BEHAVIOUR OF NONPARAMETRIC SOLUTIONS AND FREE BOUNDARY REGULARITY

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§0 Introduction

0.1 We shall study the boundary behaviour of solutions of the nonparametric least area, or area-type problems in this paper. The main strategy is to reduce the problems to some free boundary problems so that the known regularity theory for free boundaries applies. We will restrict ourselves to the two-dimension case. Since the asymptotic behaviour of free boundaries at their singular points are well-understood in two-dimension, see [C,R], the proofs for this case are particularly simple.

0.2 Let us consider first the Dirichlet problem for the minimal surface equation in a bounded C^2 - domain Ω of $\mathbb{R}^2$.

$$(0.1) \qquad \begin{aligned} &\frac{\partial}{\partial x_1}\left[u_{x_1} / \sqrt{1+|Du|^2}\right] + \frac{\partial}{\partial x_2}\left[u_2 / \sqrt{1+|Du|^2}\right] = 0 \quad \text{in } \Omega \\ &u\big|_{\partial\Omega} = \varphi \qquad\qquad \text{on } \partial\Omega . \end{aligned}$$

It was well-known, see for example [GT, chapter 13], that when Ω is convex, the Dirichlet problem (0.1) has a unique solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ for every $\varphi \in C^0(\partial\Omega)$, and that if Ω is not convex, then one can construct a smooth φ for which (0.1) is not solvable. These boundary datas for nonsolvability of (0.1) have a common nature that they are higher oscillatory near a point where $\partial\Omega$

is concave. In fact, it has been studied by several authors, particularly by C. P. Lau [La] and G. Williams [W1] recently, that even when Ω is not convex the Dirichlet problem (0.1) is solvable for all boundary datas which are close to zero or linear in Lipschitz norm. See also [L,§4] for the related discussions for the minimal surface system.

0.3 In general, one studies instead the solution of the following variational problem:

$$\min I(v) \quad \text{for} \quad v \in BV(\Omega) , \tag{0.2}$$
$$I(v) = \int_\Omega \sqrt{1+|Dv|^2} + \int_{\partial\Omega} |v-\varphi| .$$

Here $BV(\Omega)$ denotes the set of functions v having bounded variation on Ω in the sense that

$$\limsup_i \int_n |Dv_i| < \infty$$

for some sequence v_i of Lipschitz functions converge to v in the $L_1(\Omega)$ norm.

Geometrically, in (case v is a Lipschitz function on $\bar{\Omega}$, $I(v)$ is just the 2-dimensional area of the Lipschitz surface obtained by taking the union of the graph of v over Ω and that part of the boundary cylinder $\partial\Omega \times \mathbb{R}$ which is enclosed by the curves $\{(x,\varphi(x)) : x \in \partial\Omega\}$ and $\{(x,v(x)) : x \in \partial\Omega \}$.

In [G1] it is shown that one can always find a unique solution $u \in C^2(\Omega) \cap BV(\Omega)$, which satisfies the minimal surface equation in Ω, of the problem (0.2). Furthermore, if we decompose $\partial\Omega$ into

$$\partial\Omega = \partial_+(\Omega) \cup \partial_-(\Omega) \cup \Gamma , \tag{0.3}$$

where
$$\partial_+(\Omega) = \text{interior of } \{x \in \partial\Omega : H_{\partial\Omega}(x) \geq 0\} ,$$
$$\partial_-(\Omega) = \{x \in \partial\Omega : H_{\partial\Omega}(x) < 0 \} \text{ and}$$
$$\Gamma = \partial\Omega \sim (\partial_+(\Omega) \cup \partial_-(\Omega)) ,$$

$H_{\partial\Omega}(x)$ = mean curvature of $\partial\Omega$ with respect to inward normal at $x \in \partial\Omega$. Then $u = \varphi$ on $\partial_+(\Omega)$, and u is Hölder continuous at each point of $\partial_+(\Omega)$ provided φ is Lipschitz continuous.

