boldsymbol{\alpha}).
\end{aligned}$$

Towards showing that if Φ is $\mathbf{\Delta}^1_r$, then so is Φ^*, we first prove an analogue of Theorem III.3.16.

4.9 Definition. For any decomposable inductive operator Γ over ${}^{k,l}\omega$, Γ is *Φ-positive* iff P_Γ belongs to the smallest set X of relations such that:

(i) for any closed-open relation R, $\{(\mathbf{m}, \boldsymbol{\alpha}, \delta) : \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha})\} \in X$;

(ii) for any continuous functional F, $\{(\mathbf{m}, \boldsymbol{\alpha}, \delta) : \delta(\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha})) = 0\} \in X$;

(iii) X is closed under (countable) $\bigcup$, $\bigcap$, Φ, and Φ°.

4.10 Theorem. *For any positive analytic operation Φ and any $\mathsf{R} \in \mathbf{\Pi}_1^{\Phi^*}$, there exists a decomposable Φ-positive inductive operator Γ such that for all $\mathbf{m}$ and $\boldsymbol{\alpha}$,*

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \langle \mathbf{m}, \langle\ \rangle\rangle \in \bar{\Gamma}_{\boldsymbol{\alpha}}.$$

Proof. Let R be any relation in $\mathbf{\Pi}_1^{\Phi^*}$ and $\langle \mathsf{P}_s : s \in \omega\rangle$ a family of closed-open relations such that

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (\mathbf{m}, \boldsymbol{\alpha}) \in \Phi^{*\circ}\langle \mathsf{P}_s : s \in \omega\rangle.$$

We may clearly assume that if $s \subseteq t$, then $\mathsf{P}_s \subseteq \mathsf{P}_t$. Let Γ be the Φ-positive operator defined by:

$$\langle \mathbf{m}, s\rangle \in \Gamma_{\boldsymbol{\alpha}}(D) \leftrightarrow \mathsf{P}_s(\mathbf{m}, \boldsymbol{\alpha}) \vee (\forall A \in \mathsf{B}(\Phi))(\exists p \in A)(\exists B \in \mathsf{B}(\Phi))$$
$$(\forall q \in B)[\langle \mathbf{m}, s * \langle p, q\rangle\rangle \in D].$$

We claim that for all s of even length and all $\mathbf{m}$ and $\boldsymbol{\alpha}$,

$$\langle \mathbf{m}, s\rangle \in \bar{\Gamma}_{\boldsymbol{\alpha}} \leftrightarrow (\mathbf{m}, \boldsymbol{\alpha}) \in \Phi^{*\circ}\langle \mathsf{P}_{s*t} : t \in \omega\rangle \tag{9}$$

which for $s = \langle\ \rangle$ is the desired result. To establish the claim, let, for each α,

$$C_{\boldsymbol{\alpha}} = \{\langle \mathbf{m}, s\rangle : \lg(s) \text{ is even} \wedge (\mathbf{m}, \boldsymbol{\alpha}) \in \Phi^{*\circ}\langle \mathsf{P}_{s*t} : t \in \omega\rangle\}.$$

We follow the proof of Theorem III.3.2 and first show $\Gamma_{\boldsymbol{\alpha}}(C_{\boldsymbol{\alpha}}) \subseteq C_{\boldsymbol{\alpha}}$, which implies $\bar{\Gamma}_{\boldsymbol{\alpha}} \subseteq C_{\boldsymbol{\alpha}}$ and thus the implication $(\rightarrow)$ of (9). Suppose $\langle \mathbf{m}, s\rangle \in \Gamma_{\boldsymbol{\alpha}}(C_{\boldsymbol{\alpha}})$. If $\mathsf{P}_s(\mathbf{m}, \boldsymbol{\alpha})$, then II may play any strategy and win the game $\mathscr{G}^{\sim}_{\mathbf{m}, \boldsymbol{\alpha}, s}$ associated with the family $\langle \sim\mathsf{P}_{s*t} : t \in \omega\rangle$, so $\langle \mathbf{m}, s\rangle \in C_{\boldsymbol{\alpha}}$. Otherwise we have

$$(\forall A \in \mathsf{B}(\Phi))(\exists p \in A)(\exists B \in \mathsf{B}(\Phi))(\forall q \in B)[\langle \mathbf{m}, s * \langle p, q\rangle\rangle \in C_{\boldsymbol{\alpha}}].$$

By use of (8) this easily implies $\langle \mathbf{m}, s\rangle \in C_{\boldsymbol{\alpha}}$.

To show $C_{\boldsymbol{\alpha}} \subseteq \bar{\Gamma}_{\boldsymbol{\alpha}}$, we suppose $\langle \mathbf{m}, s\rangle \notin \bar{\Gamma}_{\boldsymbol{\alpha}}$ and show that in this case I has a winning strategy in $\mathscr{G}^{\sim}_{\mathbf{m},\alpha,s}$. Let $D_\alpha = \{t : \langle \mathbf{m}, s * t\rangle \notin \bar{\Gamma}_\alpha\}$. By assumption $\langle\ \rangle \in D_\alpha$. Since $\Gamma_\alpha(\bar{\Gamma}_\alpha) = \bar{\Gamma}_\alpha$, we have for any t,

$$\text{(10)}\quad t \in D_\alpha \to (\exists A \in \mathsf{B}(\Phi))(\forall p \in A)(\forall B \in \mathsf{B}(\Phi))(\exists q \in B)[t * \langle p, q\rangle \in D_\alpha].$$

Now I may win $\mathscr{G}^{\sim}_{\mathbf{m},\alpha,s}$ by the following strategy. Applying (10) to $t = \langle\ \rangle$ she picks $A_0 \in \mathsf{B}(\Phi)$ such that for any choice of $p_0 \in A_0$ and $A_1 \in \mathsf{B}(\Phi)$ there is a $p_1 \in A_1$ which she may choose such that $\langle p_0, p_1\rangle \in D_\alpha$. Applying (10) now to $t = \langle p_0, p_1\rangle$, there is a proper choice of $A_2 \in \mathsf{B}(\Phi)$ such that for any $p_2 \in A_2 \cdots \langle p_0, p_1, p_2, p_3\rangle \in D_\alpha$. Thus I has a strategy which ensures that for all even n, $\langle p_0, \ldots, p_{n-1}\rangle \in D_\alpha$ and thus in particular that $\sim\mathsf{P}_{\langle p_0,\ldots,p_{n-1}\rangle}(\mathbf{m}, \boldsymbol{\alpha})$. By the initial assumption on the P_r, this implies that for *all* n, $\sim\mathsf{P}_{\langle p_0,\ldots,p_{n-1}\rangle}(\mathbf{m}, \boldsymbol{\alpha})$ and thus I wins $\mathscr{G}^{\sim}_{\mathbf{m},\alpha,s}$. $\square$

4.11 Corollary. *For all $r \geq 2$ and any Δ^1_r positive analytic operation Φ,*
(i) *Φ^* is also Δ^1_r;*
(ii) $\nabla(\Phi) \subsetneq \Delta^1_r$.

