BOUNDARY VALUE PROBLEMS FOR FULLY NONLINEAR ELLIPTIC EQUATIONS

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We describe here some recent estimates and existence theorems for classical solutions of nonlinear, second order elliptic boundary value problems of the general form,

$$(1) \qquad F[u] = F(x,u,Du,D^2u) = 0 \quad \text{in} \quad \Omega ,$$

$$(2) \qquad G[u] = G(x,u,Du) = 0 \quad \text{on} \quad \partial\Omega ,$$

where Ω is a bounded domain in Euclidean n space, $\mathbb{R}^n$, F, G are real valued functions on the sets $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$, $\Gamma' = \partial\Omega \times \mathbb{R} \times \mathbb{R}^n$ respectively and $\mathbb{S}^n$ denotes the linear space of $n \times n$ symmetric real matrices. Letting $X = (x,z,p,r)$, $X' = (x,z,p)$ denote points in Γ, Γ', we adopt the following definitions of ellipticity and obliqueness for functions, F,G differentiable with respect to r, p respectively. Namely, the operator F is *elliptic* at $X \in \Gamma$ if the matrix

$$F_r = [F^{ij}] = [\frac{\partial F}{\partial r_{ij}}]$$

is positive at X; while the operator G is *oblique* at $X' \in \Gamma'$ if

$$\chi = G_p = \nu \cdot G_p$$

is positive at X', where ν is the unit inner normal to $\partial\Omega$, (which for this definition we assume sufficiently smooth). Letting λ, Λ denote the minimum and maximum eigenvalues of F_r, we shall call F *uniformly elliptic* with respect to a subset $\mathcal{U} \subset \Gamma$ if F is elliptic on $\mathcal{U}$ and the ratio Λ/λ is bounded there. These definitions may be extended to non-differentiable F and G by replacing λ, Λ, χ by the quantities

$$\lambda(X) = \liminf_{\eta \to 0^+} \frac{F(x,z,p,r+\eta) - F(x,z,p,r)}{\text{trace } \eta} ,$$

$$\Lambda(X) = \limsup_{\eta \to 0^+} \frac{F(x,z,p,r+\eta) - F(x,z,p,r)}{\text{trace } \eta} ,$$

$$\chi(X') = \liminf_{t \to 0^+} \frac{G(x,z,p,+t\nu) - G(x,z,p)}{t}$$

where $\eta \in \mathbb{S}^n$, and $t \in \mathbb{R}$ are positive. Accordingly F is elliptic and G oblique when λ and χ are positive.

We shall confine attention here to operators which are uniformly elliptic (for bounded z) and either Dirichlet or oblique boundary operators. Important examples are furnished by the Bellman (or rather Hamilton-Jacobi-Bellman) equations of stochastic control theory. If $\mathcal{L} = \{L_\nu, \nu \in V\}$ is a family of linear elliptic operators on Ω, indexed by a parameter ν belonging to an index set V, and $\mathcal{F} = \{f_\nu\}$ is a corresponding family of functions on Ω, then the Bellman equation corresponding to $\mathcal{L}, \mathcal{F}$ is given by

$$F[u] = \inf_{\nu \in V} (L_\nu u - f_\nu) = 0 \tag{4}$$

Writing each L_ν in the form ,

(5) $$L_\nu u = a^{ij}_\nu(x)\, D_{ij}u + b^i_\nu(x)\, D_i u + c_\nu(x)u \ ,$$

we find that F is uniformly elliptic on Γ if there exist positive constants λ_0, Λ_0 such that

(6) $$\lambda_0|\xi|^2 \leq a^{ij}_\nu(x)\ \xi_i\xi_j \leq \Lambda_0|\xi|^2$$

for all $\xi \in \mathbb{R}^n$, $x \in \Omega$, $\nu \in V$. The connection between the Bellman equation and stochastic control theory is treated in Krylov's pioneering book [12]. The Bellman equations are also significant in the development of the theory of fully nonlinear equations as their study, notably by Krylov, Lions, Evans among others, paved the way for the more general theory that we outline below.

