BOUNDEDNESS OF MAXIMAL OPERATORS AND MAXIMAL COMMUTATORS ON NON-HOMOGENEOUS SPACES

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ABSTRACT. Let (X, μ) be a non-homogeneous space in the sense that X is a metric space equipped with an upper doubling measure μ. The aim of this paper is to study the endpoint estimate of the maximal operator associated to a Calderón-Zygmund operator T and the L^p boundedness of the maximal commutator with RBMO functions

1. Introduction

Let (X, d, μ) be a geometrically doubling regular metric space and have an upper doubling measure, that is, μ is dominated by a function λ (see Section 2 for precise definition). A kernel $K(\cdot,\cdot) \in L^1_{\mathrm{loc}}(X \times X\backslash\{(x,y) : x = y\})$ is called a Calderón-Zygmund kernel if the following two conditions hold:

(i) K satisfies the estimates

$$|K(x,y)| \leq C \min\left\{\frac{1}{\lambda(x, d(x,y))}, \frac{1}{\lambda(y, d(x,y))}\right\}; \tag{1}$$

(ii) there exists $0 < \delta \leq 1$ such that

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C \frac{d(x,x')^\delta}{d(x,y)^\delta \lambda(x, d(x,y))} \tag{2}$$

whenever $d(x,x') \leq d(x,y)/2$.

In what follows, by the associate kernel of a linear operator T, we shall mean the function $K(\cdot,\cdot)$ defined off-diagonal $\{(x,y) \in X \times X : x \neq y\}$ so that

$$Tf(x) = \int_X K(x,y) f(y) d\mu(y),$$

holds for all $f \in L^\infty(\mu)$ with bounded support and $x \notin supp f$.

A linear operator T is called a Calderón-Zygmund operator if its associate kernel $K(\cdot,\cdot)$ satisfies (1) and (2).

In [1] the authors studied the boundedness of Calderón-Zygmund operators and their commutators with RBMO functions. It was proved that if the Calderón-Zygmund operator T is bounded on $L^2(\mu)$ then T is of weak type $(1,1)$ and hence T is bounded on $L^p(\mu)$ for all $1 < p < \infty$. Moreover, L^p boundedness of the commutators of Calderón-Zygmund operators

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with RBMO functions for $1 < p < \infty$ was also obtained in [1]. The obtained results in [1] can be viewed as extensions of those in [9] to spaces of non-homogenous type.

In this paper, we consider the maximal operator T_* associated with the Calderón-Zygmund operator T defined by

$$T_*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|,$$

where $T_\epsilon f(x) = \int_{d(x,y)\geq\epsilon} K(x,y)f(y)d\mu(y)$. Note that in [1], thanks to Cotlar inequality, it was proved that the maximal operator T_* is bounded on $L^p(\mu)$ for all $1 < p < \infty$. The aim of this paper is to prove the following results:

- T_* is of weak type $(1,1)$;
- The commutator of T_* with an RBMO function is bounded on $L^p(\mu)$ for $1 < p < \infty$.

Note that since the kernel $K_\epsilon(x,y) = K(x,y)\chi_{\{d(x,y)>\epsilon\}}(x,y)$ may not satisfy the condition (2), the Calderón-Zygmund theory may not be applicable to this situation. To overcome this problem, we use the smoothing technique in [8] by replacing $K_\epsilon(x,y)$ by some new "smooth" kernels. For detail, we refer to Section 3.2.

The organization of our paper as follows. Section 2 recalls the concept of RBMO space and the Calderón-Zygmund decomposition. Section 3 will be devoted to study the boundedness of the maximal operator T_* and the maximal commutator of T_* with an RBMO function. It will be shown that T_* is of type weak $(1,1)$ and the maximal commutator $T_{*,b}$ is bounded on $L^p(\mu)$ for all $1 < p < \infty$.

2. RBMO(μ) and Calderón-Zygmund decomposition

Let (X,d) be a metric space. We first review two concepts introduced in [2].

Geometrically doubling regular metric spaces. (X,d) is geometrically doubling if there exists a number $N \in \mathbb{N}$ such that every open ball $B(x,r) = \{y \in X : d(y,x) < r\}$ can be covered by at most N balls of radius $r/2$. We use this somewhat non-standard name to clearly differentiate this property from other types of doubling properties. If there is no specification, the ball B means the ball center x_B with radius r_B. Also, we set $n = \log_2 N$, which can be viewed as (an upper bound for) a geometric dimension of the space.

Upper doubling measures. A metric measure space (X,d,μ) is said to be upper doubling measure if there exists a dominating function λ with the following properties:

(i) $\lambda : X \times (0,\infty) \mapsto (0,\infty)$;
(ii) for $x \in X$, $r \mapsto \lambda(x,r)$ is increasing;
(iii) there exists a constant $C_\lambda > 0$ such that

$$\lambda(x,2r) \leq C_\lambda \lambda(x,r)$$

for all $x \in X$ and $r > 0$;

(iv) and the following inequality holds

$$\mu(x,r) \leq \lambda(x,r)$$

for all $x \in X$ and $r > 0$, where $\mu(x,r) = \mu(B(x,r))$.

(v) $\lambda(x,r) \approx \lambda(y,r)$ for all $r > 0; x, y \in X$ and $d(x,y) \leq r$.

Throughout the paper, we always assume that (X, d, μ) is geometrically doubling regular metric spaces and the measure μ is upper doubling measures.

For $\alpha, \beta > 1$, a ball $B \subset X$ is called (α, β)-doubling if $\mu(\alpha B) \leq \beta\mu(B)$. The following result asserts the existence of a lot of small and big doubling balls.