Proof. Suppose $r \geq 2$ and Φ is a Δ^1_r positive analytic operation. It follows directly from the definitions (and Theorem III.2.5) that any Φ-positive inductive operation Γ is Δ^1_r and hence from the boldface version of Theorem III.3.18(i) that $\bar{\Gamma} \in \Delta^1_r$. Hence by the preceding theorem, $\Pi^{\Phi^*}_1 \subseteq \Delta^1_r$ and thus also $\Sigma^{\Phi^*}_1 \subseteq \Delta^1_r$. In particular, $\mathsf{B}(\Phi^*) \in \Sigma^{\Phi^*}_1$ so $\mathsf{B}(\Phi^*) \in \Delta^1_r$ and thus Φ^* is a Δ^1_r operation. That $\nabla(\Phi) \subseteq \Delta^1_r$ is now immediate from Theorem 4.8. That this inclusion is proper follows by a standard diagonal argument: if

$$\mathsf{V}(\langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle, \beta) \leftrightarrow (\exists A \in \mathsf{B}(\Phi^*))(\forall p \in A)\, \mathsf{U}^0_1(\langle\mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle, (\beta)^p).$$

Then V is a Δ^1_r relation universal for $\Sigma^{\Phi^*}_1$. Hence $\mathsf{V} \in \Delta^1_r \sim \Delta^{\Phi^*}_1$, so also $\mathsf{V} \in \Delta^1_r \sim \nabla(\Phi)$. $\square$

The class $\nabla(\Phi)$ may be decomposed into a hierarchy just as were the Borel relations. We set $\Sigma^\Phi_0 = \Pi^\Phi_0 =$ the class of closed-open relations and take

$$\Sigma^\Phi_\rho = \{\Phi\langle \mathsf{P}_p : p \in \omega\rangle\text{: all } \mathsf{P}_p \text{ have the same rank and belong to } \Pi^\Phi_{(\rho)}\}.$$

The classes Π^Φ_ρ, Δ^Φ_ρ, etc. are defined analogously as in Definition IV.3.4. It is immediate that all of these classes are included in $\nabla(\Phi)$ and indeed that $\nabla(\Phi) = \Delta^\Phi_{(\aleph_1)}$. The analogue of Lemma IV.3.5 holds with the same proof and the same is true for the parts of Lemma IV.3.6 which concern expansion, and composition and substitution of continuous functionals. It is not, however, true for all positive analytic Φ that Σ^Φ_ρ is closed under $\bigcup$ and $\exists^0$ (for example, if Φ is $\bigcap$ or $\Phi\langle \mathsf{P}_p : p \in \omega\rangle = \mathsf{P}_0$). The hierarchy theorem (corresponding to IV.3.11) is clearly false for the second of these examples. These results hold, however, for a restricted class of operations:

4.12 Definition. An operation Φ is *normal* iff there exist primitive recursive functions f and g such that for any family $\langle \mathsf{P}_p : p \in \omega\rangle$,

(i) $\Phi\langle\Phi\langle \mathsf{P}_{\langle p,q\rangle} : q \in \omega\rangle : p \in \omega\rangle = \Phi\langle \mathsf{P}_{f(r)} : r \in \omega\rangle$;

(ii) $\bigcup\langle \mathsf{P}_p : p \in \omega\rangle = \Phi\langle \mathsf{P}_{g(p)} : p \in \omega\rangle$.

For normal operations Φ there is no difficulty in imitating the proofs of IV.3.10–11 to show that for all ρ such that $0<\rho<\aleph_1$, $\mathbf{\Sigma}_\rho^\Phi \not\subseteq \mathbf{\Delta}_\rho^\Phi$ and $\mathbf{\Delta}_{\rho+1}^\Phi \not\subseteq \mathbf{\Sigma}_\rho^\Phi \cup \mathbf{\Pi}_\rho^\Phi$ (cf. Exercise 4.16). Of course $\bigcup$, $\bigcap$, $\mathscr{A}$, and $\mathscr{A}^\circ$ are normal and it can be verified by an elementary but tedious computation that if Φ is normal, so is Φ^* (cf. Hinman [1969]).

The characterization of Theorem 4.10 also yields some extensions of the results of § IV.5. Let Φ be a normal operation. If Γ is any Φ-positive inductive operator, it is easy to prove by induction on ρ that for all $\rho<\aleph_1$, $\Gamma^\rho \in \mathbf{\nabla}(\Phi)$. Suppose $\mathsf{A} \in \mathbf{\Pi}_1^{\Phi^*}$ and Γ is a Φ-positive inductive operator such that for all α,

$$\alpha \in \mathsf{A} \leftrightarrow \langle\ \rangle \in \bar{\Gamma}_\alpha.$$

Set

$$\mathsf{A}_\rho^t = \{\alpha : t \in \Gamma_\alpha^{(\rho)}\}.$$

Then just as in Theorem IV.5.1, $\mathsf{A} = \bigcup\{\mathsf{A}_\rho^{\langle\rangle} : \rho < \aleph_1\}$ and thus A is the union of an $\aleph_1$-sequence of sets belonging to $\mathbf{\nabla}(\Phi)$. Similarly, if

$$\mathsf{B}_\rho = \mathsf{A}_\rho^{\langle\rangle} \cup \bigcup\{\mathsf{A}_{\rho+1}^t \sim \mathsf{A}_\rho^t : t \in \omega\}.$$

Each B_ρ $(\rho<\aleph_1)$ belongs to $\mathbf{\nabla}(\Phi)$ and $\mathsf{A} = \bigcap\{\mathsf{B}_\rho : \rho < \aleph_1\}$.