NATURAL STRUCTURE CONDITIONS

Appropriate hypotheses on the operators F and G can be expressed as growth restrictions on the corresponding functions. For *quasilinear* uniformly elliptic equations, Ladyzhenskaya and Ural'tseva, (see [17]), introduced a set of conditions which they described as "natural". For elliptic, fully nonlinear operators of the form (1), we extended their conditions as follows [29] :

F1 : $\Lambda \leq \lambda\mu$, (Uniform ellipticity) ;

F2 : $|F(x,z,p,0)| \leq \lambda\mu_0(1+|p|^2)$;

F3 : $|F_x|, |F_z|, (1+|p|)\,|F_p| \leq \lambda\mu_1(1+|p|^2+|r|)$.

These conditions are required to hold on any set of the form $\mathcal{U}_K = \{X = (x,z,p,r) \in \Gamma \mid |z| \leq K\}$ and μ, μ_0, μ_1 are positive constants depending on K.

As for the quasilinear case ([17], [18], [9]), F1, F2, F3 suffice for the establishment of apriori gradient and gradient Hölder estimates under appropriate boundary conditions [29]. But the treatment of existence for fully nonlinear elliptic equations, for example by the method of continuity, requires also second derivative estimates. It turns out that a convenient condition on the second derivatives of F, which embraces both a reasonable extension of the natural conditions and smooth approximations to the Bellman equation, is

F4 :
$$F_{YY}(X) = \frac{\partial^2 F}{\partial X_i \partial X_i} Y_i Y_j$$
$$\leq \mu_2 \lambda \{(1+|r|)|Y'|+|s|\}|Y'|$$

for all $Y = (Y',s) \in (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n) \times \$^n$. Condition F4 need only hold on sets of the form $\tilde{\mathcal{U}}_K = \{X = (x,z,p,r) \in \Gamma \mid |z|+|p| \leq K\}$ and μ_2 is a further positive constant depending on K. We observe that F4 implies the concavity of F with respect to r and moreover will be satisfied if F is concave with respect to r and

$$(7) \qquad |F_{X'X'}| \ , \ (1+|r|)|F_{X'r}| \leq \bar{\mu}_2(1+|r|)$$

where $X' = (x,z,p)$ and $\bar{\mu}_2$ is a positive constant.

For oblique boundary operators G, a notion of natural conditions, modelled on the linear case

$$G[u] = Mu - g = \beta_i D_i u + \gamma u - g \,, \tag{8}$$

was introduced by Lieberman and Trudinger [21]. These conditions, corresponding to F2, F3, may be expressed as

G2 : $\quad |G(x,z,p')| \leqq \mu_0 \chi(1+|p'|)$;

G3 : $\quad |G_x|, |G_z|, \ (1+|p|)|G_p| \leqq \mu_1 \chi(1+|p|)$,

where $p' = p-(p\cdot\nu)\nu$ is the tangential projection of p. Conditions G2, G3 are required to hold for all $X' = (x,z,p) \in \Gamma'$ with $|z| \leqq K$, for any K and μ_0, μ_1 are again positive constants depending on K. An example of a nonlinear operator G satisfying G2, G3 is furnished by the *capillarity* boundary condition,

$$G[u] = \nu\cdot Du - g(x)\sqrt{1+|Du|^2} = 0 \quad \text{on} \quad \partial\Omega \,, \tag{9}$$

where $\sup|g| < 1$.

