10. RAYLEIGH-SCHRÖDINGER SERIES

Let λ_0 be a simple eigenvalue of $T_0 \in BL(X)$ and φ_0 be a corresponding eigenvector. For $V_0 \in BL(X)$, consider the family of operators $T(t) = T_0 + tV_0$, $t \in \mathbb{C}$. For suitable values of t, we develop an iterative procedure of obtaining an eigenvalue $\lambda(t)$ of $T(t)$, and a corresponding eigenvector $\varphi(t)$ starting with the initial terms λ_0 and φ_0. We give conditions on t for which this procedure is guaranteed to converge. We also discuss the question of the simplicity of $\lambda(t)$, and of its isolation from the rest of $\sigma(T(t))$. The theory of linear perturbation developed in the last section will be heavily relied on.

Since λ_0 is a *simple* eigenvalue of T_0 with a corresponding eigenvector φ_0, it follows from Theorem 8.3 that there is an eigenvector φ_0^* of T_0^* corresponding to the eigenvalue $\bar{\lambda}_0$ such that $\langle\varphi_0,\varphi_0^*\rangle = 1$, and that the spectral projection P_0 associated with T_0 and λ_0 is given by

(10.1) $$P_0 x = \langle x,\varphi_0^*\rangle\varphi_0 \ , \ x \in X .$$

The reduced resolvent S_0 associated with T_0 and λ_0 satisfies

(10.2) $$S_0 = \lim_{z\to\lambda_0} R_0(z)(I-P_0) .$$

Let Γ be a curve in $\rho(T_0)$ which isolates λ_0 from the rest of $\sigma(T_0)$. Then Corollary 9.9 shows that for all t in the disk

(10.3) $$\partial_\Gamma = \{t \in \mathbb{C} : |t| < 1/\max_{z\in\Gamma} r_\sigma(V_0R_0(z))\} ,$$

the operator $T(t)$ has only one spectral value $\lambda(t)$ inside Γ, it is a simple eigenvalue of $T(t)$, and $t \mapsto \lambda(t)$ is an analytic function on ∂_Γ . Let the Taylor expansion of $\lambda(t)$ around 0 be given by

$$(10.4) \qquad \lambda(t) = \lambda_0 + \sum_{k=1}^{\infty} \lambda_{(k)} t^k , \quad t \in \partial_\Gamma .$$

Also, for all t with $|t|$ sufficiently small,

$$(10.5) \qquad \varphi(t) = \frac{P(t)\varphi_0}{\langle P(t)\varphi_0, \varphi_0^* \rangle}$$

is an eigenvector of $T(t)$ corresponding to $\lambda(t)$ and it satisfies

$$(10.6) \qquad \langle \varphi(t), \varphi_0^* \rangle = 1 ,$$

where $P(t)$ is the spectral projection associated with $T(t)$ and Γ . Since $\varphi(t)$ is analytic on a neighbourhood of 0 , we can consider its Taylor expansion around 0 :

$$(10.7) \qquad \varphi(t) = \varphi_0 + \sum_{k=1}^{\infty} \varphi_{(k)} t^k , \quad t \text{ near } 0 .$$

(Since $P(0)\varphi_0 = P_0\varphi_0 = \varphi_0$ and $\langle \varphi_0, \varphi_0^* \rangle = 1$, we have $\varphi(0) = \varphi_0$.)

The series (10.4) and (10.7) are known as the <u>Rayleigh-Schrödinger series for</u> $T(t)$ <u>with initial terms</u> λ_0 <u>and</u> φ_0 , <u>respectively</u>.

We remark that instead of considering an eigenvector φ_0 of T_0 corresponding to λ_0 and the eigenvector φ_0^* of T_0^* corresponding to $\bar{\lambda}_0$ which satisfies $\langle \varphi_0, \varphi_0^* \rangle = 1$, we can consider any $x_0 \in X$, $x_0^* \in X^*$ with $\langle P_0 x_0, x_0^* \rangle \neq 0$ and the Taylor expansion of the analytic function

$$(10.8) \qquad x(t) = \frac{P(t)x_0}{\langle P(t)x_0, x_0^* \rangle}$$

in a neighbourhood 0 . Although this flexibility in the choice of

$x_0 \in X$ and $x_0^* \in X^*$ can be useful, we restrict ourselves to the case $x_0 = \varphi_0$ and $x_0^* = \varphi_0^*$, and leave the general case to Problem 10.1.

PROPOSITION 10.1 The coefficients in the Rayleigh-Schrödinger series (10.4) and (10.7) are iteratively given by

$$\lambda_{(k)} = \langle V_0\varphi_{(k-1)}, \varphi_0^*\rangle ,$$

$$(10.9) \qquad \begin{aligned} \varphi_{(k)} &= S_0[-V_0\varphi_{(k-1)} + \sum_{i=1}^{k} \lambda_{(i)}\varphi_{(k-i)}] \\ &= S_0(\lambda_{(1)}I-V_0)\varphi_{(k-1)} + \sum_{i=2}^{k-1} \lambda_{(i)}S_0\varphi_{(k-i)} , \end{aligned}$$

for $k = 1,2,\ldots,$ where $\varphi_{(0)} = \varphi_0$.

In case X is a Hilbert space, T_0 and V_0 are self-adjoint, and $\|\varphi_0\| = 1$, then

$$\lambda_{(1)} = \langle V_0\varphi_0, \varphi_0\rangle ,$$

$$\lambda_{(2)} = \langle V_0\varphi_{(1)}, \varphi_0\rangle ,$$

$$(10.10) \qquad \lambda_{(2k+1)} = \langle V_0\varphi_{(k)}, \varphi_{(k)}\rangle - \sum_{i=1}^{k}\sum_{j=1}^{k} \lambda_{(2k+1-i-j)}\langle\varphi_{(i)}, \varphi_{(j)}\rangle , \quad k = 1,2,\ldots .$$

$$\lambda_{(2k)} = \langle V_0\varphi_{(k-1)}, \varphi_{(k)}\rangle - \sum_{i=1}^{k-1}\sum_{j=1}^{k} \lambda_{(2k-i-j)}\langle\varphi_{(i)}, \varphi_{(j)}\rangle , \quad k = 2,3,\ldots .$$

Further, each $\lambda_{(k)}$ is a real number.

Proof Since for all t near 0, $\lambda(t)$ is an eigenvalue of $T(t)$ and $\varphi(t)$ is a corresponding eigenvector, we have

$$T(t)\varphi(t) = \lambda(t)\varphi(t) , \quad \text{i.e.,}$$

$$(T_0+tV_0)\left[\sum_{k=0}^{\infty} \varphi_{(k)}t^k\right] = \left[\sum_{k=0}^{\infty} \lambda_{(k)}t^k\right]\left[\sum_{k=0}^{\infty} \varphi_{(k)}t^k\right] .$$

with $\varphi_{(0)} = \varphi_0$ and $\lambda_{(0)} = \lambda_0$, by (10.4) and (10.7). Since T_0 and V_0 are continuous operators, we have

$$\sum_{k=0}^{\infty} T_0\varphi_{(k)}t^k = -\sum_{k=0}^{\infty} V_0\varphi_{(k)}t^{k+1} + \sum_{k=0}^{\infty} \left[\sum_{i=0}^{k} \lambda_{(i)}\varphi_{(k-i)}\right]t^k .$$