Lemma 2.1 ([2])**.** *The following statements hold:*

(i) If $\beta > C_\lambda^{\log_2 \alpha}$, then for any ball $B \subset X$ there exists $j \in \mathbb{N}$ such that $\alpha^j B$ is (α, β)-doubling.

(ii) If $\beta > \alpha^n$, here n is doubling order of λ, then for any ball $B \subset X$ there exists $j \in \mathbb{N}$ such that $\alpha^{-j} B$ is (α, β)-doubling.

For any two balls $B \subset Q$, we defined

$$K_{B,Q} = 1 + \int_{r_B \leq d(x,x_B) \leq r_Q} \frac{1}{\lambda(x_B, d(x,x_B))} d\mu(x). \tag{3}$$

We have the following properties.

Lemma 2.2. *(i) If $Q \subset R \subset S$ are balls in X, then*

$$\max\{K_{Q,R}, K_{R,S}\} \leq K_{Q,S} \leq C(K_{Q,R} + K_{R,S}).$$

(ii) If $Q \subset R$ are comparable size, then $K_{Q,R} \leq C$.

(iii) If $\alpha Q, \dots \alpha^{N-1} Q$ are non (α, β)-doubling balls (with $\beta > C_\lambda^{\log_2 \alpha}$) then $K_{Q,\alpha^N Q} \leq C$.

The proof of Lemma 2.2 is not difficult and we omit the details here.

Associated to two balls $B \subset Q$, the coefficient $K'_{B,Q}$ can be defined as follows: let $N_{B,Q}$ be the smallest integer satisfying $6^{N_{B,Q}} r_B \geq r_Q$, then we set

$$K'_{B,Q} := 1 + \sum_{k=1}^{N_{B,Q}} \frac{\mu(6^k B)}{\lambda(x_B, 6^k r_B)}. \tag{4}$$

In general, it is not difficult to show that $K_{B,Q} \leq C K'_{B,Q}$. In the particular case when λ satisfies $\lambda(x, ar) = a^m \lambda(x,r)$ for all $x \in X$ and $a, r > 0$ for some $m > 0$, we have $K_{B,Q} \approx K'_{B,Q}$.

2.1. **Definition of RBMO**(μ)**.** Adapting to definition of RBMO spaces of Tolsa in [9], T. Hytönen introduced the RBMO(μ), see [2].

Definition 2.3. *Fix a parameter* $\rho > 1$. *A function* $f \in L^1_{\mathrm{loc}}(\mu)$ *is said to be in the space* RBMO(μ) *if there exists a number* C, *and for every ball* B, *a number* f_B *such that*

$$\frac{1}{\mu(\rho B)}\int_B |f(x) - f_B| d\mu(x) \leq C \tag{5}$$

and, whenever B, B_1 *are two balls with* $B \subset B_1$, *one has*

$$|f_B - f_{B_1}| \leq C K_{B,B_1}. \tag{6}$$

The infimum of the values C *in (6) is taken to be the* RBMO *norm of* f *and denoted by* $\|f\|_{\mathrm{RBMO}(\mu)}$.

The RBMO norm $\|\cdot\|_{\mathrm{RBMO}(\mu)}$ is independent of $\rho > 1$. Moreover the John-Nirenberg inequality holds for RBMO(μ). Precisely, we have the following result, see Corollary 6.3 in [2].

Proposition 2.4. *For any* $\rho > 1$ *and* $p \in [1, \infty)$, *there exists a constant* C *so that for every* $f \in$ RBMO(μ) *and every ball* B_0,

$$\Big(\frac{1}{\mu(\rho B_0)}\int_{B_0} |f(x) - f_{B_0}|^p d\mu(x)\Big)^{1/p} \leq C\|f\|_{\mathrm{RBMO}(\mu)}.$$

2.2. **Calderón-Zygmund decomposition.** In non-doubling setting, the Calderón-Zygmund decomposition in $\mathbb{R}^n$ was first investigated by [9] and then was generalized to the general case of non-homogeneous spaces (X, μ) by [1].

Proposition 2.5. *(Calderón-Zygmund decomposition) For any* $f \in L^1(\mu)$ *and any* $\lambda > 0$ *(with* $\lambda > \beta_0 ||f||_{L^1(\mu)}/||\mu||$ *if* $||\mu|| < \infty$*) we have:*

(a) *There exists a family of finite disjoint balls* $\{6Q_i\}_i$ *such that the family of balls* $\{Q_i\}_i$ *is pairwise disjoint and*

$$\frac{1}{\mu(6^2 Q_i)}\int_{Q_i} |f| d\mu > \frac{\lambda}{\beta_0}, \tag{7}$$

$$\frac{1}{\mu(\eta^2 Q_i)}\int_{\frac{\eta}{6}Q_i} |f| d\mu \leq \frac{\lambda}{\beta_0}, \text{ for all } \eta > 6, \tag{8}$$

$$|f| \leq \lambda \text{ a.e. } (\mu) \text{ on } \mathbb{R}^d \setminus \bigcup_i 6Q_i. \tag{9}$$

(b) *For each* i, *let* R_i *be a* $(3 \times 6^2, C_\lambda^{\log_2 3\times 6^2+1})$*- doubling ball concentric with* Q_i, *with* $l(R_i) > 6^2 l(Q_i)$ *and we denote* $\omega_i = \frac{\chi_{6Q_i}}{\sum_k \chi_{6Q_k}}$. *Then there exists a family of functions* φ_i *with constant signs and supp* $(\varphi_i) \subset R_i$ *satisfying*

$$\int \varphi_i d\mu = \int_{6Q_i} f\omega_i d\mu, \tag{10}$$

$$\sum_i |\varphi_i| \leq B\lambda, \tag{11}$$

(where B is some constant), and:

$$||\varphi_i||_\infty \mu(R_i) \leq C \int_X |w_i f| d\mu. \tag{12}$$

We will end this section by the following lemma which is useful in the sequel, see [1].