Let us say that an operation Φ *preserves measurability* (*preserves the Baire property*) iff whenever all P_p $(p \in \omega)$ are measurable (have the Baire property) so is (does) $\Phi\langle \mathsf{P}_p : p \in \omega\rangle$. Obviously, if Φ preserves measurability (the Baire property) then all members of $\mathbf{\nabla}(\Phi)$ are measurable (have the Baire property). Of course $\bigcup$ preserves both properties and Theorems IV.5.3 and IV.5.10 show essentially that $\mathscr{A}$ also preserves both. To formulate this more precisely, for any class Y of relations, let $\mathbf{\Sigma}_1^\Phi(Y)$ be the class of relations of the form $\Phi\langle \mathsf{P}_p : p \in \omega\rangle$ with $\mathsf{P}_p \in Y$, and define the class of $(\Phi; Y)$-positive inductive operators by introducing Y as an initial class. Then the proof of Theorem 4.10 is easily modified to yield that for any $\mathsf{R} \in \mathbf{\Pi}_1^{\Phi^*}(Y)$ there is a decomposable $(\Phi; Y)$-positive inductive operator Γ such that for all $\mathbf{m}$ and $\boldsymbol{\alpha}$,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \langle \mathbf{m}, \langle\ \rangle\rangle \in \bar{\Gamma}_{\boldsymbol{\alpha}}.$$

4.13 Theorem. *For any positive analytic operation Φ, if Φ preserves measurability (the Baire property), then so does Φ^*.*

Proof. Suppose that Φ preserves measurability and let Y be the class of measurable sets. It will suffice to show that every $\mathsf{A} \in \Pi_1^{\Phi^*}(Y)$ is measurable. Let Γ be a $(\Phi; Y)$-positive inductive operator as above and set

$$\mathsf{A}_\rho^t = \{\alpha : t \in \Gamma_\alpha^{(\rho)}\}.$$

Since Φ, Φ°, $\bigcup$, and $\bigcap$ all preserve measurability, it is easy to prove by induction on ρ that all A_ρ^t ($\rho < \aleph_1$) are measurable. The proof concludes exactly as for Theorem IV.5.3. The proof for the Baire property is similar. □

Let $\Psi_0 = \bigcup$ and $\Psi_{r+1} = \Psi_r^*$. Then it follows that for all r, Ψ_r preserves both measurability and the Baire property and thus all sets in $\nabla(\Psi_r)$ are measurable and have the Baire property. By appropriately "joining" at limit ordinals, one can construct a sequence Ψ_ρ ($\rho < \aleph_1$) of positive analytic operations such that all members of $\bigcup \{\nabla(\Psi_\rho) : \rho < \aleph_1\}$ are measurable and have the Baire property. These are known classically as the R-sets and form a proper subclass of Δ_2^1.

4.14–4.17 Exercises

4.14. Compute $\mathsf{B}(\bigcup)$, $\mathsf{B}(\bigcap)$, $\mathsf{B}(\mathscr{A})$ and $\mathsf{B}(\mathscr{A}^\circ)$.

4.15 Verify that $\Delta_1^{\cup^*} = \Delta_1^{\mathscr{A}}$ (cf. Exercise III.3.22).

4.16. Prove the hierarchy theorem for normal operations (discussion following Definition 4.12).

4.17. For any positive analytic operation Φ and any class Y of relations, $\nabla(\Phi; Y)$ is the smallest class of relations including Y and closed under Φ and Φ°. Show that

$$\nabla(\bigcup; \Sigma_1^1 \cup \Pi_1^1) \subsetneq \Delta_1^{\mathscr{A}}(\Sigma_1^1 \cup \Pi_1^1).$$

(Show that the right side contains a relation universal for the left).

4.18 Notes. The operation $\mathscr{A}$ was defined by Suslin in 1917. Its original importance was in affording the first "constructive" method of obtaining a non-Borel set, but it also led directly to the definition and study of the analytical operations in the 1920's. Kantorovitch–Livenson [1932] and [1933] is a good survey of this work, which includes many of the results of this section. The operator $*$ is ascribed there to Kolmogorov, but he seems not to have published any account of it. The main facts were established in Ljapunov [1953]. The presentation in terms of games is new here and was also discovered independently by Aczel [1975].

The class of R-sets is not the largest class all of whose members are measurable and have the Baire property. A theorem due to Solovay and published in Fenstad–Normann [1974] asserts that all provably $\mathbf{\Delta}_2^1$ relations are measurable and have the Baire property. R-sets and some extensions discussed in Ljapunov [1953] are all provably $\mathbf{\Delta}_2^1$.

5. *Effective Hierarchies in* Δ_r^1

The effective Borel hierarchy of § IV.4 is derived from the classical Borel hierarchy by restricting the generating operation of countable union to families of relations which are recursively enumerable (relative to a given indexing). No ingenuity is needed to apply the same techniques to the hierarchies of the preceding section and obtain a class $\Delta^{\Phi}_{(\omega_1)}$ of relations "effectively generated" by any positive analytic operation Φ. Unfortunately, a more accurate analogy is obtained by a more complex procedure.

The problem is that the ordinal ω_1, the number of levels in the effective Borel hierarchy, is not only the least non-recursive ordinal, but also the least non-effective-Borel ordinal — that is, the least ordinal which is not the order type of an effective Borel ($=\Delta_1^1$) well-ordering of ω. If Φ is a more powerful operation ($\mathscr{A}$, for example), ω_1 is represented already by a Π_1^{Φ} well-ordering of ω (Theorem IV.2.11 — cf. Definition 5.2 below), and it seems natural that Σ_ρ^{Φ} should be defined for all ρ for which a well-ordering of type ρ occurs in the hierarchy. The apparent circularity in this idea is avoided by a "boot-strap" procedure: at each level of the hierarchy, Φ is applied to families which are recursively enumerable *in* a relation which occurs at some previous level. This will require that the indices and relations be generated simultaneously. It is by no means obvious that the resulting construction has the desired property and indeed this is proved most naturally by the methods of § VI.5, where we shall also give further evidence for the "naturalness" of the construction.

In the second part of the section we shall consider briefly a similar generalization of the second hierarchy of § IV.4 obtained by iterating a jump operator J over a set of notations for ordinals.