THE CLASSICAL DIRICHLET PROBLEM

Here $G = g(x) - z$ and a classical solution of the Dirichlet problem

$$F[u] = 0 \quad \text{in} \quad \Omega \,, \quad u = g \quad \text{on} \quad \partial\Omega \tag{10}$$

is a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying (10) in the usual pointwise sense. For the uniformly elliptic Bellman equations, the existence of $C^{1,1}(\bar{\Omega})$ solutions of (10), (satisfying $F[u] = 0$ almost everywhere in Ω) , was established through the work of Evans and Friedman [7], Lions and Menaldi [23], Lions [22] and Evans and Lions [8], and in special cases, namely two dimensions and two operators, the classical solvability resulted from the Morrey estimates (see [9]) and the work of Brezis and Evans [1], respectively. The general interior Hölder estimates for second derivatives which facilitated the classical solvability in the general case were discovered independently by Evans [5], [6] and Krylov [13] and are treated in [9]. For operators F satisfying the natural structure conditions, we established the following estimates in [29].

<u>THEOREM 1</u> *Let* $u \in C^2(\Omega)$ *satisfy* $F[u] = 0$ *in* Ω, *where* $F \in C^2(\Gamma)$ *satisfies* F1, F2, F3. *Then for any subdomain* $\Omega' \subset\subset \Omega$, *we have the estimate*

$$(11) \qquad |u|_{1,\alpha;\Omega'} \leq C$$

where $\alpha \in (0,1)$ *and* $C > 0$, *depend on* $n, \mu, \mu_0, \mu_1, |u|_{0;\Omega}$ *with* C *also depending on* $\mathrm{dist}(\Omega', \partial\Omega)$. *If* F4 *also holds, then*

$$(12) \qquad |u|_{2,\alpha;\Omega'} \leq C$$

for some $\alpha \in (0,1)$ *depending only on* n, μ *and* C *depending also on* μ_0, μ_1, μ_2, $\mathrm{dist}(\Omega', \partial\Omega)$.

We remark that Theorem 1 continues to hold for solutions $u \in C^{1,1}(\Omega)$ and operators F of the form

$$(13) \qquad F[u] = \inf_{\nu \in V} F_\nu[u]$$

where $\{F_\nu\}$ satisfies the natural conditions F1..F4 uniformly in ν, [31]. Furthermore conditions F3 and F4 may be relaxed slightly without significant adjustment of our proofs, so that in particular we may assume in their place :

F3* $\quad (1+|p|)|F_p|,\ \delta F \leq \mu_1 \lambda(1+|p|^2+|r|)$;

F4* $\quad F_{yy} \leq \mu_2\lambda\{(1+|r|)^3|\bar{y}|^2 + (1+|r|)|q|^2 + ((1+|r|)|\bar{y}|+|q|)|s|\}$,

where now $Y' = (\bar{y},q)$, $\bar{y} \in \Omega \times \mathbb{R}$, $q \in \mathbb{R}^n$, and

$$\delta F = F_z + \frac{p}{|p|^2}\cdot F_x .$$

Further relaxation of F3* is also possible; (see [28],[4],[21]).

The proofs of the first derivative estimates in Theorem 1 follow the analogous quasilinear situation, as treated by Ladyzhenskaya and Uralt'seva (see [9],[1]), with the exception that non-divergence results of Krylov and Safonov [16], Trudinger [28] are substituted for the divergence structure approach in these works. There are three different approaches to the second derivative bounds. A modification of Lions' technique [22], (which was extended to cover interior estimates by Lenhart [19]), is used by Caffarelli, Nirenberg and Spruck [4] while in [29] we deduced interior (and subsequently global) second derivative bounds by a completely different interpolation approach; (see also [32]). In [24], [21] a further method based on Krylov's argument [13] is provided. These three approaches all have merit; the Lions' approach permits a slight weakening of the concavity condition, the author's approach dispenses with conidition F2 while the Krylov approach turned out to be essential for the treatment of second derivative bounds for oblique boundary value problems in [24], [21].

Existence theorems follow from Theorem 1 when we adjoin a suitable hypothesis to control $\sup|u|$. In particular, taking account of our first remark above, we have the following result which includes the Bellman case.