Let us compare the coefficients of t^k on both sides. For $k = 0$, we simply get

$$T_0\varphi_0 = \lambda_0\varphi_0 .$$

This is the known fact that λ_0 and φ_0 are eigenelements of T_0. For $k = 1,2,\dots$, we have

$$(10.11) \qquad (T_0-\lambda_0 I)\varphi_{(k)} = -V_0\varphi_{(k-1)} + \sum_{i=1}^{k} \lambda_{(i)}\varphi_{(k-i)} .$$

Now, by (10.6), we see that

$$1 = \langle\varphi(t),\varphi_0^*\rangle = 1 + \sum_{k=1}^{\infty} \langle\varphi_{(k)},\varphi_0^*\rangle t^k$$

for all t near 0. Hence

$$(10.12) \qquad \langle\varphi_{(k)},\varphi_0^*\rangle = 0 , \quad k = 1,2,\dots .$$

Taking scalar products with φ_0^* on both sides of (10.11) and using (10.12), we obtain

$$-\langle V_0\varphi_{(k-1)},\varphi_0^*\rangle + \lambda_{(k)} = \langle(T_0-\lambda_0 I)\varphi_{(k)},\varphi_0^*\rangle = \langle\varphi_{(k)},(T_0^*-\bar\lambda_0 I)\varphi_0^*\rangle = 0 ,$$

since φ_0^* is an eigenvector of T_0^* corresponding to $\bar\lambda_0$. Thus,

$$\lambda_{(k)} = \langle V_0\varphi_{(k-1)},\varphi_0^*\rangle , \; k = 1,2,\dots .$$

Next, applying S_0 on both sides of (10.11), and noting that

$$S_0(T_0-\lambda_0 I)\varphi_{(k)} = (I-P_0)\varphi_{(k)} = \varphi_{(k)} - \langle\varphi_{(k)},\varphi_0^*\rangle\varphi_0 = \varphi_{(k)} ,$$

we have

$$\varphi_{(k)} = S_0\left[-V_0\varphi_{(k-1)} + \sum_{i=1}^{k} \lambda_{(i)}\varphi_{(k-i)}\right] .$$

This proves (10.9), if we note that $S_0\varphi_0 = S_0P_0\varphi_0 = 0$.

Let, now, T_0 and V_0 be self-adjoint operators on a Hilbert space X , and $\|\varphi_0\| = 1$. Then $\varphi_0^* = \varphi_0$. We claim that for $k = 3,4,\ldots$ and $m = 1,\ldots,k-2$,

$$\lambda_{(k)} = \langle V_0\varphi_{(k-m-1)},\varphi_{(m)}\rangle - \sum_{i=1}^{k-m-1}\sum_{j=1}^{m} \lambda_{(k-i-j)}\langle\varphi_{(i)},\varphi_{(j)}\rangle .$$

This relation can be proved for each fixed k by induction on m if we use (10.9) and the self-adjointness of T_0 , S_0 and V_0 . The proof simply consists of a long calculation and we omit it. (Cf. [S], Problem 14 on p.296.) Changing k to $2k+1$ and to $2k$, and putting $m = k$, we obtain (10.10).

Since T_0 is self-adjoint, its eigenvalue λ_0 is real. Let a circle Γ with centre λ_0 separate λ_0 from the rest of $\sigma(T_0)$. Then by Corollary 9.9, $\lambda(t) = \lambda_0 + \sum_{k=0}^{\infty} \lambda_{(k)}t^k$ is the only spectral value of $T(t) = T_0 + tV_0$ inside Γ for all $t \in \partial_\Gamma$. Since $\lambda(t)$ is a simple eigenvalue of $T(t)$, and the conjugate curve $\bar{\Gamma}$ coincides with Γ , it follows by Corollary 8.2(c) that $\overline{\lambda(t)} = \bar{\lambda}_0 + \sum_{k=1}^{\infty} \overline{\lambda_{(k)}}\,\bar{t}^k$ is the only spectral value of $[T(t)]^* = T_0 + \bar{t}V_0$ inside Γ for all $t \in \partial_\Gamma$. But $\lambda(\bar{t}) = \lambda_0 + \sum_{k=1}^{\infty} \lambda_{(k)}\bar{t}^k$ is the only spectral value of $T(\bar{t}) = T_0 + \bar{t}V_0$ inside Γ for all $t \in \partial_\Gamma$. Thus, $\overline{\lambda(t)} = \lambda(\bar{t})$ for all $t \in \partial_\Gamma$. This shows that $\bar{\lambda}_{(k)} = \lambda_{(k)}$ for all k , i.e., $\lambda_{(k)}$ is real. //

We note that the coefficients in the two Rayleigh-Schrödinger series with initial terms λ_0 and φ_0 can be calculated iteratively in the following order:

$$\lambda_{(1)}, \varphi_{(1)}; \lambda_{(2)}, \varphi_{(2)}; \lambda_{(3)}, \varphi_{(3)}; \dots .$$

In case T_0 and V_0 are self-adjoint, then we can, in fact, find

$$\lambda_{(1)}, \varphi_{(1)}; \lambda_{(2)}, \lambda_{(3)}, \varphi_{(2)}; \lambda_{(4)}, \lambda_{(5)}, \varphi_{(3)}; \dots$$

in this order.

The calculation of the $\lambda_{(k)}$'s involves only the scalar products, while the calculation of the $\varphi_{(k)}$'s involves finding $x = S_0\eta$, where $\eta \in X$ is such that $P_0\eta = 0$. Since $P_0S_0 = 0$ and $S_0|_{(I-P_0)X}$ is the inverse of $(T_0-\lambda_0 I)|_{(I-P_0)(X)}$, we see that x is the unique element of X such that

$$(T_0-\lambda_0 I)x = \eta , \quad P_0x = 0 .$$

For $t \in \mathbb{C}$, and $j = 1,2,\dots,$ let

$$\lambda_j(t) = \lambda_0 + \sum_{k=1}^{j} \lambda_{(k)} t^k ,$$

$$\varphi_j(t) = \varphi_0 + \sum_{k=1}^{j} \varphi_{(k)} t^k ,$$

where $\lambda_{(k)}$ and $\varphi_{(k)}$ are given by (10.9). Thus, $\lambda_j(t)$ and $\varphi_j(t)$ can be calculated in an iterative manner, and for $|t|$ sufficiently small, they converge to eigenelements $\lambda(t)$ and $\varphi(t)$ of $T(t)$ respectively, as $j \to \infty$.

It is of particular interest to know specific values of the parameter t for which $\lambda_j(t)$ and $\varphi_j(t)$ will approximate eigenelements of $T(t)$; a larger absolute value of such t implies the possibility of allowing bigger perturbations. We note that the

Rayleigh-Schrödinger series (10.4) for $\lambda(t)$ converges, i.e., $\lambda_j(t) \to \lambda(t)$, for all $t \in \partial_\Gamma$. It will then be advisable to choose a suitable curve Γ around λ_0 so that ∂_Γ is as large as possible; it would also be helpful if we know some lower bounds for the radius of such ∂_Γ. The Rayleigh-Schrödinger series (10.7) for $\varphi(t)$, however, is known to converge only in some neighbourhood of 0. This is because the denominator $\langle P(t)\varphi_0, \varphi_0^*\rangle$ of $\varphi(t)$ may have a zero at some $t_0 \in \partial_\Gamma$, and then, unless the numerator $P(t)\varphi_0$ also has a zero of the same order at t_0, we will have a pole of $\varphi(t)$ at t_0. Thus, it is useful to know the values of $t \in \partial_\Gamma$ for which the denominator does not vanish, and more generally, even if it vanishes, does not cause a singularity of $\varphi(t)$.