It would be entirely understandable if the reader were to blanch slightly at the prospect of heaping new complexities on the already complicated and somewhat tedious proofs of § IV.4. It is a sad fact that relatively clear intuitions often require masses of unpleasant calculation for their justification. We shall attempt in this section to give proofs in sufficient detail such that the average reader will be able to get an intuitive grasp of the ideas without too much pain, and the dedicated reader will be able to reconstruct complete proofs.

As a start in this direction, we consider here only relations on numbers.

5.1 Definition. For any positive analytic Φ and each k, $N^{\Phi,k}$ is the smallest subset of ω such that for all $a \in N^{\Phi,k}$, there exist relations $P_a^\Phi \subseteq {}^k\omega$ which satisfy the following conditions: for all a, b, and c,

(i) if $(c)_1 = k$ and $(c)_2 = 0$, then $\langle 7, c\rangle \in N^{\Phi,k}$ and $P^\Phi_{\langle 7,c\rangle} = \mathrm{Dm}\{c\}$;

(ii) if $b \in N^{\Phi,k}$ and for all p, $\{a\}(p, P_b^\Phi) \in N^{\Phi,k}$, then $\langle a, b\rangle \in N^{\Phi,k}$ and $P^\Phi_{\langle a,b\rangle} = \Phi\langle \sim P^\Phi_{\{a\}(p, P_b^\Phi)} : p \in \omega\rangle$.

It is immediate that for $a \in N^{\Phi,k}$, P_a^Φ denotes a unique k-ary relation. The definition may be viewed as a single inductive definition of the relation

$$V^{\Phi,k} = \{(a, i, \mathbf{m}) : a \in N^{\Phi,k} \wedge ([i = 0 \wedge P_a^\Phi(\mathbf{m})] \vee [i = 1 \wedge \sim P_a^\Phi(\mathbf{m})])\}.$$

We denote by $\omega_1[\Phi]$ the closure ordinal of this inductive definition. The sets $N^{\Phi,k}_{(\rho)}$, N^Φ_ρ, N^Φ, etc. are defined as in § IV.4.

5.2 Definition. For all $\rho < \omega_1[\Phi]$,

(i) $\Sigma_\rho^\Phi = \{P_a^\Phi : a \in N_\rho^\Phi\}$;

(ii) $\Pi_\rho^\Phi = \{\sim P_a : a \in N_\rho^\Phi\}$;

(iii) $\Delta_\rho^\Phi = \Sigma_\rho^\Phi \cap \Pi_\rho^\Phi$;

(iv) $\nabla(\Phi) = \{P_a^\Phi : a \in N^\Phi\}$.

The reader may find it curious that we have arranged things so that Σ_1^Φ consists of relations obtained by applying Φ to families of co-semi-recursive relations rather than to families of recursive relations. The reasons for this are purely technical and are of no concern except in the proof of the hierarchy theorem where it is essential that the sets of indices corresponding to the various levels are relatively simple — see Theorem IV.4.14.

It is not in general true that $\Sigma_\rho^\cup$ coincides with Σ_ρ^0 as defined in § IV.4. It is relatively easy to prove that for all ρ, $\Sigma_\rho^0 \subseteq \Sigma_\rho^\cup$ and it follows from Theorem 5.3 below that $\nabla(\bigcup) \subseteq \Delta_1^1$, so the classes $\Sigma_\rho^\cup$ form an alternative hierarchy on Δ_1^1.

We shall call Φ a Δ_r^1 *operation* iff (equivalently) $\mathsf{B}(\Phi) \in \Delta_r^1$ or for any $\mathsf{R} \in \Delta_r^1$, also $\Phi\mathsf{R} \in \Delta_r^1$ (cf. (4) of § 4 and Exercise 5.9).

5.3 Theorem. *For all $r \geq 1$ and any positive analytic Δ_r^1 operation Φ, $\nabla(\Phi) \subseteq \{R : R \in \Delta_r^1\}$, and if $r \geq 2$ the inclusion is proper.*

Proof. It will suffice to show that if Φ is Δ_r^1, then the relations $V^{\Phi,k}$ defined above belong to Π_1^1, if $r = 1$, and Δ_r^1, if $r \geq 2$. This establishes the inclusion and a standard diagonal argument shows $V^{\Phi,k} \notin \nabla(\Phi)$. By Theorems III.3.1 and 10, it furthermore suffices to construct a monotone operator $\Gamma \in \Delta^1_r$ such that $\bar{\Gamma} = V^{\Phi,k}$. For any R, let

$$N_R = \{b : \forall \mathbf{m}\,(\exists i \leq 1)\, R(b, i, \mathbf{m})\};$$

$$R_b = \{\mathbf{m} : R(b, 0, \mathbf{m})\};$$

$$Ix_R(a, b) \leftrightarrow b \in N_R \wedge \forall p . \{a\}(p, R_b) \in N_R.$$

If $R \subseteq V^{\Phi,k}$, then $N_R \subseteq N^{\Phi,k}$, and for any $b \in N_R$, $R_b = P^{\Phi}_b$. We now define Γ similarly as in the proof of Lemma IV.4.9: for any $R \subseteq {}^{k+2}\omega$, all i, and all $\mathbf{m} \in {}^k\omega$,

(i) if $(c)_1 = k$ and $(c)_2 = 0$, then
  (1) if $\{c\}(\mathbf{m})\downarrow$, then $(\langle 7, c\rangle, 0, \mathbf{m}) \in \Gamma(R)$;
  (2) if $\{c\}(\mathbf{m})\uparrow$, then $(\langle 7, c\rangle, 1, \mathbf{m}) \in \Gamma(R)$;

(ii) if $Ix_R(a, b)$, then
  (1) if $(\exists A \in \mathrm{B}(\Phi))(\forall p \in A)[(\{a\}(p, R_b), 1, \mathbf{m}) \in R]$, then $(\langle a, b\rangle, 0, \mathbf{m}) \in \Gamma(R)$;
  (2) if $(\forall A \in \mathrm{B}(\Phi))(\exists p \in A)[(\{a\}(p, R_b), 0, \mathbf{m}) \in R]$, then $(\langle a, b\rangle, 1, \mathbf{m}) \in \Gamma(R)$.

We leave it to the reader to check that this suffices. □

The properties of the hierarchy corresponding to IV.4.5–7 require in general that Φ be normal, but under this hypothesis the proofs are similar to the earlier ones. (For closure under finite intersection use Exercise 5.7.) The hierarchy theorem (corresponding to Theorem IV.4.15) also holds for all normal Φ, although its proof is substantially more complicated.