<u>THEOREM 2</u>. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with boundary* $\partial\Omega$ *satisfying an exterior cone condition,* $g \in C^0(\partial\Omega)$ *and suppose that* F *is of the form* (13) *where* V *is countable and* $\{F_\nu\}$ *satisfy* F1..F4 *uniformly in* ν *, together with the condition*

$$(14) \qquad (\operatorname{sign} z)\, F_\nu(x,z,p,0) \leq \bar{\mu}\lambda_\nu(1+|p|)$$

for all $X \in \Gamma, |z| \geq M_0$ *for some positive constants* $\bar{\mu}, M_0$. *Then the Dirichlet problem* (10) *is classically solvable with solution* $u \in C^0(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$ *for some* $\alpha > 0$ *depending on* n *and* μ .

The solution of (10) is unique if $F_{\nu z} \leq 0$ for all $\nu \in V$. In fact the proof of Theorem 2 proceeds through this case, (which can be handled by the method of continuity and approximation near $\partial\Omega$ [29]) , using the Leray-Schauder method. The exterior cone condition yields the necessary equicontinuity at the boundary for solutions of approximating problems but can be replaced by weaker barrier conditions.

Theorems 1 and 2 can be extended to parabolic equations of the form ,

$$(15) \qquad \frac{\partial u}{\partial t} = F[u] = F(t,x,u,Du,D^2u)$$

in cylinders $D = (0,T] \times \Omega$ where now F is defined on the set $\Gamma_T = (0,T] \times \Gamma$ for some positive T. The natural conditions F1..F4 can be carried over in the form :

$\tilde{F}1$: $\qquad \lambda \geq \lambda_0$, $\Lambda \leq \mu\lambda_0$ (Uniform parabolicity) ;

$\tilde{F}2$: $\qquad |F(t,x,z,p,0)| \leq \lambda_0\mu_0(1+|p|^2)$;

$\tilde{F}3$: $\quad (1+|p|)|F_p|,|F_z|,|F_x| \leqq \lambda_0\mu_1(1+|p|^2+|r|)$;

$\tilde{F}4$: $\quad F_{yy}(X) \leqq \lambda_0\mu_2\ \{(1+|r|)|Y'|+|s|\}\ |Y'|,|F_t| \leqq \lambda_0\mu_2(1+|r|)$,

with Γ replaced by Γ_T and λ_0 a further positive constant (depending on K). Conditions F3*, F4* may be extended analogously with $|F_t| \leqq \lambda_0\mu_2(1+|r|)^3$ in F4*. For a classical solution u of (15), with $D^2u, \frac{\partial u}{\partial t} \in C(D)$, we then obtain, in place of (11), (12), the estimates

(16) $$|Du|_{0,\alpha;D'} \leqq C ,$$

(17) $$|D^2u, \frac{\partial u}{\partial t}|_{0,\alpha;D'} \leqq C$$

where α and C are as before except that they also depend on λ_0, and $\Omega',\Omega,\partial\Omega$ are replaced by $D',D,\partial'D$ respectively where $\partial'D = (\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})$.

Extending the existence result, Theorem 2, we now deduce the classical solvability of the first initial boundary value problem ,

(18) $$\frac{\partial u}{\partial t} = F[u] = \inf_{\nu \in V} F_\nu(t,x,u,Du,D^2u) \quad \text{in} \quad D,$$

$$u = g \quad \text{on} \quad \partial'D ,$$

where $\partial\Omega$ satisfies an exterior cone condition, $g \in C^0(\partial'D)$ and F_ν satisfy (for countable V) the amended structure conditions $\tilde{F}1..\tilde{F}4$ uniformly in ν , together with a condition ensuring an apriori sup bound such as

$$(19) \qquad z F_\nu(x,z,0,0) \leq \bar{\mu} z^2$$

for $|z| \geq M_0$.

GLOBAL REGULARITY

When the boundary data are sufficiently smooth, the solutions of the Dirichlet problem (10), (whose existence are asserted in Theorem 2) are correspondingly globally smooth. In fact, by combining the global derivative bounds in [29] and Krylov's boundary Hölder estimates [14], we can deduce the following global regularity result.