Before we address ourselves to the above questions, we remark that if r is the radius of convergence of the series (10.7) for $\varphi(t)$, then for every t with $|t| < r$, the series (10.7) and hence the series (10.4) (since $\lambda_{(k)} = \langle V_0\varphi_{(k-1)}, \varphi_0^*\rangle$) converge to, say, $\Phi(t)$ and $\Lambda(t)$. Then we must have

$$T(t)\Phi(t) = \Lambda(t)\Phi(t)\ , \quad \langle\Phi(t), \varphi_0^*\rangle = 1\ , \quad \text{for all } |t| < r\ .$$

This is because the analytic functions

$$f(t) = T(t)\Phi(t) - \Lambda(t)\Phi(t) \quad \text{and} \quad g(t) = \langle\Phi(t), \varphi_0^*\rangle$$

are equal to 0 and 1, respectively, on a neighbourhood of 0, and hence must equal 0 and 1, respectively, in any domain in which they are analytic and which contains 0. This is an immediate consequence of the identity theorem. (See Problem 4.2.) The above argument also shows that if both the functions $\lambda(t)$ and $\varphi(t)$ have analytic continuations to a domain D, which may be larger than the disk of convergence of (10.7), then the continuations represent eigenelements of

$T(t)$, and the scalar product of the eigenvector with φ_0^* is equal to 1 . This is often possible if one knows the singularities of the limit function on the circle of convergence.

By repeatedly shifting the origin to points where $\lambda(t)$ and $\varphi(t)$ are analytic, we obtain Taylor expansions for $\Lambda(t)$ and $\Phi(t)$ such as

$$\Lambda(t) = \sum_{k=0}^{\infty} \Lambda_{(k)}(t-t_0)^k ,$$
$$\Phi(t) = \sum_{k=0}^{\infty} \Phi_{(k)}(t-t_0)^k ,$$

where Λ_k and Φ_k can be calculated in terms of $\lambda_{(k)}$ and $\varphi_{(k)}$. These series converge very rapidly near the new origin t_0 .

Examples

We now consider simple examples to get an idea of what is involved in finding the disk ∂_Γ and in calculating the coefficients $\lambda_{(k)}$ and $\varphi_{(k)}$ by the formulae (10.9). Other examples will be treated numerically in Section 19.

(i) Let $X = \mathbb{C}^2$,

$$T_0 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad V_0 = \begin{bmatrix} 0 & 1/16 \\ 4 & 0 \end{bmatrix} .$$

Let $\lambda_0 = 2$ and $\varphi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \varphi_0^*$. If $z \neq 0, 2$, then

$$R_0(z) = \begin{bmatrix} -1/z & 0 \\ 0 & 1/(2-z) \end{bmatrix} , \quad V_0R_0(z) = \begin{bmatrix} 0 & 1/16(2-z) \\ -4/z & 0 \end{bmatrix} .$$

Since $\det(V_0R_0(z)-\mu I) = \mu^2 + 1/4z(2-z)$, we see that $\sigma(V_0R_0(z)) = \{\pm 1/2\sqrt{z(2-z)}\}$. Let Γ_ϵ denote the circle with centre $\lambda_0 = 2$ and radius $\epsilon < 2$. Then for $z \in \Gamma_\epsilon$, we have $|z-2| = \epsilon$, so that

$$r_\sigma(V_0R_0(z)) = 1/2\sqrt{\epsilon|z|} ,$$

$$\max_{z\in\Gamma_\epsilon} r_\sigma(V_0R_0(z)) = 1/2\sqrt{\epsilon(2-\epsilon)} .$$

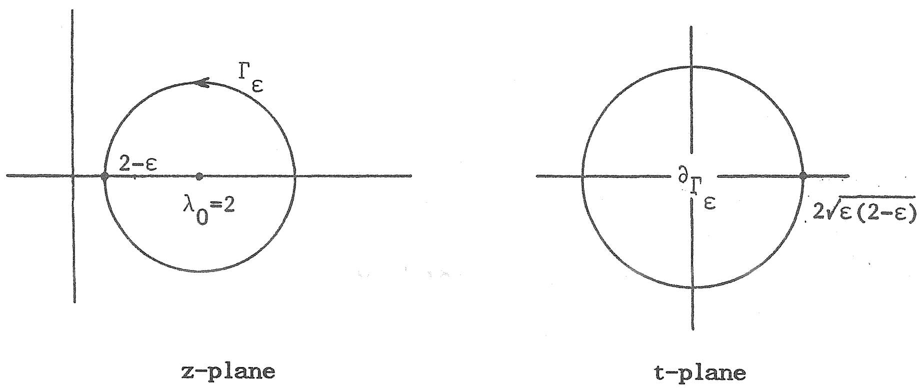

Figure 10.1

Thus, $\partial_{\Gamma_\epsilon} = \{t \in \mathbb{C} : |t| < 2\sqrt{\epsilon(2-\epsilon)}\}$. We find that the radius of $\partial_{\Gamma_\epsilon}$ is largest when $\epsilon = 1$, i.e., when $\epsilon = \mathrm{dist}(\lambda_0,\sigma(T_0)\backslash\{\lambda_0\})/2$, and the radius of $\partial_{\Gamma_\epsilon}$ then equals 2 . We have

$$P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S_0 = -\begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} .$$

(Cf. $P_{\lambda(t)}$ and $S_{\lambda(t)}$ for $t = 0$ in the example which illustrates (7.8).) Now,

$$\lambda_{(1)} = \langle V_0\varphi_0,\varphi_0^*\rangle = [0,1]\begin{bmatrix} 1/16 \\ 0 \end{bmatrix} = 0 ,$$

$$\varphi_{(1)} = -S_0V_0\varphi_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/16 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/32 \\ 0 \end{bmatrix} ,$$

$$\lambda_{(2)} = \langle V_0\varphi_{(1)},\varphi_0^*\rangle = [0,1]\begin{bmatrix} 0 \\ 1/8 \end{bmatrix} = 1/8 ,$$

$$\varphi_{(2)} = S_0(\lambda_{(1)}-V_0)\varphi_{(1)} = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1/8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,$$

$$\lambda_{(3)} = \langle V_0\varphi_{(2)},\varphi_0^*\rangle = 0 ,$$

$$\varphi_{(3)} = S_0(\lambda_{(1)}-V_0)\varphi_{(2)} + \lambda_{(2)}S_0\varphi_{(1)}$$

$$= -1/8 \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/32 \\ 0 \end{bmatrix} = - \begin{bmatrix} 1/512 \\ 0 \end{bmatrix} ,$$

$$\lambda_{(4)} = \langle V_0\varphi_{(3)}, \varphi_0^* \rangle = -[0,1] \begin{bmatrix} 0 \\ 1/128 \end{bmatrix} = -1/128 .$$