Lemma 2.6. *For any two concentric balls $Q \subset R$ such that there are no (α, β)-doubling balls $\beta > C_\lambda^{\log_2 \alpha}$ of the form $\alpha^k Q, k \in \mathbb{N}$ such that $Q \subset \alpha^k Q \subset R$, we have*

$$\int_{R\setminus Q} \frac{1}{\lambda(x_Q, d(x_Q, x))} d\mu(x) \leq C.$$

3. Boundedness of maximal operator T_* and maximal commutator

3.1. **The weak type of $(1,1)$ of T_*.** In [1], the Cotlar inequality is obtained. More precisely, we have the following result.

Theorem 3.1. *Let T be a L^2 bounded Calderón-Zygmund operator. Then there exist $C > 0$ and $0 < \eta < 1$ such that for any bounded function with bounded support f and $x \in X$ we have*

$$T_* f(x) \leq C\Big(M_{\eta,6}(Tf)(x) + M_{(6)} f(x)\Big).$$

where

$$M_{(\rho)} = \sup_{Q \ni x} \frac{1}{\mu(\rho Q)} \int_Q |f| d\mu$$

and

$$M_{p,\rho} f(x) = \sup_{Q \ni x} \Big(\frac{1}{\mu(\rho Q)} \int_Q |f|^p d\mu\Big)^{1/p}.$$

For the proof we refer the reader to [1, Theorem 6.6].

Therefore, from the boundedness on $L^p(\mu)$ of $M_{(\rho)}$ and $M_{p,\rho}$, the boundedness of T_* on $L^p(\mu)$ follows. The endpoint estimate of T_* will be asserted in the following theorem.

Theorem 3.2. *Let T be a Calderón-Zygmund operator. If T is bounded on $L^2(\mu)$ then the maximal operator T_* is of weak type $(1,1)$.*

Proof. To do this, we will claim that there exists $C > 0$ such that for any $\lambda > 0$ and $f \in L^1(\mu) \cap L^2(\mu)$ we have

$$\mu\{x : |T_*(x)| > \lambda\} \leq \frac{C}{\lambda} \|f\|_{L^1(\mu)}.$$

We can assume that $\lambda > \beta_0 \|f\|_{L^1(\mu)}/\|\mu\|$. Otherwise, there is nothing to prove. We use the same notations as in Proposition 2.5 with R_i which is chosen as the smallest $(3 \times 6^2, C_\lambda^{\log_2 3\times 6^2+1})$- doubling ball of the family $\{3 \times 6^2 Q_i\}_i$. Then we can write $f = g + b$, with

$$g = f\chi_{X\setminus \cup_i 6Q_i} + \sum_i \varphi_i$$

and

$$b := \sum_i b_i = \sum_i (w_i f - \varphi_i).$$

Taking into account (7), one has

$$\mu(\cup_i 6^2 Q_i) \leq \frac{C}{\lambda} \sum_i \int_{Q_i} |f| d\mu \leq \frac{C}{\lambda} \int_X |f| d\mu$$

where in the last inequality we use the pairwise disjoint property of the family $\{Q_i\}_i$.

We need only to show that

$$\mu\{x \in X \backslash \cup_i 6^2 Q_i : |T_* f(x)| > \lambda\} \leq \frac{C}{\lambda} \int_X |f| d\mu.$$

We have

$$\begin{aligned}\mu\{x \in X \backslash \cup_i 6^2 Q_i : |T_* f(x)| > \lambda\} &\leq \mu\{x \in X \backslash \cup_i 6^2 Q_i : |T_* g(x)| > \lambda/2\} \\ &\quad + \mu\{x \in X \backslash \cup_i 6^2 Q_i : |T_* b(x)| > \lambda/2\} \\ &:= I_1 + I_2.\end{aligned}$$

Note that $|g| \leq C\lambda$. Therefore, the first term I_1 is dominated by

$$\frac{C}{\lambda^2} \int |g|^2 d\mu \leq \frac{C}{\lambda} \int |g| d\mu.$$

On the other hand,

$$\begin{aligned}\int |g| d\mu &\leq \int_{X \backslash \cup_i 6 Q_i} |f| d\mu + \sum_i \int_{R_i} |\varphi_i| d\mu \\ &\leq \int_X |f| d\mu + \sum_i \mu(R_i) \|\varphi_i\|_{L^\infty(\mu)} \\ &\leq \int_X |f| d\mu + C \sum_i \int_X |f w_i| d\mu \leq C \int_X |f| d\mu.\end{aligned}$$

Therefore,

$$\mu\{x \in X \backslash \cup_i 6^2 Q_i : |T_* g(x)| > \lambda/2\} \leq \frac{C}{\lambda} \int |f| d\mu.$$

For I_2, we have

$$\begin{aligned}I_2 &\leq \mu\{x \in X \backslash \cup_i 6^2 Q_i : \sum_i \chi_{X \backslash 2R_i} |T_* b_i(x)| > \lambda/6\} \\ &\quad + \mu\{x \in X \backslash \cup_i 6^2 Q_i : \sum_i \chi_{2R_i \backslash 6^2 Q_i} |T_* \varphi_i(x)| > \lambda/6\} \\ &\quad + \mu\{x \in X \backslash \cup_i 6^2 Q_i : \sum_i \chi_{2R_i \backslash 6^2 Q_i} |T_*(w_i f)(x)| > \lambda/6\} \\ &:= K_1 + K_2 + K_3.\end{aligned}$$