In [SL1], L. Simon proved the following remarkable theorem.
Theorem (L. Simon) If $\partial\Omega$ is C^4 then u is Hölder continuous at each point of $\partial_-(\Omega)$, and the restriction of u to $\partial_-(\Omega)$ is locally Lipschitz continuous provided φ is also Lipschitz continuous there.

The behaviour of u near Γ has also been studied, see for examples [SL2] and [W2].

0.4 Several years ago, Lau and I [LL] had made an observation that the nonparametric solution of (0.2) is actually the unique solution to a parametric obstacle problem. This enabled us to reduce the study of the boundary behaviour of the trace of u on $\partial\Omega$ to the study of the free boundary in a variational inequality for the minimal surface operator. By combining the regularity theory for the free boundary, see [C], and the above cited theorem of L. Simon, we show the higher regularity of the trace u over the part of $\partial_-(\Omega)$ where $u \neq \varphi$ and the $C^{1/2}$ - Hölder estimate of u near such points.

These studies suggest the following two general problems:

(A) It is desirable to have a proof of L. Simon's theorem (or the main result of [LL]) by the free boundary method.

(B) The technique of L. Simon may suggest a new method for studying the free boundary problems. Of particular interest is when the free boundary intersects the prescribed boundary.

0.5 The present work is a preliminary report of our studies along these lines. In section 1, we show that if $\partial_-(\Omega)$ is $C^{2,\alpha}$ (concerning the minimum smoothness hypothesis on $\partial\Omega$, [SL1] seems to require at least $C^{3,\alpha}$) then the restriction of u to the part of $\partial_-(\Omega)$ where $\varphi \neq u$ is locally $C^{1,\alpha}$. The higher regularity follows as in [LL] provided that $\partial\Omega$ is of a higher smoothness class.

In section 2, we consider the equation for surfaces with prescribed mean curvature in a C^2, bounded domain $\Omega \subseteq \mathbb{R}^2$

$$(0.4) \qquad \frac{\partial}{\partial x_1}\left[u_{x_1} / \sqrt{1+|Du|^2}\right] + \frac{\partial}{\partial x_2}\left[u_{x_2} / \sqrt{1+|Du|^2}\right] = H(x) .$$

A necessary condition for (0.4) to have a solution $u \in C^2(\Omega)$ is that

$$\left|\int_A H dx\right| < H_1(\partial A) \tag{0.5}$$

for every Caccioppoli set $A \subset \Omega$, $\emptyset < A \neq \Omega$, see [G2].

In [G2], E. Giusti studied the extremal case, i.e., H satisfies (0.5) and

$$\int_\Omega H dx = H_1(\partial\Omega) . \tag{0.6}$$

His main result can be roughly described as follows.

If Ω is a C^2 , bounded domain, and if H is a Lipschitz function on $\bar{\Omega}$ which satisfies (0.5), (0.6). Then there is a unique solution u of (0.4) (up to an additive constant) such that

(i)
$$\lim_{x\to x_0} \frac{Du}{\sqrt{1+|Du|^2}} = \nu(x_0)$$

uniformly for $x_0 \in \partial\Omega$, $\nu(x_0)$ being the exterior unit normal to $\partial\Omega$ at x_0 ;

(ii) if in a neighbourhood of $x_0 \in \partial\Omega$ we have
$$H_{\partial\Omega}(\infty) < H(x) ,$$
then $u(x)$ is bounded there; and

(iii) if instead we have
$H_{\partial\Omega}(x) \equiv H(x)$ in an open arc $\ell \subset \partial\Omega$, then

$$\lim_{x \to x_0} u(x) = + \infty$$

uniformly in x_0 in any compact set K of ℓ .

It should be noted that (0.5) and (0.6) already imply that

$$H_{\partial\Omega}(x) \leq H(x) \quad \text{on} \quad \partial\Omega \ . \tag{0.7}$$

By the similar method as in section 1 of this paper, we will recover a part of the Giusti's theorem. Furthermore we will show the smoothness of u restricted to the part of $\partial\Omega$ where $H_{\partial\Omega}(x) < H(x)$ provided H and $\partial\Omega$ are smooth there.