In two more steps, we would obtain $\varphi_{(4)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\lambda_{(5)} = 0$, and $\varphi_{(5)} = \begin{bmatrix} 1/4096 \\ 0 \end{bmatrix}$, $\lambda_{(6)} = 1/1024$. Thus, we have

$$\lambda(t) = 2 + t^2/8 - t^4/128 + t^6/1024 \ldots ,$$

$$\varphi(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} t/32 \\ 0 \end{bmatrix} - \begin{bmatrix} t^3/512 \\ 0 \end{bmatrix} + \begin{bmatrix} t^5/4096 \\ 0 \end{bmatrix} \ldots$$

$$= \begin{bmatrix} t/32 - t^3/512 + t^5/4096 \ldots \\ 1 \end{bmatrix} .$$

If we calculate sufficiently many terms of the above series, we can see that the series for $\lambda(t)$ is none other than the Taylor series for

$$1 + \sqrt{1+t^2/4} = \frac{1}{2}\left[2 + \sqrt{4+t^2}\right] ,$$

where $\sqrt{4+t^2}$ denotes the principal branch of the square root of $4+t^2$. Similarly, $\varphi(t)$ has 1 as the second component while the first component is given by the Taylor series for

$$-\frac{1}{4t}\left[1 - \sqrt{1+t^2/4}\right] = -\mu(t)/4t ,$$

where $\mu(t) = \frac{1}{2}\left[2 - \sqrt{4+t^2}\right]$. Thus

$$\varphi(t) = \begin{bmatrix} -\mu(t)/4t \\ 1 \end{bmatrix} .$$

It is easy to check that these results agree with the direct calculation of the eigenvalues $\lambda(t)$ and $\mu(t)$ of

$$T(t) = T_0 + tV_0 = \begin{bmatrix} 0 & t/16 \\ 4t & 2 \end{bmatrix},$$

and the eigenvector $\varphi(t)$ corresponding to $\lambda(t)$ satisfying $\langle\varphi(t),\varphi_0^*\rangle = 1$. (Cf. the example which illustrates (7.8).)

It can be seen that both the series for $\lambda(t)$ and $\varphi(t)$ converge for $|t| < 2$ and hence represent eigenelements of $T(t)$. Moreover, they have analytic continuations across every point on the circle of convergence $|t| = 2$, except for the points $t = \pm 2i$, and will continue to represent eigenelements there.

(ii) Suppose that the operator T_0 is diagonalizable, i.e., T_0 can be represented by a diagonal matrix

$$\operatorname{diag}(\lambda_0, \mu_1, \mu_2, \ldots),$$

with respect to a Schauder basis φ_0 , x_1 , x_2 ,... for X .Further assume that there is a Schauder basis $\varphi_0^*, x_1^*, \ldots$ of X^* which is adjoint to $\varphi_0, x_1, \ldots$, i.e., $\langle x_i, x_j^*\rangle = \delta_{i,j}$, $\langle\varphi_0, x_i^*\rangle = 0 = \langle x_i, \varphi_0^*\rangle$, and $\langle\varphi_0, \varphi_0^*\rangle = 1$. With respect to this basis T_0^* is represented by the diagonal matrix

$$\operatorname{diag}(\bar{\lambda}_0, \bar{\mu}_1, \bar{\mu}_2, \ldots).$$

Suppose that $\operatorname{dist}(\lambda_0, \{\mu_1, \mu_2, \ldots\}) > 0$, i.e., λ_0 is a simple eigenvalue of T_0 . Then P_0 and S_0 are represented by the matrices $\operatorname{diag}(1, 0, 0, \ldots)$ and $\operatorname{diag}\left[0, \frac{1}{\mu_1-\lambda_0}, \frac{1}{\mu_2-\lambda_0}, \ldots\right]$, respectively. Also,

$$\lambda_{(1)} = \langle V_0\varphi_0, \varphi_0^*\rangle ,$$

$$\varphi_{(1)} = -S_0V_0\varphi_0 = -\sum_{k=1}^{\infty} \frac{1}{\mu_k - \lambda_0} \langle V_0\varphi_0, x_k^*\rangle x_k , \quad \text{and}$$

$$\lambda_{(2)} = \langle V_0\varphi_{(1)}, \varphi_0^*\rangle = -\sum_{k=1}^{\infty} \frac{1}{\mu_k - \lambda_0} \langle V_0\varphi_0, x_k^*\rangle\langle V_0x_k, \varphi_0^*\rangle .$$

The above formulae are often found in textbooks on quantum mechanics where X is assumed to be a Hilbert space and T_0 is a (usually unbounded) self-adjoint operator, so that we have $\varphi_0^* = \varphi_0$ and $x_k^* = x_k$, $k = 1,2,\dots$. (Cf. [S], p.247.) Note that $\lambda_1 = \lambda_0 + \lambda_{(1)}$ and $\lambda_2 = \lambda_0 + \lambda_{(1)} + \lambda_{(2)}$ give the first order and the second order approximations to the eigenvalue λ of $T = T_0 + V_0$. It should be noticed that in the expressions for $\varphi_{(1)}$ and $\lambda_{(2)}$, the terms for which $|\mu_k-\lambda_0|$ is small dominate; these come from the eigenvalues of T_0 which are closest to λ_0 . In practice, only these terms are considered to obtain approximations of $\varphi_{(1)}$ and $\lambda_{(2)}$.

We return to the consideration of the values of $t \in \partial_\Gamma$ for which the function $\varphi(t) = P(t)\varphi_0/\langle P(t)\varphi_0, \varphi_0^*\rangle$ is analytic. The first result in this regard gives conditions on t under which the denominator does not vanish. See [N], Theorem 2.3.1 for a similar result. We introduce the following notations: For a curve Γ in $\rho(T_0)$, let

$$\begin{aligned} a_0 &= \max_{z\in\Gamma} \|(V_0R_0(z))^2\| , \\ b_0 &= \max_{z\in\Gamma} \|V_0R_0(z)\| , \\ c_0 &= \ell(\Gamma)\|V_0\varphi_0\|\ \|\varphi_0^*\| \,/\, 2\pi[\mathrm{dist}(\lambda_0,\Gamma)]^2 , \end{aligned} \tag{10.13}$$

where $\ell(\Gamma)$ is the length of Γ . Note that the constants a_0 , b_0 and c_0 depend on the curve Γ .

PROPOSITION 10.2 Let

$$a_0 + (a_0+b_0)c_0 < 1 . \tag{10.14}$$

Then for all t with $|t| \le 1$, we have $t \in \partial_\Gamma$ and $\langle P(t)\varphi_0, \varphi_0^* \rangle \ne 0$, so that the two Rayleigh-Schrödinger series (10.4) and (10.7) converge to eigenelements of $T(t)$.

Proof Since

$$\max_{z\in\Gamma} r_\sigma(V_0R_0(z)) \le \max_{z\in\Gamma} \|[V_0R_0(z)]^2\|^{1/2} = a_0^{1/2} < 1 ,$$

it follows that $t \in \partial_\Gamma$ for all t with $|t| \le 1$. Consider the Kato-Rellich perturbation series (9.15)

$$P(t) = P_0 + \sum_{k=1}^{\infty} P_{(k)} t^k .$$

Now, $P_0\varphi_0 = \varphi_0$, and for $k = 1,2,\dots$, we have by (9.16),

$$P_{(k)}\varphi_0 = \frac{(-1)^{k+1}}{2\pi i} \int_\Gamma R_0(z)[V_0R_0(z)]^{k-1}V_0R_0(z)\varphi_0 dz$$

$$= \frac{(-1)^{k+1}}{2\pi i} \int_\Gamma \frac{R_0(z)[V_0R_0(z)]^{k-1}V_0\varphi_0}{\lambda_0 - z} dz ,$$

since $R_0(z)\varphi_0 = \varphi_0/(\lambda_0-z)$ for $z \in \rho(T_0)$. Also, for $x \in X$,

$$\begin{aligned} \langle R_0(z)x, \varphi_0^* \rangle &= \langle R_0(z)x, P_0^*\varphi_0^* \rangle = \langle P_0R_0(z)x, \varphi_0^* \rangle \\ &= \langle R_0(z)P_0x, \varphi_0^* \rangle = \langle x, \varphi_0^* \rangle/(\lambda_0-z) , \end{aligned}$$

so that

$$\langle P_{(k)}\varphi_0, \varphi_0^* \rangle = \frac{(-1)^{k+1}}{2\pi i} \int_\Gamma \frac{\langle [V_0R_0(z)]^{k-1}V_0\varphi_0, \varphi_0^* \rangle}{(\lambda_0-z)^2} dz .$$