It is easy to estimate the terms K_2 and K_3. Indeed, we have

$$\begin{aligned} K_2 &\leq \frac{C}{\lambda}\sum_i \int_{2R_i\backslash 6^2Q_i} |T_*\varphi_i| d\mu \\ &\leq \frac{C}{\lambda}\sum_i \int_{2R_i} |T_*\varphi_i| d\mu \\ &\leq \frac{C}{\lambda}\sum_i \Big(\int_{2R_i} |T_*\varphi_i|^2 d\mu\Big)^{1/2}(\mu(R_i))^{1/2}. \end{aligned}$$

Using the L^2 boundedness of T_*, we get that

$$\begin{aligned} K_2 &\leq \frac{C}{\lambda}\sum_i \Big(\int_{2R_i} |\varphi_i|^2 d\mu\Big)^{1/2}(\mu(R_i))^{1/2} \\ &\leq \frac{C}{\lambda}\sum_i \|\varphi_i\|_{L^\infty(\mu)}\mu(R_i) \\ &\leq \frac{C}{\lambda}\sum_i \int_X |w_i f| d\mu = \frac{C}{\lambda}\int_X |f| d\mu. \end{aligned}$$

and

$$\begin{aligned} K_3 &\leq \frac{C}{\lambda}\sum_i \int_{2R_i\backslash 6^2Q_i} \sup_{\epsilon>0}\Big|\int_{d(x,y)>\epsilon} K(x,y)(w_i f)(y)d\mu(y)\Big| d\mu(x) \\ &\leq \frac{C}{\lambda}\sum_i \int_{2R_i\backslash 6^2Q_i}\int_X |K(x,y)||(w_i f)(y)| d\mu(y)d\mu(x) \\ &\leq \frac{C}{\lambda}\sum_i \int_{2R_i\backslash 6^2Q_i}\int_{6Q_i} \frac{1}{\lambda(y,d(x,y))}|(w_i f)(y)| d\mu(y)d\mu(x) \\ &\leq \frac{C}{\lambda}\sum_i \int_{2R_i\backslash 6^2Q_i}\int_X \frac{1}{\lambda(x_{Q_i},d(x,x_{Q_i}))}|(w_i f)(y)| d\mu(y)d\mu(x) \\ &\leq \frac{C}{\lambda}\sum_i \int_{2R_i\backslash 6^2Q_i} \frac{1}{\lambda(x_{Q_i},d(x,x_{Q_i}))} d\mu(x)\int_X |(w_i f)(y)| d\mu(y) \\ &\leq \frac{C}{\lambda}\sum_i \int_X |(w_i f)(y)| d\mu(y) \quad \text{(due to Lemma 2.6)} \\ &\leq \frac{C}{\lambda}\int_X |f| d\mu. \end{aligned}$$

We now take care of the term K_1. For each i and $x \in X\backslash 2R_i$, we consider three cases:

Case 1. $\epsilon < d(x,R_i)$**:** We have,

$$|T_\epsilon b_i(x)| = \Big|\int_{R_i} K(x,y)b_i(y)d\mu(y)\Big|.$$

Case 2. $\epsilon > d(x,R_i) + 2r_{R_i}$**:** In this situation, it is easy to see that $|T_\epsilon b_i(x)| = 0$.

Case 3. $d(x,R_i) \le \epsilon \le d(x,R_i) + 2r_{R_i}$**:** It can be verified that for $y \in R_i$ we have $d(x,y) \ge d(x,R_i) \ge \frac{1}{3}(d(x,R_i) + 2r_{R_i}) \ge \frac{\epsilon}{3}$. Therefore, one has, by (1)

$$\begin{aligned}|T_\epsilon b_i(x)| &\le \Big|\int_{R_i} K(x,y)b_i(y)d\mu(y)\Big| + \Big|\int_{d(x,y)\le\epsilon} K(x,y)b_i(y)d\mu(y)\Big| \\ &\le \Big|\int_{R_i} K(x,y)b_i(y)d\mu(y)\Big| + \int_{d(x,y)\le\epsilon} \frac{C}{\lambda(x,d(x,y))}|b_i(y)|d\mu(y).\end{aligned}$$

Since $\lambda(x,\cdot)$ is increasing and $d(x,y) \ge \frac{\epsilon}{3}$, we can write

$$\begin{aligned}|T_\epsilon b_i(x)| &\le \Big|\int_{R_i} K(x,y)b_i(y)d\mu(y)\Big| + \int_{B(x,\epsilon)} \frac{C}{\lambda(x,\frac{\epsilon}{3})}|b_i(y)|d\mu(y) \\ &\le \Big|\int_{R_i} K(x,y)b_i(y)d\mu(y)\Big| + \int_{B(x,\epsilon)} \frac{C}{\lambda(x,6\epsilon)}|b_i(y)|d\mu(y) \\ &\le \Big|\int_{R_i} K(x,y)b_i(y)d\mu(y)\Big| + \frac{C}{\mu(x,6\epsilon)}\int_{B(x,\epsilon)} |b_i(y)|d\mu(y)\end{aligned}$$

Hence, in general, we have, for each i and $x \in X\backslash 2R_i$,

$$|T_\epsilon b_i(x)| \le \Big|\int_{R_i} K(x,y)b_i(y)d\mu(y)\Big| + \frac{C}{\mu(x,6\epsilon)}\int_{B(x,\epsilon)} |b_i(y)|d\mu(y).$$

It follows that

$$\begin{aligned}\sum_i \chi_{X\backslash 2R_i}|T_\epsilon b_i(x)| &\le \sum_i \chi_{X\backslash 2R_i}\Big|\int_{R_i} K(x,y)b_i(y)d\mu(y)\Big| \\ &\qquad + \sum_i \frac{C}{\mu(x,6\epsilon)}\int_{B(x,\epsilon)} |b_i(y)|d\mu(y) \\ &\le \sum_i \chi_{X\backslash 2R_i}\Big|\int_{R_i} K(x,y)b_i(y)d\mu(y)\Big| \\ &\qquad + CM_{(6)}(\sum_i |b_i|)(x) \le A_1 + A_2\end{aligned}$$

uniformly in $\epsilon > 0$.