The definition of the class of *effective Φ-positive* inductive operators Γ is obtained from Definition 4.9 of the preceding section by replacing "closed-open" and "continuous" by "semi-recursive" and "recursive". Then the proof of Theorem 4.10 is easily adapted to show that for any $R \in \Pi^{\Phi^*}_1$, there is an effective Φ-positive inductive operator Γ such that for all m,

$$R(\mathbf{m}) \leftrightarrow \langle \mathbf{m}, \langle\ \rangle\rangle \in \bar{\Gamma}.$$

If Φ is normal, it is also true that for any inductive operator Γ which is effective Φ-positive, $\bar{\Gamma} \in \Pi^{\Phi^*}_1$ so we have that for all R, $R \in \Pi^{\Phi^*}_1$ iff $R \leqslant \bar{\Gamma}$ for some effective Φ-positive Γ. In particular, the operator Γ defined in the proof of Theorem 5.3 is effective Φ-positive, so that if Φ is normal, $V^{\Phi,k} \in \Pi^{\Phi^*}_1$. From this it follows that $\nabla(\Phi) \subseteq \Delta^{\Phi^*}_1$, the effective analogue of Theorem 4.8 (see Theorems VI.6.14–19 below).

In § VI.5 we shall introduce a notion of recursion relative to a positive analytic operation Φ, It turns out that for normal Φ, $\nabla(\Phi)$ consists exactly of the relations recursive in Φ and $\omega_1[\Phi]$ is the least ordinal not the order-type of a well-ordering of ω recursive in Φ, hence the least non-$\nabla(\Phi)$ ordinal as discussed in the introduction to this section. In § VI.6 we consider another notion of recursion relative to Φ (recursion in $\Phi^{\#}$). The set of relations recursive in Φ in this second sense is exactly $\Delta_1^{\Phi^*}$. Thus for $\Phi = \bigcup$ the two senses coincide but for more powerful $\Phi(\mathscr{A}$, for example), $\nabla(\Phi)$ is a proper subclass of $\Delta_1^{\Phi^*}$.

We now turn to the generalization of the second hierarchy of § IV.4. We started there with a set O of notations for the recursive ordinals and constructed sets D_u by applying the jump operator iteratively. Our generalization will consist in replacing the ordinary jump operator by a general jump operator J and extending the set of notations by allowing for recursions relative to previously generated sets much as we extended the set of indices N in Definition 5.1.

5.4 Definition. A *jump operator* is a function $\mathsf{J}: {}^\omega\omega \to {}^\omega 2$ such that there exists an index d and a primitive recursive function h such that for all a, α, and β,

(i) α^{oJ} is recursive in $\mathsf{J}(\alpha)$ with index d;

(ii) if α is recursive in β with index a, then $\mathsf{J}(\alpha)$ is recursive in $\mathsf{J}(\beta)$ with index $h(a)$.

Of course, oJ is a jump operator. For other examples, consider for any $i \leq 1$ and $r > 0$:

$$\mathsf{J}_r^i(\alpha)(\langle a, \mathbf{m}\rangle) = \begin{cases} 0, & \text{if} \quad \mathsf{U}_r^i(a, \langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle); \\ 1, & \text{otherwise.} \end{cases}$$

J_1^0 is the ordinary jump, oJ, and J_1^1 is the *hyperjump*, hJ, (Exercise IV.2.32). Any jump operator has a natural extension to sets defined by $\mathsf{J}(A) = \{m : \mathsf{J}(\mathsf{K}_A)(m) = 0\}$. Propertities (i) and (ii) of the definition hold also with α and β replaced by A and B. To avoid confusion we shall now write $\mathsf{oJ}(A)$ instead of A^{oJ}.

5.5 Definition. For any jump operator J, $<^{\mathsf{J}}$ is the smallest subset of $\omega \times \omega$ such that for all w in the field of $<^{\mathsf{J}}$ there exist sets D_w^{J} which satisfy the following conditions:

(i) $1 <^{\mathsf{J}} 2$, $D_1^{\mathsf{J}} = \{0\}$, and $D_2^{\mathsf{J}} = \mathsf{J}(D_1^{\mathsf{J}})$;

(ii) if $u <^{\mathsf{J}} v$, then $v <^{\mathsf{J}} 2^v$ and $D_{2^v}^{\mathsf{J}} = \mathsf{J}(D_v^{\mathsf{J}})$;

(iii) if $u \in \mathrm{Fld}(<^{\mathsf{J}})$, $\{a\}(0, D_u^{\mathsf{J}}) \simeq u$, and for all p, $\{a\}(p, D_u^{\mathsf{J}}) <^{\mathsf{J}} \{a\}(p+1, D_u^{\mathsf{J}})$, then for all p, $\{a\}(p, D_u^{\mathsf{J}}) <^{\mathsf{J}} 3^a 5^u$ and

$$D_{3^a 5^u}^{\mathsf{J}} = \{\langle m, p\rangle : m \in D_{\{a\}(p, D_u^{\mathsf{J}})}^{\mathsf{J}}\};$$

(iv) if $u <^{\mathsf{J}} v$ and $v <^{\mathsf{J}} w$, then $u <^{\mathsf{J}} w$.

We write O^{J} for the field of $<^{\mathsf{J}}$ and assign ordinals $|u|^{\mathsf{J}}$ to $u \in O^{\mathsf{J}}$ as in § IV.4. We set

$$\nabla(\mathsf{J}) = \{R : R \text{ is recursive in } D_u^{\mathsf{J}}, \text{ for some } u \in O^{\mathsf{J}}\}.$$

Again O^{oJ} and D_u^{oJ} are not identical with O and D_u as defined in § IV.4, but are in a sense equivalent to them. First, it is not (too) hard to show by effective transfinite recursion that there are primitive recursive functions f and g such that for all $u \in O$, D_u is recursive in $D_{f(u)}^{\mathsf{oJ}}$ with index $g(u)$. Hence, by Theorem IV.4.21, $\Delta_1^1 \subseteq \nabla(\mathsf{oJ})$. The converse inclusion follows from the next theorem.

We call J a Δ_r^1 *jump operator* iff the relation $\mathsf{P_J}(m, A) \leftrightarrow m \in \mathsf{J}(A)$ is a Δ_r^1 relation.