Putting $k = 1$, we see that $\langle P_{(1)}\varphi_0, \varphi_0^*\rangle = 0$, and for $k = 2,3,\ldots$,

$$|\langle P_{(k)}\varphi_0, \varphi_0^*\rangle| \leq \frac{\ell(\Gamma)\|V_0\varphi_0\| \ \|\varphi_0^*\|}{2\pi[\mathrm{dist}(\lambda_0,\Gamma)]^2} \max_{z\in\Gamma} \|[V_0R_0(z)]^{k-1}\|$$

$$= c_0 \max_{z\in\Gamma} \|[V_0R_0(z)]^{k-1}\| .$$

Thus,

$$|\langle P(t)\varphi_0, \varphi_0^*\rangle| = |1 + \sum_{k=2}^{\infty} \langle t^k P_{(k)}\varphi_0, \varphi_0^*\rangle|$$

$$\geq 1 - c_0|t| \sum_{k=2}^{\infty} \max_{z\in\Gamma} \|[tV_0R_0(z)]^{k-1}\| .$$

We show that for $|t| \leq 1$,

$$c_0|t| \sum_{k=2}^{\infty} \max_{z\in\Gamma}\|[tV_0R_0(z)]^{k-1}\| < 1$$

to conclude $\langle P(t)\varphi_0, \varphi_0^*\rangle \neq 0$. Since

$$\sum_{k=2}^{\infty} \max_{z\in\Gamma}\|[tV_0R_0(z)]^{k-1}\| \leq \left[|t|\max_{z\in\Gamma}\|V_0R_0(z)\| + |t|^2\max_{z\in\Gamma}\|[V_0R_0(z)]^2\|\right]$$

$$\times \sum_{j=0}^{\infty} \left[|t|^2\max_{z\in\Gamma}\|[V_0R_0(z)]^2\|\right]^j$$

$$\leq \frac{|t|(b_0+|t|a_0)}{1 - |t|^2a_0} ,$$

we see that for $|t| \leq 1$,

$$c_0|t| \sum_{k=2}^{\infty} \max_{z\in\Gamma}\|[tV_0R_0(z)]^{k-1}\| \leq \frac{c_0(b_0+a_0)}{1 - a_0} ,$$

which is less than 1 by assumption. This completes the proof. //

We shall later show that in many practical situations the hypothesis $a_0 + (a_0+b_0)c_0 < 1$ is, in fact, satisfied. (See Remark 14.10.)

COROLLARY 10.3 If

$$b_0[b_0+(1+b_0)c_0] < 1 \text{ , or } b_0 + c_0 < 1 \text{ ,} \tag{10.15}$$

then the conclusions of Proposition 10.2 hold.

Proof Since $a_0 \le b_0^2$, we see that

$$a_0 + (a_0+b_0)c_0 \le b_0[b_0+(1+b_0)c_0] \text{ .}$$

Hence the result follows in the first case; as for the second, note that if $b_0 + c_0 < 1$, then $b_0 < 1$ and

$$b_0[b_0+(1+b_0)c_0] < b_0^2 + (1+b_0)c_0 = b_0(b_0+c_0) + c_0 < b_0 + c_0 < 1 \text{ .} \qquad //$$

We remark that the conditions given in (10.15) are, in general, less stringent than the condition

$$b_0[1+ \frac{\ell(\Gamma)}{2\pi} \|\varphi_0^*\| \max_{z\in\Gamma} \|R_0(z)\|] < 1$$

stated on p.143 of [C], since $1 \le \text{dist}(\lambda_0,\Gamma) \max_{z\in\Gamma} \|R_0(z)\|$ and $\|V_0\varphi_0\| \le \text{dist}(\lambda_0,\Gamma)b_0$, when $\|\varphi_0\| = 1$. See Problem 10.2 for a concrete illustration.

In order to estimate the domain of analyticity of the function $\lambda(t)$, we wish to find a lower bound for the radius of ∂_Γ , at least for some particular curve Γ . As far as the function $\varphi(t) = P(t)\varphi_0$ / $\langle P(t)\varphi_0,\varphi_0^*\rangle$ is concerned, both the numerator and the denominator are analytic on ∂_Γ . However, $\varphi(t)$ would have a pole at t_0 if the denominator has a zero of a higher order than the order of the zero of the numerator. We do not know whether, in fact, $\varphi(t)$ can have a pole in ∂_Γ. We shall, therefore, content ourselves by finding a disk (with centre 0) in ∂_Γ which is pole-free. Our results are in terms of the following quantities:

$$\begin{aligned} &\eta_0 = \|V_0\varphi_0\| , \quad p_0 = \|\varphi_0^*\| , \quad s_0 = \|S_0\| , \\ (10.16) \qquad &\alpha_0 = \|V_0S_0\| , \quad \gamma_0 = \max\{\eta_0p_0s_0, \alpha_0\} . \end{aligned}$$

Let $\Gamma_\epsilon(t) = \lambda_0 + \frac{\epsilon}{s_0} e^{it}$, $0 \lesssim t \leq 2\pi$, where $0 < \epsilon < 1$. Since

$$r_\sigma(S_0) = \frac{1}{\operatorname{dist}(\lambda_0,\sigma(T_0)\setminus\{\lambda_0\})} \leq \|S_0\| = s_0$$

by (7.3), we see that the circle Γ_ϵ lies in $\rho(T_0)$ and separates λ_0 from the rest of the spectrum of T_0 . We note that the quantities given in (10.16) do not depend on the curve Γ_ϵ .

LEMMA 10.4 Let $0 < \epsilon < 1$.

(a) If $|t| < \epsilon(1-\epsilon)/\gamma_0$, then $t \in \partial_{\Gamma_\epsilon}$.

(b) If $|t| < 1/2\alpha_0$, and we let $Z_0 = (I-P_0)(X)$ then

$$\{z \in \mathbb{C} : |z-\lambda_0| \leq 1/2s_0\} \subset \rho((I-P_0)T(t)|_{Z_0}) .$$

Proof (a) For $0 < |z-\lambda_0| < \operatorname{dist}(\lambda_0,\sigma(T_0)\setminus\{\lambda_0\})$, we have by (7.8),

$$R_0(z) = \sum_{k=0}^{\infty} S_0^{k+1}(z-\lambda_0)^k - \frac{P_0}{z - \lambda_0} ,$$

since λ_0 is simple, so that $R_0(z)$ has a simple pole at $z = \lambda_0$. Hence if $|z-\lambda_0| = \epsilon/s_0$, we have

$$\|V_0R_0(z)\| \le \frac{\|V_0S_0\|}{1-\epsilon} + \frac{\|V_0P_0\|s_0}{\epsilon}$$
$$= [\epsilon\alpha_0+(1-\epsilon)\eta_0p_0s_0]/\epsilon(1-\epsilon) ,$$

since $\|V_0P_0\| = \|V_0\varphi_0\| \; \|\varphi_0^*\| = \eta_0p_0$. But $\alpha_0 \le \gamma_0$ as well as $\eta_0p_0s_0 \le \gamma_0$, by the definition of γ_0 , so that

$$\max_{z\in\Gamma_\epsilon} r_\sigma(V_0R_0(z)) \le \max_{z\in\Gamma_\epsilon} \|V_0R_0(z)\| \le \gamma_0/\epsilon(1-\epsilon) .$$

Since $\partial_{\Gamma_\epsilon} = \{t \in \mathbb{C} : |t| < 1/\max_{z\in\Gamma_\epsilon} r_\sigma(V_0R_0(z))\}$, we see that $|t| < \epsilon(1-\epsilon)/\gamma_0$ implies $t \in \partial_{\Gamma_\epsilon}$.