THEOREM 3 *Let* u *be a solution of the classical Dirichlet problem* (10) *where* $\partial\Omega \in C^3$, $g \in C^3(\Omega)$ *and* F *satisfies* F1..F4 . *Then* $u \in C^{2,\alpha}(\bar{\Omega})$ *for some positive* α *depending on* n, μ *and*

$$(20) \qquad |\mu|_{2,\alpha;\Omega} \leq C$$

where C *depends on* $n, \mu, \mu_0, \mu_1, \mu_2, \partial\Omega, g$ *and* $|u|_{0;\Omega}$.

We remark that we can permit $\partial\Omega \in C^{2,\beta}$, $g \in C^{2,\beta}(\Omega)$ for some $\beta > 0$, in the above theorem and moreover if only $\partial\Omega \in C^{1,\beta}$, $g \in C^{1,\beta}(\bar{\Omega})$ and F1..F3 hold, we conclude $u \in C^{1,\alpha}(\bar{\Omega})$, [15]. In these situations, α depends also on β . The key to the boundary regularity is a remarkable result of Krylov [14] concerning linear, uniformly elliptic equations,

$$(21) \qquad Lu = a^{ij}(x) D_{ij}u = f ,$$

with principal coefficients satisfying

$$\lambda_0 |\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda_0 |\xi|^2 ,$$

for all $\xi \in \mathbb{R}^n$, for some positive λ_0, Λ_0. If $u \in C^0(B^+ \cup T) \cap C^2(B^+)$ satisfies (21) in $B^+ = B_1^+$ and $u = 0$ on $T = T_1$, where B_ρ^+ is the half ball $\{|x| < \rho, x_n > 0\}$ and T_ρ the flat boundary portion $\{|x| < \rho, x_n = 0\}$, then the function $v = u/x_n \in C^\alpha(B^+ \cup T)$ for some $\alpha = \alpha(n, \Lambda_0/\lambda_0) > 0$ and moreover, for any $\rho < 1$,

$$(22) \qquad [v]_{\alpha; B_\rho^+} \leqq C(|u|_0 + |f/\lambda_0|_0),$$

where C depends on $n, \Lambda_0/\lambda_0$ and ρ. Krylov's original proof was simplified by Caffarelli; a version is presented in [21]. An alternative approach to boundary regularity, also relying on an interesting linear result, was given by Caffarelli, Nirenberg and Spruck [3],[4]. For the assertions of Theorem 3, it suffices to only have the Hölder estimate (22) for $v = D_n u$ on T. However the following consequence of the full strength of Krylov's estimate is crucial for more general regularity considerations.

<u>THEOREM 4</u> *Let* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfy* (21) *in* Ω *with* $u = g$ *on* $\partial\Omega$ *where* $\partial\Omega \in C^{1,\beta}, g \in C^{1,\beta}(\bar{\Omega})$ *for some* $\beta > 0$. *Suppose that for any* $\Omega' \subset\subset \Omega$,

$$(23) \qquad [Du]_{\beta;\Omega'}, \quad |f/\lambda_0|_{0;\Omega'} \leqq K(\mathrm{dist}(\Omega', \partial\Omega))^{-\beta}$$

Then $u \in C^{1,\alpha}(\bar{\Omega})$, *for some* $\alpha = \alpha(n, \Lambda_0/\lambda_0, \beta)$ *and*

$$(24) \qquad |u|_{1,\alpha;\Omega} \leqq C\,(|u|_{0;\Omega} + K + |g|_{1,\beta;\Omega}),$$

where C *depends on* $n, \Lambda_0/\lambda_0, \beta$ *and* $\partial\Omega$.