5.6 Theorem. *For all $r \geq 1$ and any Δ_r^1 jump operator* J, $\nabla(\mathsf{J}) \subseteq \{R : R \in \Delta_r^1\}$ *and if $r \geq 2$, the inclusion is proper.*

Proof. Suppose J is a Δ_r^1 jump operator. As in the proof of Lemma IV.4.18, modified similarly as for Theorem 5.3, we can define a Δ_r^1 monotone operator Γ such that

$$(u, i, m) \in \bar{\Gamma} \leftrightarrow u \in O^{\mathsf{J}} \wedge ([i = 0 \wedge m \in D_u^{\mathsf{J}}] \vee [i = 1 \wedge m \notin D_u^{\mathsf{J}}] \vee [i = 2 \wedge u <^{\mathsf{J}} m]).$$

If $r = 1$, then $\bar{\Gamma} \in \Pi_1^1$ so all $D_u^{\mathsf{J}} \in \Delta_1^1$; if $r \geq 2$, then $\bar{\Gamma} \in \Delta_r^1$ and all $D_u^{\mathsf{J}} \in \Delta_r^1$, but a diagonal argument shows that $\bar{\Gamma} \notin \nabla(\mathsf{J})$. □

In § VI.5 we shall also introduce a notion of recursion relative to a jump operator and show that $\nabla(\mathsf{J})$ consists exactly of the relations recursive in J. If Φ is a positive analytic operation such that $\mathsf{P_J}$ is Φ-positive, then J will be recursive in Φ and thus $\nabla(\mathsf{J}) = \{R : R \text{ recursive in } \mathsf{J}\} \subseteq \{R : R \text{ recursive in } \Phi\} = \nabla(\Phi)$.

5.7–5.9 Exercises

5.7. Show that for any positive analytic operator Φ and any families $\langle \mathsf{P}_p \rangle$ and $\langle \mathsf{Q}_q \rangle$, $\Phi\langle \mathsf{P}_p \rangle \cap \Phi\langle \mathsf{Q}_q \rangle = \Phi\langle \Phi\langle \mathsf{P}_p \cap \mathsf{Q}_q \rangle\rangle$.

5.8. Show that for any positive analytic Φ, $\nabla(\Phi)$ is closed under the quantifier Φ.

5.9. Adapt the proof of Theorem 4.6 to show that for $r \geq 1$, $\mathsf{B}(\Phi) \in \Delta_r^1$ iff Δ_r^1 is closed under the quantifier Φ.

5.10 Notes. The idea of generating sets and indices simultaneously originates with Kleene [1963]. The method was exploited in Clarke [1964] and Enderton [1964]. The constructions discussed here are essentially those of Hinman [1966] and [1969] and Shoenfield [1968].

It is not only for reasons of simplicity that we have restricted attention here to hierarchies of sets of numbers. If we attempt to generalize Definition 5.1 to be parallel to Definition IV.4.1–2, we come to expressions $\{a\}(p, \mathsf{P}_b^{\Phi})$ which at the present stage of the theory are not even defined. Although such recursions relative to functionals are defined in Chapter VI, the resulting construction is not a natural one. The reason is that among the relations P_b^{Φ} for b in some early stage of $N^{\Phi,k,l}$ there is one that is recursively equivalent to Φ (as a functional). Thus all enumerating functions recursive in Φ are already available at this stage and the "boot-strap" nature of the construction is lost. This objection can be overcome by restricting the enumerating functions to be recursive in the relations on numbers P_b^{Φ} for $b \in N^{\Phi,k,0}$. In either case, however, the hierarchy does not in general exhaust the class of relations recursive in Φ. This is discussed further in Hinman [1969] and proved in Hinman [1966].

6. A Hierarchy for Δ_2^1

In the preceding section we have seen that most of the characterizations of Δ_1^1 in terms of simpler relations do not have any natural extensions to characterizations of Δ_r^1 for $r \geqslant 2$. There is, however, one characterization which does extend, that of Corollary IV.2.22:

$$R \in \Delta_1^1 \leftrightarrow R \in \Delta_1^0[\gamma], \text{ for some implicitly } \Pi_1^0 \text{ function } \gamma.$$

The extension leads to some interesting results on implicitly Π_1^1 functions as well as a pleasant hierarchy of the Δ_2^1 relations on numbers.

6.1 Theorem. *For all $R \subseteq {}^k\omega$,*

$$R \in \Delta_2^1 \leftrightarrow R \in \Delta_1^1[\gamma], \textit{ for some implicitly } \Pi_1^1 \textit{ function } \gamma,$$

Proof. If R satisfies the right side, then $R \in \Delta_2^1$ by Examples III.2.3 and Corollary III.2.13. For the converse implication, suppose $R \in \Delta_2^1$ and let $\mathsf{K}'_R(m) = \mathsf{K}_R((m)_0, \ldots, (m)_{k-1})$. Then $\mathsf{K}'_R \in \Delta_2^1$ so there exists a Π_1^1 relation R such that

$$\mathsf{K}'_R(m) = n \leftrightarrow \exists\beta\, \mathsf{R}(m, n, \beta).$$

Hence for any α,

$$\alpha = \mathsf{K}'_R \leftrightarrow \exists\beta\, \forall m\, \forall n\, [\alpha(m) = n \rightarrow \mathsf{R}(m, n, (\beta)^{\langle m,n\rangle})].$$

If we denote the right side of this equivalence by $\exists\beta\, \mathsf{S}(\alpha, \beta)$, then $\mathsf{S} \in \Pi_1^1$ so we may apply the Uniformization Theorem to obtain a relation $\mathsf{S}' \in \Pi_1^1$ such that

$$\alpha = \mathsf{K}_R' \leftrightarrow \exists!\, \beta\, \mathsf{S}'(\alpha, \beta) \leftrightarrow \exists\beta\, \mathsf{S}'(\alpha, \beta).$$

Then S' holds of a unique pair of functions (K_R', β) and thus the function $\gamma = \langle \mathsf{K}_R', \beta\rangle$ is implicitly Π_1^1. Clearly R is $\Delta_1^1[\gamma]$. □

Note that we actually proved that every Δ_2^1 relation is recursive in some implicitly Π_1^1 function. Before constructing the hierarchy for Δ_2^1, we establish some facts about the implicitly Π_1^1 functions which are interesting in their own right. First, it follows immediately from the theorem that $\{\alpha : \alpha$ is recursive in some $B \in \Sigma_1^1\}$ is not a basis for even the class of Π_1^1 singletons (cf. Theorem III.4.7).