(b) Let $|t| < 1/2\alpha_0$ and $|z-\lambda_0| \le 1/2s_0$. Then z lies inside the circle Γ_1 with centre at λ_0 and radius $1/s_0$. Now, with $Z_0 = (I-P_0)X$,

$$\sigma(T_0|_{Z_0}) = \sigma(T_0) \cap \operatorname{Ext} \Gamma_1 .$$

This shows that $z \in \rho(T_0|_{Z_0})$. To show $z \in \rho((I-P_0)T(t)|_{Z_0})$, it is then enough to prove that $r_\sigma(A(z)) < 1$, where

$$A(z) = \left[T_0|_{Z_0}-(I-P_0)T(t)|_{Z_0}\right]\left[T_0|_{Z_0}-zI|_{Z_0}\right]^{-1} .$$

(See Theorem 9.1.) As $T(t) = T_0 + tV_0$, we have

$$A(z) = -t(I-P_0)V_0\left[T_0|_{Z_0}-zI|_{Z_0}\right]^{-1} = -t(I-P_0)V_0R_0(z)|_{Z_0} .$$

Hence by (5.11) and (5.12),

$$r_\sigma(A(z)) = |t|r_\sigma((I-P_0)V_0R_0(z)(I-P_0))$$
$$= |t|r_\sigma(V_0R_0(z)(I-P_0)) .$$

But by the expression for $R_0(z)$ given in the proof of (a),

$$V_0R_0(z)(I-P_0) = \sum_{k=0}^{\infty} V_0S_0^{k+1}(z-\lambda_0)^k ,$$

for all $z \in \mathbb{C}$ with $|z\lambda_0| \leq 1/2s_0$, so that

$$\|V_0R_0(z)(I-P_0)\| \leq \|V_0S_0\|/(1-1/2) = 2\alpha_0 .$$

Thus, $r_\sigma(A(z)) \leq |t|2\alpha_0 < 1$, and the proof is complete. //

THEOREM 10.5 (Cf. [LR], Theorem 2.3.) The disk

$$D = \{t \in \mathbb{C} : |t| < 1/4\gamma_0\}$$

is contained in $\partial_{\Gamma_{1/2}}$ and the function $\varphi(t) = P(t)\varphi_0/\langle P(t)\varphi_0,\varphi_0^*\rangle$ is analytic on D . For $|t| < 1/4\gamma_0$, the two Rayleigh-Schrödinger series (10.4) and (10.7) converge respectively to a simple eigenvalue $\lambda(t)$ and a corresponding eigenvector $\varphi(t)$ of $T(t)$ which satisfies $\langle\varphi(t),\varphi_0^*\rangle = 1$. Further,

$$|\lambda(t)-\lambda_0| \leq \frac{1 - \sqrt{1-4|t|\gamma_0}}{2s_0} , \tag{10.17}$$

and $\lambda(t)$ is the only spectral value of $T(t)$ in the disk

$$\left\{z \in \mathbb{C} : |z-\lambda_0| < \frac{1 + \sqrt{1-4|t|\gamma_0}}{2s_0}\right\} . \tag{10.18}$$

Proof Letting $\epsilon = 1/2$ in Lemma 10.4(a), we see that $|t| < 1/4\gamma_0$ implies $t \in \partial_{\Gamma_{1/2}}$. Thus, $D \subset \partial_{\Gamma_{1/2}}$. To show that $\varphi(t)$ is analytic on D we argue as follows. Let $t_0 \in D$. Since $t \mapsto P(t)\varphi_0 \in X$ is analytic on $\partial_{\Gamma_{1/2}}$, we have

$$P(t)\varphi_0 = (t-t_0)^k x(t) ,$$

for t near t_0 , where the function $t \mapsto x(t)$ is analytic on a neighbourhood N of t_0 and does not vanish there. Since,

$$\langle P(t)\varphi_0,\varphi_0^*\rangle = (t-t_0)^k\langle x(t),\varphi_0^*\rangle \ , \ t \in N \ , \ t \neq t_0 \ ,$$

the only possible singularity of $\varphi(t)$ at $t = t_0$ is a pole, and this happens only if $\langle x(t_0),\varphi_0^*\rangle = 0$. Also, for $t \in N$, $t \neq t_0$, $x(t) = P(t)\varphi_0/(t-t_0)^k$ is an eigenvector of $T(t)$ corresponding to the eigenvalue $\lambda(t)$. But by the continuity of $P(t)$ and $x(t)$ at $t = t_0$, we have

$$P(t_0)x(t_0) = \lim_{t\to t_0} P(t)x(t) = \lim_{t\to t_0} x(t) = x(t_0) \ ,$$

i.e., $x(t_0) \neq 0$ is an eigenvector of $T(t_0)$ corresponding to the eigenvalue $\lambda(t_0)$. Let $\langle x(t_0),\varphi_0^*\rangle = 0$. Since $x(t_0) \in Z(P_0) = Z_0$, we see that $\lambda(t_0) \in \sigma((I-P_0)T(t_0)|_{Z_0})$. But since $\lambda(t_0)$ lies inside $\Gamma_{1/2}$, we have $|\lambda(t_0)-\lambda_0| < 1/2s_0$, and since $|t_0| < 1/4\gamma_0 < 1/2\alpha_0$, Lemma 10.4(b) shows that $\lambda(t_0) \in \rho((I-P_0)T(t_0)|_{Z_0})$. This contradiction allows us to conclude the analyticity of $\varphi(t)$ at $t = t_0$. Hence for $|t| < 1/4\gamma_0$, the functions $\lambda(t)$ and $\varphi(t)$ are analytic, and as such have convergent Taylor expansions around 0 . That $\lambda(t)$ is a simple eigenvalue of $T(t)$, $\lambda(t)$ lies inside $\Gamma_{1/2}$, i.e., $|\lambda(t)-\lambda_0| < 1/2s_0$ and it is the only spectral value of $T(t)$ inside $\Gamma_{1/2}$ follows directly from Corollary 9.9. But we now give better estimates.