We indicate the proof when $\partial\Omega \in C^2$, $f \in L^\infty(\Omega)$ and $g = 0$. The general case follows by modification similar to the barrier arguments in [10], [20]. By locally flattening the boundary $\partial\Omega$, we can reduce to the situation described above in the half ball B^+. Extending u by odd reflection to the entire ball $\{|x| < 1\}$, we consider the function

$$w(x,y) = u(x+y) + u(x-y) - 2u(x)$$

for $|x|, |y| < {}^1/4$. Using (22), we estimate

$$|w| = |(x_n+y_n)v(x+y)+(x_n-y_n)v(x-y)-2x_n v(x)|$$

$$\leqq C(|x_n|+|y_n|)|y|^\alpha (|u|_0 + |f/\lambda_0|_0)$$

$$\leqq C |y|^{1+\gamma} (|u|_0 + |f/\lambda_0|_0)$$

provided $0<\gamma<\alpha$ and $|x_n| \leqq |y|^{1+\gamma-\alpha}$. Next, by the assumed interior estimate (23), we have

$$|w| \leqq 2K|y|^{1+\beta}/(|x_n|-|y_n|)^\beta$$

$$\leqq C(\alpha,\gamma)K |y|^{1+\beta(\alpha-\gamma)}$$

for $|x_n| \geqq |y|^{1+\gamma-\alpha}$. Choosing γ appropriately, say $\gamma = \alpha\beta/(1+\beta)$ we now conclude $u \in C^{1,\gamma}(\bar{B}^+_{1/4})$ by virtue of [27, Ch,5,Prop.8] and Theorem 4 follows. The above considerations also extend to parabolic problems. Theorem 4 also extends an earlier result of Lieberman which requires the interior estimate for Du in (23) to have a special form.

OBLIQUE BOUNDARY VALUE PROBLEMS

When G is oblique, we call u a classical solution of the boundary value problem (1),(2) if $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ satisfies (1) and (2) pointwise. For linear boundary conditions of the form (8), the classical solvability of the uniformly elliptic Bellman equation, under appropriate coefficient and boundary smoothness was established by Lions and Trudinger [24]. This work was extended to operators F and G satisfying the natural structure conditions by Lieberman and Trudinger [21] resulting in the following theorem.

THEOREM 5 *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with boundary* $\partial\Omega \in C^{2,\beta}$ *for some* $\beta > 0$. *Suppose that* F *satisfies the hypotheses of Theorem 2 and that* $G \in C^{0,1}(\Gamma')$ *satisfies the structure conditions* G2,G3 *together with the condition*

$$(25) \qquad (\operatorname{sign} z)\, G(x,z,p) < 0$$

for $|z| \geq M_0(|p|)$ *for some increasing function* M_0 . *Then the boundary value problem* (1),(2) *is classically solvable with solution* $u \in C^{1,\alpha}(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$ *for some* $\alpha > 0$.

Alternative conditions to (25), (14) are possible [21]. Furthermore in the papers [24],[21] global second derivative bounds are established under further smoothness of $\partial\Omega$ and G and global second derivative Hölder estimates are derived in the case of one operator F. These latter estimates can be improved by means of Theorem 4 so that they also embrace the uniformly elliptic Bellman equation and facilitate the derivation of the second derivative bounds by the interpolation argument of [29]. Corresponding to Theorem 3, we now have the following global estimates [24].

THEOREM 6 *Let* u *be a classical solution of the boundary value problem* (1),(2) *where* $\partial\Omega \in C^4$ *and* $F \in C^2(\Gamma)$, $G \in C^2(\Gamma')$ *satisfy* F1..F4,G2,G3. *Then* $u \in C^{2,\alpha}(\bar{\Omega})$ *for some positive* α *depending on* n,μ *and*

(26) $$|u|_{2,\alpha;\Omega} \leq C$$

where C *depends on* $n,\mu,\mu_0,\mu_1,\mu_2,\partial\Omega,|u|_{0;\Omega}$ *and* D^2G.