If γ is implicitly Π_1^1, then there are (infinitely many) recursive relations P such that for any α,

$$\alpha = \gamma \leftrightarrow \forall\beta\, \exists n\, \mathsf{P}(\bar{\beta}(n), \alpha) \leftrightarrow \mathsf{F}_\mathsf{P}[\alpha] \in \mathsf{W},$$

where F_P is the functional F constructed in the proof of Theorem IV.1.1. We temporarily call such a P a *matrix* for γ. Note that for each matrix P for γ, $\mathsf{F}_\mathsf{P}[\gamma] \in \mathsf{W}$.

6.2 Definition. For any implicitly Π_1^1 function γ,

$$\chi(\gamma) = \inf\{\|\mathsf{F}_\mathsf{P}[\gamma]\| : \mathsf{P} \text{ is a matrix for } \gamma\}.$$

It is clear that $\chi(\gamma)$ is always a countable ordinal — in fact:

6.3 Lemma. *For any implicitly Π_1^1 function γ,*

$$\chi(\gamma) < \omega_1[\gamma] < \delta_2^1.$$

Proof. For any matrix P for γ, $\mathsf{F}_\mathsf{P}[\gamma]$ is clearly the order-type of a well-ordering recursive in γ so that $\|\mathsf{F}_\mathsf{P}[\gamma]\| < \omega_1[\gamma]$. Since $\gamma \in \Delta_2^1$, $\omega_1[\gamma] < \delta_2^1$ by Exercise IV.2.31. □

6.4 Theorem. *For any β and any implicitly Π_1^1 function γ,*

$$\gamma \in \Delta_1^1[\beta] \leftrightarrow \chi(\gamma) < \omega_1[\beta].$$

Proof. The implication $(\rightarrow)$ is immediate from the preceding Lemma and a relativized version of Theorem IV.2.11. For $(\leftarrow)$, suppose that $\chi(\gamma) < \omega_1[\beta]$ and let P be a matrix for γ such that $\mathsf{F_P}[\gamma] = \chi(\gamma)$. Choose $\delta \in \mathsf{W}$, δ recursive in β, such that $\|\delta\| = \chi(\gamma)$. Then for any α,

$$\alpha = \gamma \leftrightarrow \mathsf{F_P}[\alpha] \in \mathsf{W} \leftrightarrow \mathsf{F_P}[\alpha] \leqslant_\Sigma^{\mathsf{W}} \delta \leftrightarrow \mathsf{F_P}[\alpha] \leqslant_\Pi^{\mathsf{W}} \delta.$$

Thus $\{\gamma\} \in \Delta_1^1[\beta]$, so also $\gamma \in \Delta_1^1[\beta]$ by the relativized version of Corollary III.2.7(vii). □

6.5 Corollary. *For any implicitly Π_1^1 functions γ and δ,*

(i) $\chi(\gamma) \leqslant \chi(\delta) \rightarrow \gamma \in \Delta_1^1[\delta]$;

(ii) $\gamma \in \Delta_1^1[\delta]$ *or* $\delta \in \Delta_1^1[\gamma]$.

Proof. (i) is immediate from 6.3 and 6.4, and (ii) is immediate from (i). □

To interpret these results most succinctly, we return to the notion of hyperdegree introduced in § IV.2 following Theorem IV.2.12, and the ordering $\leqslant_1^1$ on them. We call a hyperdegree x *implicitly* Π_1^1 iff some implicitly Π_1^1 function belongs to x. We may extend the function χ to implicitly Π_1^1 hyperdegrees x by setting

$$\chi(x) = \inf\{\chi(\gamma) : \gamma \in x \text{ and } \gamma \text{ is implicitly } \Pi_1^1\}.$$

6.6 Corollary. *The relation $\leqslant_1^1$ restricted to implicitly Π_1^1 hyperdegrees is a well-ordering.*

Proof. That the ordering is linear is exactly 6.5(ii). From the contrapositive of 6.5(i) we have for any implicitly Π_1^1 functions γ and δ,

$$\delta <_1^1 \gamma \rightarrow \chi(\delta) < \chi(\gamma)$$

from which it follows that for any implicitly Π_1^1 hyperdegrees x and y,

$$y <_1^1 x \rightarrow \chi(y) < \chi(x).$$

Since $\chi(x)$ and $\chi(y)$ are ordinals, this implies that the ordering is well-founded. □

We return now to the construction of hierarchy for the Δ_2^1 relations on numbers. The most obvious choice for the levels of the hierarchy is the sequence of sets

$$X(\sigma) = \{R : R \in \Delta_1^1[\gamma], \text{ for some implicitly } \Pi_1^1 \text{ function } \gamma \text{ such that } \chi(\gamma) < \sigma\}.$$

However, it is immediate from Theorem 6.4 that for some $\sigma < \omega_1$, $X(\sigma) = X(\tau) = X(\omega_1) = \Delta_1^1$ for all τ, $\sigma \leqslant \tau \leqslant \omega_1$. To obtain a properly increasing sequence of sets we select only certain of the $X(\sigma)$.

For any hyperdegree x and any $\alpha, \beta \in x$, $\Delta_1^1[\alpha] = \Delta_1^1[\beta]$ and then from Theorem IV.2.14, $\omega_1[\alpha] = \omega_1[\beta]$. We denote these common values by $\Delta_1^1[x]$ and $\omega_1[x]$.

6.7 Definition. For all $\sigma > 0$,

(i) $x_0 = \{\alpha : \alpha \in \Delta_1^1\}$ (the zero hyperdegree);

(ii) if there exists an implicitly Π_1^1 hyperdegree x such that for all $\tau < \sigma$, $x_\tau <_1^1 x$, then x_σ is the $\leqslant_1^1$-least such x; otherwise $x_\sigma = x_0$;

(iii) $Z_\sigma = X(\omega_1[x_\sigma])$;

(iv) $\kappa = (\text{least } \sigma > 0)\ [x_\sigma = x_0]$.

Note that by Theorem 6.4, for $\sigma < \kappa$, $Z_\sigma = \{R : R \in \Delta_1^1[x_\sigma]\}$.