For $0 < \epsilon < 1$, we see that $|t| < \epsilon(1-\epsilon)/\gamma_0$ if and only if $r_1(t) < \epsilon < r_2(t)$, where

$$r_1(t) = \frac{1 - \sqrt{1-4|t|\gamma_0}}{2} \quad \text{and} \quad r_2(t) = \frac{1 + \sqrt{1-4|t|\gamma_0}}{2} \ .$$

Lemma 10.4(a) now shows that $t \in \partial_{\Gamma_\epsilon}$ for every ϵ with $r_1(t) < \epsilon < r_2(t)$. Again by Corollary 9.9, we note that (i) $\lambda(t)$ lies inside Γ_ϵ, i.e., $|\lambda(t)-\lambda_0| < \epsilon/s_0$ and (ii) it is the only spectral point of $T(t)$ inside Γ_ϵ. Letting $\epsilon \to r_1(t)$ in (i) and $\epsilon \to r_2(t)$ in (ii) we see that $|\lambda(t)-\lambda_0| \leq r_1(t)/s_0$, and that $\lambda(t)$ is the only spectral point of $T(t)$ in $\{z \in \mathbb{C} : |z-\lambda_0| < r_2(t)/s_0\}$. Thus, (10.17) and (10.18) hold. //

We illustrate Theorem 10.5 schematically as follows

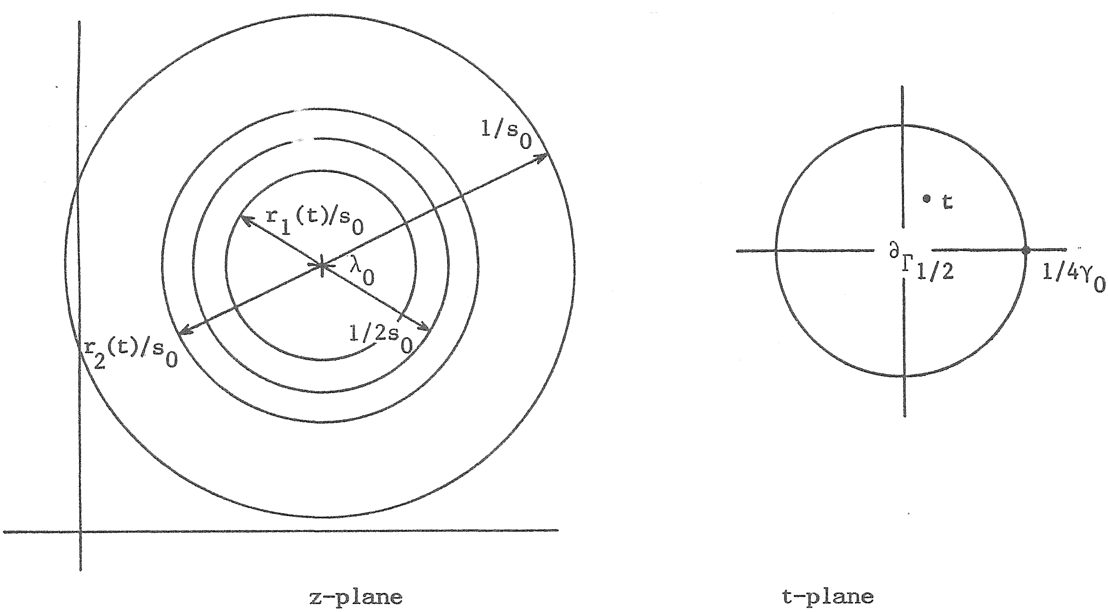

Figure 10.2

Note that for $|t| < 1/4\gamma_0$, we have

$$0 \leq r_1(t) \leq 1/2 \leq r_2(t) \leq 1 ;$$

$r_1(t) \downarrow 0$ and $r_2(t) \uparrow 1$ as $|t| \to 0$, while $r_1(t) \uparrow 1/2$ and $r_2(t) \downarrow 1/2$ as $|t| \to 1/4\gamma_0$. As $|t|$ becomes smaller, we get a better estimate for $|\lambda(t)-\lambda_0|$ and a larger region of isolation for $\lambda(t)$.

Since $\gamma_0 \leq \|V_0\| \|S_0\| \|P_0\|$, the above theorem shows that if the norms of the spectral projection P_0 and the reduced resolvent S_0 associated with T_0 and λ_0 are small, then we can allow a large perturbation V_0 and obtain eigenelements λ and φ of $T = T_0 + V_0$, as long as we have $\|V_0\| \|P_0\| \|S_0\| < 1/4$. We now consider a special case where $\|P_0\| = 1$ (the smallest possible value) and $\|S_0\| = \text{dist}(\lambda_0, \sigma(T_0)\setminus\{\lambda_0\})$.

THEOREM 10.6 (Cf. [LR], Theorem 3.5.) Let T_0 be a normal operator on a Hilbert space X, let λ_0 be a simple eigenvalue of T_0 and let $d_0 = \text{dist}(\lambda_0, \sigma(T_0)\setminus\{\lambda_0\})$. If $0 \neq V_0 \in BL(X)$ and $|t| < d_0/2\|V_0\|$, then $T(t) = T_0 + tV_0$ has a simple eigenvalue $\lambda(t)$ such that

$$|\lambda(t)-\lambda_0| \leq \|V_0\| |t| ,$$

and $\lambda(t)$ is the only spectral value of $T(t)$ lying in the disk

$$\{z \in \mathbb{C} : |z-\lambda_0| < d_0 - \|V_0\| |t|\} .$$

Also, the Rayleigh-Schrödinger series (10.4) and (10.7) converge to eigenelements of $T(t)$ for $|t| < d_0/2\|V_0\|$.

Proof Since T_0 is normal, we have for $z \in \rho(T_0)$,

$$\|R_0(z)\| = 1/\text{dist}(z,\sigma(T_0)) \quad \text{and} \quad \|S_0\| = 1/d_0 ,$$

by (8.13) and (8.14). Let $0 < \epsilon < 1$, and $\Gamma_\epsilon(t) = \lambda_0 + \epsilon d_0 e^{it}$, $0 \leq t \leq 2\pi$. Then

$$\begin{aligned} \max_{z\in\Gamma_\epsilon} r_\sigma(V_0R_0(z)) &\leq \|V_0\| \max_{z\in\Gamma_\epsilon} \|R_0(z)\| \\ &= \|V_0\| \max_{z\in\Gamma_\epsilon} \frac{1}{\text{dist}(z,\sigma(T_0))} \\ &= \|V_0\|/d_0 \min\{\epsilon, 1-\epsilon\} . \end{aligned}$$

Thus, $|t| < d_0 \min\{\epsilon, 1-\epsilon\}/\|V_0\|$ implies that $t \in \partial_{\Gamma_\epsilon}$, so that $T(t)$ has a simple eigenvalue $\lambda(t)$ inside Γ_ϵ and it is the only spectral value of $T(t)$ inside Γ_ϵ. Now for $0 < \epsilon < 1$, we note that $|t| < d_0 \min\{\epsilon, 1-\epsilon\}/\|V_0\|$ if and only if $r_1(t) < \epsilon < r_2(t)$, where

$$r_1(t) = \|V_0\| |t|/d_0 \quad \text{and} \quad r_2(t) = 1 - \|V_0\| |t|/d_0 .$$

Letting $\epsilon \to r_1(t)$ and $\epsilon \to r_2(t)$ we obtain the statements regarding $\lambda(t)$.

Finally, if $|t_0| < d_0/2\|V_0\|$, then since $d_0 = 1/\|S_0\|$, we have

$$|t_0| < 1/2\|V_0S_0\| = 1/2\alpha_0 .$$

By Lemma 10.4(b), we conclude that $\lambda(t_0) \in \rho((I-P_0)T(t_0)|_{Z_0})$. The proof of Theorem 10.5 now shows that $\varphi(t) = P(t)\varphi_0/\langle P(t)\varphi_0, \varphi_0^*\rangle$ cannot have a singularity at $t = t_0$. Thus, the Rayleigh-Schrödinger series (10.4) and (10.7) converge for $|t| < d_0/2\|V_0\|$. //

We see from the above result that if the simple eigenvalue λ_0 of a normal operator T_0 is well separated from the rest of the spectrum of T_0, i.e., if d_0 is large, then even for a large perturbation V_0, we can obtain eigenelements of $T_0 + V_0$ by the Rayleigh-Schrödinger procedure.