The smoothness hypotheses on $\partial\Omega$, F and G can be relaxed slightly. The effect of Theorem 4 above is to eliminate a restriction (6.3) (involving Hölder continuity of F_r) in the corresponding estimates in [24], Theorems 1.1, 6.2. Both Theorems 3 and 6 extend to operators of the form (13), thereby extending Krylov's $C^{2,\alpha}(\bar{\Omega})$ regularity for the Dirichlet problem for the uniformly elliptic Bellman equation to oblique boundary conditions satisfying the natural conditions G2,G3. We note here that in the papers [21], [24] only $C^{1,1}(\bar{\Omega})$ regularity is proved in this case; accordingly $C^{2,\alpha}(\bar{\Omega})$ may be substituted for $C^{1,1}(\bar{\Omega})$ in Theorem 1.2 of [24] and Theorem 7.11 of [21].

BELLMAN OBSTACLE PROBLEMS

Obstacle problems for the Bellman equation are related to the optimal stopping of controlled diffusion processes [13]. If ψ is a smooth function on Ω we may consider the following types of problems:

I $\quad \inf(F[u],\psi-u) = 0$ in Ω, $G[u] = 0$ on $\partial\Omega$;

II $\quad F[u] = 0$ in Ω, $\inf(G[u],\psi-u) = 0$ on $\partial\Omega$;

III $\quad \inf(F[u],\psi-u) = 0$ in Ω, $\inf(G[u],\psi-u) = 0$ on $\partial\Omega$.

Problems I, II, III will be referred to as obstacle, Signorini, Signorini-obstacle respectively and in the last two cases the operator G should be oblique. Let us assume here that F is a uniformly elliptic Bellman operator (4) with coefficients a_ν^{ij}, b_ν^i, c_ν, f_ν, $\nu = 1,\ldots\infty$, $\in C^{1,1}(\bar{\Omega})$ with uniformly bounded $C^{1,1}(\bar{\Omega})$ norms, and $c_\nu \leqq 0 \; \forall \nu$. The classical Dirichlet problem, $G = g-z$, for I was treated by Lenhardt [19] and more recently by Perthame [26] who established global $C^{1,1}(\bar{\Omega})$ regularity, following Jensen's work [11] for the linear operator case. Oblique boundary value problems are treated by Lions and Trudinger [25], utilizing the key linear equation result of Caffarelli, Nirenberg and Spruck [3]. We may formulate some resultant existence theorems as follows, for operators G of the form

(27) $$G[u] = \inf_\nu (M_\nu u - g_\nu)$$

where the linear operators

(28) $$M_\nu u = \beta^i_\nu D_i u + \gamma_\nu u \;,$$

satisfy :

(29) $$0 < \lambda_0 \leqq \beta_\nu \cdot \nu \;;$$

$\beta^i_\nu, \gamma_\nu, g_\nu \in C^{1,1}(\bar{\Omega})$ with uniformly bounded $C^{1,1}(\bar{\Omega})$ norms ;

$$\gamma_\nu \leqq 0 \;\; \forall \nu \in V \,, \qquad \sup \gamma_\nu + \sup c_\nu < 0$$

<u>THEOREM 7</u> *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with boundary* $\partial\Omega \in C^4$, *and* $\psi \in C^{1,1}(\bar{\Omega})$. *Then the above problems* I,II,III *are uniquely solvable with solutions* $u \in C^{0,1}(\bar{\Omega}) \cap C^{1,1}(\Omega)$, $G[u] \in C^0(\bar{\Omega})$, *provided in case* I *we also have* $G[\psi] \leq 0$ *on* $\partial\Omega$. *In this case* $u \in C^1(\bar{\Omega}) \cap C^{1,1}(\Omega)$ *and moreover if* $(M_\nu, g_\nu) = (M,g)$ *is constant in* ν, *then* $u \in C^{1,1}(\bar{\Omega})$.

These results also extend to parabolic operators and to more general F[25]. It is known from the linear case [2], that one cannot expect global regularity beyond $C^{1,\alpha}(\bar{\Omega})$, for certain $\alpha < 1$, for the Signorini problems II and III.

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