6.8 Theorem. (i) *For all* $\sigma < \tau < \kappa$, $Z_\sigma \subsetneq Z_\tau \subseteq \Delta_2^1$;

(ii) *for any* R, $R \in \Delta_2^1$ *iff* $R \in Z_\sigma$ *for some* $\sigma < \kappa$.

Proof. (i) is immediate from the remark preceding the theorem. For (ii), suppose $R \in \Delta_2^1$. By Theorem 6.1 choose an implicitly Π_1^1 function γ such that $R \in \Delta_1^1[\gamma]$. Then there exists a $\sigma < \kappa$ such that $\text{hydg}(\gamma) \leqslant_1^1 x_\sigma$ and thus $R \in \Delta_1^1[x_\sigma]$ so $R \in Z_\sigma$. $\square$

It remains only to evaluate the length κ of this hierarchy. Since any γ which is implicitly Π_1^1 is Δ_2^1, $\sigma < \omega_1[x_\sigma] < \delta_2^1$ for all $\sigma < \kappa$, and thus $\kappa \leqslant \delta_2^1$.

6.9 Theorem. $\kappa = \delta_2^1$.

Proof. Suppose to the contrary that $\kappa < \delta_2^1$. If κ were a successor ordinal $\lambda + 1$, then x_λ would be $<_1^1$-greatest among all implicitly Π_1^1 hyperdegrees. By Theorem 6.4 this implies that $\chi(\gamma) < \omega_1[x_\lambda]$ for all implicitly Π_1^1 functions γ and this leads easily to the conclusion that the class of Δ_2^1 functions is Δ_2^1, contrary to Theorem 1.19(v). Hence κ is a limit ordinal.

We shall derive a contradiction by constructing an implicitly Π_1^1 function γ such that $x_\sigma <_1^1 \text{hydg}(\gamma)$ for all $\sigma < \kappa$. By the Uniformization Theorem, there exists a Π_1^1 relation V such that for all a and γ,

$$\mathsf{V}(a, \gamma) \rightarrow \sim \mathsf{U}_1^1(a, \langle\ \rangle, \langle \gamma \rangle), \quad \text{and}$$

$$\exists! \gamma\, \mathsf{V}(a, \gamma) \leftrightarrow \exists \gamma \sim \mathsf{U}_1^1(a, \langle\ \rangle, \langle \gamma \rangle).$$

Thus γ is implicitly Π_1^1 iff for some a, γ is the unique function satisfying $\mathsf{V}(a, \gamma)$. Let ε be a Δ_2^1 well-ordering such that $\|\varepsilon\| = \kappa$ and consider the following set of functions:

$$\mathsf{A} = \{\alpha : \forall p \in \mathrm{Fld}(\varepsilon)\, \exists \gamma\, (\mathsf{V}(\alpha(p), \gamma) \wedge (\forall q <_\varepsilon p)\, \exists \beta\, [\mathsf{V}(\alpha(q), \beta) \wedge \beta <_1^1 \gamma])\}.$$

A is Σ_2^1 and if $\alpha \in \mathsf{A}$, then for all $p \in \mathrm{Fld}(\varepsilon)$, if $\sigma = |p|_\varepsilon$, then the unique γ such that $\mathsf{V}(\alpha(p), \gamma)$ satisfies $x_\sigma \leqslant_1^1 \mathrm{hydg}(\gamma)$. Furthermore, since each $|p|_\varepsilon$ is less than κ, there always is such a γ and thus $\mathsf{A} \neq \varnothing$. By the Basis Theorem choose a fixed $\alpha \in \mathsf{A} \cap \Delta_2^1$.

Let δ be an implicitly Π_1^1 function such that both α and ε are recursive in δ, say with indices a and e, respectively. Let

$$\begin{aligned}
\mathsf{B} = \{\gamma : &(\gamma)^0 = \delta\ \wedge \\
&\forall p\, [p \in \mathrm{Fld}(\lambda m\,.\{e\}(m, (\gamma)^0)) \to \mathsf{V}(\{a\}(p, (\gamma)^0), (\gamma)^{p+1})] \wedge \\
&\forall r\, [(r \notin \mathrm{Sq} \vee \lg(r) \neq 2 \vee ((r)_0 \dot{-} 1) \notin \mathrm{Fld}(\lambda m\,.\{e\}(m, (\gamma)^0))) \to \gamma(r) = 0]\}.
\end{aligned}$$

Clearly B is a Π_1^1 set with a unique member γ, which is thus implicitly Π_1^1. For any $\tau < \kappa$, there exists a $p \in \mathrm{Fld}(\varepsilon)$ such that $|p|_\varepsilon = \tau + 1$ and thus

$$x_\tau <_1^1 x_{\tau+1} \leqslant_1^1 \mathrm{hydg}((\gamma)^{p+1}) \leqslant_1^1 \mathrm{hydg}(\gamma).$$

It follows that $x_\kappa \neq x_0$, a contradiction. $\square$

6.10–6.17 Exercises

6.10. Show that not all Δ_2^1 functions are implicitly Π_1^1 (use the existence of incomparable hyperdegrees from Exercise 2.13).

6.11. Prove

(i) if β is implicitly Π_1^1 and β and γ are each Δ_1^1 in the other, then also γ is implicitly Π_1^1;

(ii) if β is implicitly Π_1^1, then so is the hyperjump of β;

(iii) in Definition 6.7, $x_{\sigma+1}$ is the hyperjump of x_σ (use Theorems 6.4 and IV.2.14).

6.12. Show that for any implicitly Π_1^1 functions γ and δ,

$$\gamma <_1^1 \delta \leftrightarrow \omega_1[\gamma] < \omega_1[\delta].$$

6.13. Show that for any implicitly Π_1^1 function γ, γ is implicitly Π_1^0 iff $\chi(\gamma) \leqslant 1$.

6.14. Show that

$$\sup{}^+\{\|\gamma\| : \gamma \in \mathsf{W} \wedge \gamma \text{ is implicitly } \Pi_1^1\} = \delta_2^1.$$

6.15. Use the relation V defined in the proof of Theorem 6.9 together with Theorem 6.1 to give a new proof that $\{\alpha : \alpha \in \Delta_2^1\} \in \Sigma_2^1$.

6.16. Combine the results of the preceding two exercises to contruct a Σ_2^1 well-ordering of length δ_2^1.

6.17. What parts of this section may be generalized to Δ_r^1 under either of the hypotheses $V = L$ or PD?

6.18 Notes. The results of this section are due to Suzuki [1964].