So, we can write

$$\begin{aligned}K_1 &= \mu\{x \in X\backslash \cup_i 6^2Q_i : \sum_i \chi_{X\backslash 2R_i}|T_* b(x)| > \lambda/6\} \\ &\le \mu\{x \in X\backslash \cup_i 6^2Q_i : A_1 > \lambda/12\} + \mu\{x \in X\backslash \cup_i 6^2Q_i : A_2 > \lambda/12\} \\ &\le K_{11} + K_{12}.\end{aligned}$$

For the term K_{11}, using $\int b_i d\mu = 0$ and (2), we have

$$\begin{aligned}
K_{11} &\leq \frac{C}{\lambda}\sum_i \int_{X\backslash 2R_i} \Big|\int_{R_i} K(x,y)b_i(y)d\mu(y)\Big| dx \\
&\leq \frac{C}{\lambda}\sum_i \int_{X\backslash 2R_i} \Big|\int_{R_i} (K(x,y)-K(x,x_{R_i}))b_i(y)d\mu(y)\Big| d\mu(x) \\
&\leq \frac{C}{\lambda}\sum_i \int_{X\backslash 2R_i} \int_{R_i} |(K(x,y)-K(x,x_{R_i}))b_i(y)|d\mu(y)d\mu(x) \\
&\leq \frac{C}{\lambda}\sum_i \int_{X\backslash 2R_i} \int_{R_i} \frac{d(y,x_{R_i})^\delta}{d(x,y)^\delta \lambda(x,d(x,y))}|b_i(y)|d\mu(y)d\mu(x) \\
&\leq \frac{C}{\lambda}\sum_i \int_{X\backslash 2R_i} \int_{R_i} \frac{r_{R_i}^\delta}{d(x,x_{R_i})^\delta \lambda(x,d(x,x_{R_i}))}|b_i(y)|d\mu(y)d\mu(x) \\
&\leq \frac{C}{\lambda}\sum_i \int_{X\backslash 2R_i} \frac{r_{R_i}^\delta}{d(x,x_{R_i})^\delta \lambda(x,d(x,x_{R_i}))}d\mu(x) \int_{R_i} |b_i(y)|d\mu(y).
\end{aligned}$$

By decomposing $X\backslash 2R_i$ into the annuli associated to the ball R_i, we can show that

$$\int_{X\backslash 2R_i} \frac{r_{R_i}^\delta}{d(x,x_{R_i})^\delta \lambda(x,d(x,x_{R_i}))}d\mu(x) \leq C$$

for all i.

Therefore, we can dominate the term K_{11} by

$$\begin{aligned}
K_{11} &\leq \frac{C}{\lambda}\sum_i \int_{R_i} |b_i(y)|d\mu(y) \\
&\leq \frac{C}{\lambda}\sum_i \int_{R_i} |\varphi_i|d\mu(y) + \frac{C}{\lambda}\sum_i \int_X |w_i f|d\mu(y) \\
&\leq \frac{C}{\lambda}\sum_i \int_X |w_i f|d\mu(y) \leq \frac{C}{\lambda}\int_X |f|d\mu.
\end{aligned}$$

We now proceed with K_{12}. Since $M_{(6)}(\cdot)$ is of type weak $(1,1)$, we have

$$\begin{aligned}
K_{12} &\leq \frac{C}{\lambda}\sum_i \int_X |b_i|d\mu \\
&\leq \frac{C}{\lambda}\sum_i \Big(\int_X |\varphi_i|d\mu + \int_X |w_i f|d\mu\Big) \\
&\leq \frac{C}{\lambda}\sum_i \int_X |w_i f|d\mu \leq \frac{C}{\lambda}\int_X |f|d\mu.
\end{aligned}$$

This completes our proof. □

3.2. **Boundedness of the maximal commutators.** In this section we restrict ourself to consider the spaces (X,μ) in which $\lambda(x,ar) = a^m\lambda(x,r)$ for all $x \in X$ and $a, r > 0$ for some m. Recall that in such spaces (X,μ), $K_{B,Q} \approx K'_{B,Q}$ for all balls $B \subset Q$.

For $b \in \mathrm{RBMO}(\mu)$, we defined the maximal commutator $T_{*,b}$ by

$$T_{*,b}f(x) = \max_{\epsilon>0} \Big| T_{\epsilon,b}f(x) \Big| = \max_{\epsilon>0} \Big| \int_{d(x,y)>\epsilon} (b(x)-b(y))K(x,y)f(y)d\mu(y) \Big|.$$

As mentioned earlier, one problem in studying the boundedness of the maximal commutators is that the kernel of T_* may not be a Calderón-Zygmund kernel. This causes certain difficulties in estimating maximal commutators $T_{*,b}$. To overcome this problem, we will exploit the ideas in [8].