Remark 10.7 We conclude this section by remarking that Theorems 10.5 and 10.6 would prove to be useful for finding eigenelements of $T_0 + V_0$ only if $\alpha_0 = \|V_0S_0\|$ is small: $\alpha_0 < 1/4$ or $\alpha_0 < 1/2$. If this were not so, one has to look for sharper estimates of $r_\sigma(V_0R_0(z))$ for z near λ_0, such as $\|(V_0R_0(z))^2\|^{1/2}$.

Theorem 10.5 holds if we replace γ_0 by $\sqrt{\delta_0}$, where

$$\text{(10.19)} \quad \begin{aligned} \delta_0 &= \max\{\eta_0 p_0 s_0 \gamma_0, \ \tau_0\} \quad (\le \gamma_0^2) , \\ \tau_0 &= \sup\{\|V_0 S_0^k V_0 S_0\| / s_0^{k-1} : k = 1,2,\dots\} \quad (\le \alpha_0^2) . \end{aligned}$$

We leave the proof of this result to Problem 10.4. See also [LR], Theorem 2.3.

Theorems 10.5, 10.6 and the above result say that if the perturbation V_0 is small in some sense (e.g., $\sqrt{\delta_0} < 1/4$) , then not only the Rayleigh-Schrödinger series with initial terms as the eigenelements (λ_0, φ_0) of T_0 converge to eigenelements (λ, φ) of $T = T_0 + V_0$, but the eigenvalue λ is simple, and it is the unique spectral point of T which is nearest to λ_0 . If no conditions on V_0 are put, then the Rayleigh-Schrödinger series with initial term λ_0 may neither converge to a *simple* eigenvalue of T , nor to the *nearest* eigenvalue of T . (See Problems 10.6 and 10.7.)

Problems

10.1 Let $x(t)$ be given by (10.8). Then for $|t|$ small,

$$x(t) = \tilde{x}_0 + \sum_{k=1}^{\infty} x_{(k)} t^k ,$$

$$\lambda(t) = \lambda_0 + \sum_{k=1}^{\infty} \lambda_{(k)} t^k ,$$

where $\tilde{x}_0 = P_0 x_0 / \langle P_0 x_0, x_0^* \rangle$ is an eigenvector of T_0 corresponding to λ_0 , and for $k = 1,2,\dots$,

$$x_{(k)} = (I-Q_0) S_0 [-V_0 x_{(k-1)} + \sum_{i=1}^{k-1} \lambda_{(i)} x_{(k-i)}] ,$$

$$\lambda_{(k)} = \langle T_0 x_{(k)} + V_0 x_{(k-1)}, \ x_0^* \rangle ,$$

and the projection Q_0 is given by $Q_0x = \langle x, x_0^* \rangle \tilde{x}_0$, $x \in X$. If $\tilde{S}_0 = (I-Q_0)S_0$, then $\tilde{S}_0|_{Q_0(X)} \equiv 0$ and $\tilde{S}_0|_{(I-Q_0)(X)}$ is the inverse of $(I-Q_0)(T_0-\lambda_0 I)|_{(I-Q_0)(X)}$. Let $\tilde{\eta}_{(k)} = -V_0x_{(k-1)} + \langle V_0x_{(k-1)}, x_0^* \rangle \tilde{x}_0 + \sum_{i=1}^{k-1} \lambda_{(i)}x_{(k-i)}$. Then $x_{(k)}$ is the unique solution of $(I-Q_0)(T_0-\lambda_0 I)x = \tilde{\eta}_{(k)}$, $\langle x, x_0^* \rangle = 0$.

10.2 Let $X = \mathbb{C}^2$ with the p-norm, $1 \le p \le \infty$, $T_0 = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix}$, $V_0 = \begin{bmatrix} 0 & 0 \\ \epsilon & 0 \end{bmatrix}$, $0 < \epsilon < |\lambda_0-\lambda_1|(\sqrt{3}-1)/2$, $\Gamma = \{z : |z-\lambda_0| = |\lambda_0-\lambda_1|/2\}$. Then in Corollary 10.3, $b_0 = c_0 = 2\epsilon/|\lambda_0-\lambda_1|$. Also, $\max_{z\in\Gamma} \|R_0(z)\| = 2/|\lambda_0-\lambda_1|$.

10.3 Let T_0 and T_0^* be diagonable as in Example (ii). Then

$$\varphi_{(3)} = \sum_{k=1}^{\infty} \left[\sum_{m=1}^{\infty} \frac{\langle V_0\varphi_0, x_m^* \rangle \langle V_0x_m, x_k^* \rangle}{(\mu_k-\lambda_0)(\mu_m-\lambda_0)} - \frac{\langle V_0\varphi_0, \varphi_0^* \rangle \langle V_0\varphi_0, x_k^* \rangle}{(\mu_k-\lambda_0)^2} \right] x_k .$$

10.4 ([LR], Theorems 2.1 and 2.3.) Let τ_0 and δ_0 be defined by (10.19). Then Lemma 10.4 can be improved as follows: (a) If $|t| < \epsilon(1-\epsilon)/\sqrt{\delta_0}$, then $t \in \partial_{\Gamma_\epsilon}$. (b) If $|t| < 1/2\sqrt{\tau_0}$, then $\{z \in \mathbb{C} : |z-\lambda_0| \le 1/2s_0\} \subset \rho((I-P_0)T(t)|_{Z_0})$. Hence Theorem 10.5 holds if we replace γ_0 by $\sqrt{\delta_0}$.

10.5 ([N]) If X is 2-dimensional, then $\langle P(t)\varphi_0, \varphi_0^* \rangle \ne 0$ for every $t \in \partial_\Gamma$. If X is finite dimensional, then for $t \in \partial_\Gamma$ we have $\langle P(t)\varphi_0, \varphi_0^* \rangle \ne 0$ if and only if $\lambda(t)$ is an eigenvalue of $(I-P_0)T(t)|_{(I-P_0)(X)}$.

10.6 Let $T_0 = \begin{bmatrix} a & o \\ 0 & b \end{bmatrix}$, $a \neq b$, and $\lambda_0 = a$. If $V_0 = \frac{1}{2}\begin{bmatrix} b-a & 0 \\ 0 & a-b \end{bmatrix}$, then $\lambda = \lambda_0 + \lambda_{(1)} = \frac{a+b}{2}$ is a double eigenvalue of $T = T_0 + V_0 = \frac{1}{2}\begin{bmatrix} a+b & 0 \\ 0 & a+b \end{bmatrix}$. If $V_0 = \begin{bmatrix} b-a & 0 \\ 0 & a-b \end{bmatrix}$, then $\lambda = \lambda_0 + \lambda_{(1)} = b$ is an eigenvalue of $T = T_0 + V_0 = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$, but it is not the nearest eigenvalue of T to a .

10.7 If a curve Γ which separates the simple eigenvalue λ_0 from the rest of $\sigma(T_0)$ is a *circle with centre* λ_0 , and if $\max_{z\in\Gamma} r_\sigma(V_0R_0(z)) < 1$, then the Rayleigh-Schrödinger series with initial term λ_0 converges to a simple eigenvalue λ of $T = T_0 + V_0$, which is the nearest spectral point of T to λ_0 .