Let ϕ and ψ be C^∞ non-negative functions such that $\phi'(t) \leq \frac{C}{t}, \psi'(t) \leq \frac{C}{t}$ and $\chi_{[2,\infty)} \leq \phi \leq \chi_{[1,\infty)}, \chi_{[0,1/2)} \leq \psi \leq \chi_{[0,3)}$. Associated to ϕ, ψ and T, we introduced the maximal operators:

$$T_*^\phi f(x) = \sup_{\epsilon>0} \Big| T_\epsilon^\phi f(x) \Big| = \sup_{\epsilon>0} \Big| \int_X K(x,y)\phi\Big(\frac{d(x,y)}{\epsilon}\Big) f(y)d\mu(y) \Big|$$

and

$$T_*^\psi f(x) = \sup_{\epsilon>0} \Big| T_\epsilon^\psi f(x) \Big| = \sup_{\epsilon>0} \Big| \int_X K(x,y)\psi\Big(\frac{d(x,y)}{\epsilon}\Big) f(y)d\mu(y) \Big|.$$

It is not difficult to show that

$$\max\{T_\epsilon^\phi f(x), T_\epsilon^\psi f(x)\} \leq T_* f(x) + CM_{(5)}f(x).$$

Hence T_*^ϕ and T_*^ψ are bounded on $L^p(\mu), 1 < p < \infty$.

Define the maximal commutators associated with T_ϵ^ϕ and T_ϵ^ψ by setting

$$\begin{aligned} T_{*,b}^\phi f(x) &= \max_{\epsilon>0} \Big| T_{\epsilon,b}^\phi f(x) \Big| \\ &= \max_{\epsilon>0} \Big| \int_X (b(x)-b(y))K(x,y)\phi\Big(\frac{d(x,y)}{\epsilon}\Big) f(y)d\mu(y) \Big| \end{aligned}$$

and

$$\begin{aligned} T_{*,b}^\psi f(x) &= \max_{\epsilon>0} \Big| T_{\epsilon,b}^\psi f(x) \Big| \\ &= \max_{\epsilon>0} \Big| \int_X (b(x)-b(y))K(x,y)\psi\Big(\frac{d(x,y)}{\epsilon}\Big) f(y)d\mu(y) \Big| \end{aligned}$$

It is not hard to show that

$$T_{*,b}f \leq T_{*,b}^\phi f + T_{*,b}^\psi f. \tag{13}$$

We are now in position to establish the boundedness of the maximal commutator $T_{*,b}$.

Theorem 3.3. *Let T be a Calderón-Zygmund operator. If T is bounded on $L^2(\mu)$ then the maximal commutator $T_{*,b}$ is bounded on $L^p(\mu)$ for all $1 < p < \infty$. More precisely, there exists a constant $C > 0$ such that*

$$\|T_{*,b}f\|_{L^p(\mu)} \leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^p(\mu)}$$

for all $f \in L^p(\mu)$.

Proof. We will show that there exists a constant $C > 0$ such that

$$\|T_{*,b}f\|_{L^p(\mu)} \leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^p(\mu)}$$

for all $f \in L^p(\mu)$.
From (13), we need only to show that for $p > 1$, we have

$$\|T^{\phi}_{*,b}f\|_{L^p(\mu)} \leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^p(\mu)} \tag{14}$$

and

$$\|T^{\psi}_{*,b}f\|_{L^p(\mu)} \leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^p(\mu)}. \tag{15}$$

The proofs of (14) and (15) are completely analogous. So, we only deal with (14).

For each ball $B \subset X$, we denote

$$h_B := -m_B(T^{\phi}_*((b - b_B)f\chi_{X\setminus \frac{6}{5}B}).$$

As in the proof of [9, Thorem 9.1] (see also [1, Theorem 5.9]), it suffices to claim that for all balls $x \in Q \subset R$

$$\frac{1}{\mu(6Q)}\int_Q |T^{\phi}_{*,b}f - h_Q|d\mu \leq C\|b\|_{\mathrm{RBMO}}(M_{p,5}f(x) + M_{p,6}T^{\phi}_*f(x)) \tag{16}$$

for all x and B with $x \in B$, and

$$|h_Q - h_R| \leq C\|b\|_{\mathrm{RBMO}}(M_{p,5}f(x) + T^{\phi}_*f(x))K^2_{Q,R}. \tag{17}$$

To estimate (16), we write

$$\begin{aligned}|T^{\phi}_{*,b}f - h_Q| &= |(b - b_Q)T^{\phi}_*f - T^{\phi}_*((b - b_Q)f) - h_Q|\\ &\leq |(b - b_Q)T^{\phi}_*f| + |T^{\phi}_*((b - b_Q)f_1)| + |T^{\phi}_*((b - b_Q)f_2) + h_Q|\end{aligned}$$

where $f_1 = f\chi_{\frac{6}{5}Q}$ and $f_2 = f - f_1$. For the first term, by Hölder inequality, we have

$$\begin{aligned}\frac{1}{\mu(6Q)}\int_Q |(b - b_Q)T^{\phi}_*f|d\mu &\leq \Big(\frac{1}{\mu(6Q)}\int_Q |(b - b_Q)|^{p'}d\mu\Big)^{1/p'}\\ &\quad\times \Big(\frac{1}{\mu(6Q)}\int_Q |T^{\phi}_*f|^p d\mu\Big)^{1/p}\\ &\leq C\|b\|_{\mathrm{RBMO}(\mu)}M_{(6)}T^{\phi}_*f(x).\end{aligned}$$

For the second term, by Hölder inequality and the uniform boundedness of T^{ϕ}_* on $L^p(\mu)$, we have

$$\frac{1}{\mu(6Q)}\int_Q |T^{\phi}_*((b - b_Q)f_1)|d\mu \leq C\|b\|_{\mathrm{RBMO}(\mu)}M_{p,5}f(x).$$

Let us take care of the third term. For $x, y \in Q$ and $\epsilon > 0$, we write

$$\begin{aligned}
&|T_\epsilon^\phi((b-b_Q)f_2)(x) - T_\epsilon^\phi((b-b_Q)f_2)(y)| \\
&= \Big| \int_{X\setminus\frac{6}{5}Q} (K(x,z)\phi\Big(\frac{d(x,z)}{\epsilon}\Big) - K(y,z)\phi\Big(\frac{d(y,z)}{\epsilon}\Big))(b(z)-b_Q)f(z)d\mu(z)\Big| \\
&\leq \Big| \int_{X\setminus\frac{6}{5}Q} (K(x,z) - K(y,z))\phi\Big(\frac{d(x,z)}{\epsilon}\Big)(b(z)-b_Q)f(z)d\mu(z)\Big| \\
&+ \ \Big| \int_{X\setminus\frac{6}{5}Q} K(y,z)\Big(\phi\Big(\frac{d(y,z)}{\epsilon}\Big) - \phi\Big(\frac{d(x,z)}{\epsilon}\Big)\Big)(b(z)-b_Q)f(z)d\mu(z)\Big| \\
&\leq A_1 + A_2.
\end{aligned}$$

For the term A_1, by (2), we have

(18)

$$\begin{aligned}
A_1 &\leq \int_{X\setminus\frac{6}{5}Q} |K(x,z)-K(y,z)||(b(z)-b_Q)f(z)|d\mu(z) \\
&\leq C\int_{X\setminus\frac{6}{5}Q} \frac{d(x,y)^\delta}{d(x,z)^\delta\lambda(x,d(x,y))}|(b(z)-b_Q)f(z)|d\mu(z) \\
&\leq C\sum_{k=0}^\infty \int_{6^{k+1}Q\setminus 6^kQ} \frac{d(x,y)^\delta}{d(x,z)^\delta\lambda(x,d(x,y))}|(b(z)-b_Q)f(z)|d\mu(z) \\
&\leq C\sum_{k=0}^\infty 6^{-k\delta}\int_{6^{k+1}Q} \frac{1}{\lambda(x_Q,6^kr_Q)}|(b(z)-b_Q)f(z)|d\mu(z) \\
&\leq C\sum_{k=0}^\infty 6^{-k\delta}\int_{6^{k+1}Q} \frac{1}{\lambda(x_Q,6^kr_Q)}|(b(z)-b_Q)f(z)|d\mu(z) \\
&\leq C\sum_{k=0}^\infty 6^{-k\delta}\frac{1}{\mu(5\times 6^kQ)}\int_{6^{k+1}Q} |(b(z)-b_{6^{k+1}Q})f(z)|d\mu(z) \\
&\quad + C\sum_{k=0}^\infty 6^{-k\delta}\frac{1}{\mu(5\times 6^kQ)}\int_{6^{k+1}Q} |(b_{6^{k+1}Q}-b_Q)f(z)|d\mu(z) \\
&\leq C\sum_{k=0}^\infty 6^{-k\delta}\|b\|_{\mathrm{RBMO}(\mu)}M_{(5)}f(x) + C\sum_{k=0}^\infty (k+1)6^{-k\delta}\|b\|_{\mathrm{RBMO}(\mu)}Mf(x) \\
&= C\|b\|_{\mathrm{RBMO}(\mu)}M_{(5)}f(x).
\end{aligned}$$

Since $\phi'(t) \leq \frac{C}{t}$, for $z \in 6^{k+1}\frac{6}{5}Q\setminus 6^k\frac{6}{5}Q$ and $x, y \in Q$,

$$\phi\Big(\frac{d(y,z)}{\epsilon}\Big) - \phi\Big(\frac{d(x,z)}{\epsilon}\Big) \leq C\frac{d(x,y)}{d(z,x_Q)} \leq C6^{-(k+1)}.$$

From this estimate, we obtain that

$$A_2 \le \sum_{k=0}^{\infty} \int_{6^{k+1}\frac{6}{5}Q\backslash 6^k\frac{6}{5}Q} \Big|K(y,z)\Big(\phi\Big(\frac{d(y,z)}{\epsilon}\Big) - \phi\Big(\frac{d(x,z)}{\epsilon}\Big)\Big)\Big| \times |(b(z)-b_Q)f(z)|d\mu(z)$$

$$\le C\sum_{k=0}^{\infty} 6^{-k} \int_{6^{k+1}\frac{6}{5}Q\backslash 6^k\frac{6}{5}Q} \frac{1}{\lambda(y,d(y,z))}(b(z)-b_Q)f(z)|d\mu(z)$$

$$\le C\sum_{k=0}^{\infty} 6^{-k} \int_{6^{k+1}\frac{6}{5}Q\backslash 6^k\frac{6}{5}Q} \frac{1}{\lambda(x_Q,6^k r_Q)}(b(z)-b_Q)f(z)|d\mu(z).$$

At this stage, repeating the argument as in (18), we also obtain that $A_2 \le C\|b\|_{\mathrm{RBMO}(\mu)}M_{(5)}f(x)$. This together with (18) gives for all $x, y \in Q$

$$|T^\phi_\epsilon((b-b_Q)f_2)(x) - T^\phi_\epsilon((b-b_Q)f_2)(y)| \le C\|b\|_{\mathrm{RBMO}(\mu)}M_{p,5}f(x)$$

uniformly in ϵ. Taking the mean value inequality above over the ball Q with respect to y, we have

$$\frac{1}{\mu(6Q)}\int_Q |T^\phi_*((b-b_Q)f_2)+h_Q|d\mu \le C\|b\|_{\mathrm{RBMO}(\mu)}M_{(5)}f(x).$$

for all $\epsilon > 0$. Therefore, the proof of (16) is complete.

It remains to check (17). For two balls $Q \subset R$, let N be an integer number such that $(N-1)$ is the smallest number satisfying $r_R \le 6^{N-1}r_Q$. Then, we break the term $|h_Q - h_R|$ into five terms:

$$\begin{aligned}
&|m_Q(T^\phi_*((b-b_Q)f\chi_{X\backslash\frac{6}{5}Q}) - m_R(T^\phi_*((b-b_R)f\chi_{X\backslash\frac{6}{5}R})| \\
&\le |m_Q(T^\phi_*((b-b_Q)f\chi_{6Q\backslash\frac{6}{5}Q})| + |m_Q(T^\phi_*((b_Q-b_R)f\chi_{X\backslash 6Q})| \\
&+ |m_Q(T^\phi_*((b-b_R)f\chi_{6^NQ\backslash 6Q})| \\
&+ |m_Q(T^\phi_*((b-b_R)f\chi_{X\backslash 6^NQ}) - m_R(T^\phi_*((b-b_R)f\chi_{X\backslash 6^NQ})| \\
&+ |m_R(T^\phi_*((b-b_R)f\chi_{6^NQ\backslash\frac{6}{5}R}) \\
&= M_1 + M_2 + M_3 + M_4 + M_5.
\end{aligned}$$

Let us estimate M_1 first. For $y \in Q$ we have, by Proposition 3.2

$$\begin{aligned}
&|T^\phi_*((b-b_Q)f\chi_{6Q\backslash\frac{6}{5}Q})(x)| \\
&\le \frac{C}{\lambda(x,r_Q)}\int_{6Q}|b-b_Q||f|d\mu \\
&\le \frac{\mu(30Q)}{\lambda(x,30r_Q)}\Big(\frac{1}{\mu(5\times 6Q)}\int_{6Q}|b-b_Q|^{p'}d\mu\Big)^{1/p'} \times\Big(\frac{1}{\mu(5\times 6Q)}\int_{6Q}|f|^p d\mu\Big)^{1/p} \\
&\le C\|b\|_{\mathrm{RBMO}}M_{p,5}f(x).
\end{aligned}$$

Likewise, $M_5 \le \|b\|_{\mathrm{RBMO}}M_{p,5}f(x)$. Hence, we have

$$M_1 + M_5 \le C\|b\|_{\mathrm{RBMO}}M_{p,5}f(x).$$

For the term M_2, it is verified that for $x, y \in Q$

$$|T_*^\phi f\chi_{X\setminus 6Q}(y)| \leq T_*^\phi f(x) + CM_{p,5}f(x).$$

This implies

$$|m_Q(T_*^\phi((b_Q - b_R)f\chi_{X\setminus 6Q})| \leq CK_{Q,R}(T_*^\phi f(x) + M_{p,5}f(x)).$$

As in estimates A_1 and A_2, one gets that

$$M_4 \leq C\|b\|_{\mathrm{RBMO}} M_{p,5}f(x).$$

For the last term M_3, we have, for $y \in Q$,

$$|T_\epsilon^\phi((b - b_R)f\chi_{6^N Q\setminus 6Q}(y)| \leq C\sum_{k=1}^{N-1}\frac{1}{\lambda(y, 6^k r_Q)}\int_{6^{k+1}Q\setminus 6^k Q}|b - b_R||f|d\mu. \tag{19}$$

Since $|b - b_R| \leq |b - b_{6^{k+1}Q}| + |b_R - b_{6^{k+1}Q}|$, further going we have

$$\begin{aligned}
&|T_\epsilon^\phi((b - b_R)f\chi_{6^N Q\setminus 6Q}(y)| \\
&\leq C\sum_{k=1}^{N-1}\frac{1}{\lambda(y, 6^k r_Q)}\Big[\int\limits_{6^{k+1}Q\setminus 6^k Q}|b - b_{6^{k+1}Q}||f|d\mu \\
&\qquad + \int\limits_{6^{k+1}Q\setminus 6^k Q}|b_R - b_{6^{k+1}Q}||f|d\mu\Big] \\
&\leq C\sum_{k=1}^{N-1}\frac{\mu(5\times 6^{k+1}Q)}{\lambda(x_Q, 6^k r_Q)}\Big[\frac{1}{\mu(6^{k+2}Q)}\int\limits_{6^{k+1}Q\setminus 6^k Q}|b - b_{6^{k+1}Q}||f|d\mu \\
&\qquad + \frac{1}{\mu(5\times 6^{k+1}Q)}\int\limits_{6^{k+1}Q\setminus 6^k Q}|b_R - b_{6^{k+1}Q}||f|d\mu\Big]
\end{aligned} \tag{20}$$

By Hölder inequality and the similar argument in estimate the term M_4 we have

$$\frac{1}{\mu(5\times 6^{k+2}Q)}\int\limits_{6^{k+1}Q\setminus 6^k Q}|b - b_{6^{k+1}Q}||f|d\mu \leq \|b\|_{\mathrm{RBMO}} M_{p,5}f(x)$$

and

$$\frac{1}{\mu(5\times 6^{k+1}Q)}\int\limits_{6^{k+1}Q\setminus 6^k Q}|b_R - b_{6^{k+1}Q}||f|d\mu \leq CK_{Q,R}\|b\|_{\mathrm{RBMO}} M_{p,5}f(x).$$

These two above estimates together with (19) give

$$|T_\epsilon^\phi((b - b_R)f\chi_{6^N Q\setminus 6Q}(y)| \leq CK_{Q,R}^2\|b\|_{\mathrm{RBMO}} M_{p,5}f(x)$$

uniformly in $\epsilon > 0$.

It follows that $M_3 \leq CK_{Q,R}^2\|b\|_{\mathrm{RBMO}} M_{p,5}f(x)$. From the estimates of M_1, M_2, M_3, M_4 and M_5, (17) follows. This completes our proof. □

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