In section 3, we generalize to some problems for a general class of elliptic equations which comes from a parametric elliptic integral.

L. Simon has informed us that his proof in [SL1] can be generalized to the equation of prescribed mean curvature with some additional technical complexity. Besides various curvature integral estimates, his proof relies crucially on a Sobolev - type inequality on these surfaces. This inequality does not seem to be available for surfaces which minimize an elliptic parametric integral.

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§1 The Minimal Surface Case

1.1 Let Ω be a bounded C^2 domain in $\mathbb{R}^2$ and $\varphi \in C^0(\partial\Omega)$. We are here interested in the behaviour of the trace of the unique solution u of the nonparametric least area problem (0.2). The trace of u over $\partial\Omega$ is defined by the requirement that

$$\lim_{\rho\to 0^+} \rho^{-2}\int_{\Omega\,\cap\,\{\xi:\,|\xi-x|\,<\,\rho\}} |u(\xi) - u(x)|\,d\xi = 0 \tag{1.1}$$

for all points $x \in \partial\Omega$ where such a value $u(x)$ exists.

Let $\gamma = \gamma_- \cup \gamma_+$, where

$$\begin{cases} \gamma_+ = \{x \in \partial_-(\Omega) : \varphi(x) > u(x)\} \\ \gamma_- = \{x \in \partial_-(\Omega) : \varphi(x) < u(x)\} \end{cases} . \tag{1.2}$$

Up to linear measure zero sets, γ_+ , γ_- , and γ are well-defined. Suppose $x_0 \in \gamma_+(\gamma_-)$, and $\partial\Omega \cap B_{r_0}(x_0) \sim \gamma_+$

($\partial\Omega \cap B_{r_0}(x_0) \sim \gamma_-$, respectively) is a set of H_1 - measure zero the aim of this section is to show the following.

Theorem 1 *Let* Ω *be a* $C^{2,\alpha}$, $0 < \alpha < 1$, *domain in* $\mathbb{R}^2$, *and let* φ , u , γ_+ (γ_-) *be as above. Suppose* $x_0 \in \gamma_+(\gamma_-)$ *so that* $\partial\Omega \cap B_{r_0}(x_0) \sim \gamma_+$ ($\partial\Omega \cap B_{\gamma_0}(x_0) \sim \gamma_-$, *respectively) is a set of* H_1*-measure zero, for some* $\gamma_0 > 0$. *Then* u *restricted to* $\partial\Omega \cap B_{\gamma_{0/2}}(x_0)$ *is a* $C^{1,\alpha}$ *function. Furthermore its* $C^{1,\alpha}$ *- norm depends only on* $C^{2,\alpha}$ *- norm of* $\partial\Omega$, $L_\infty(\partial\Omega)$ *- norm of* φ , γ_0 , *and* $\inf\{|H_{\partial\Omega}(x)| : x \in \partial\Omega \cap B_{\gamma_0}(x_0)\}$.

Corollary 1 *Under the same hypothesis is in Theorem 1,* u *is Hölder continuous in* $B_{\gamma_{0/2}}(x_0) \cap \bar{\Omega}$ *with Hölder exponent exactly equal to* $\frac{1}{2}$. *If, in addition,* $\partial\Omega$ *is of class* $C^{k,\alpha}$ *or analytic, then* u *restricted to* $B_{\gamma_{0/2}}(x_0) \cap \partial\Omega$ *is* $C^{k-1,\alpha}$ *or analytic, respectively, for* $k = 3,4,\ldots,$ *and* $0 < \alpha < 1$.

Corollary 1 follows from Theorem 1 as in [LL].

1.2 We need some preliminary reductions. Let φ, u, Ω be as above. We refer to [L,§4] for the proof of the following.

LEMMA 1 *Let* Q *be the multiplicity one integral 2-current in* $\mathbb{R}^3$ *with* $\operatorname{spt}(Q) \subseteq \partial\Omega \times \mathbb{R}$ *and* $\partial Q = T - Tu$, *where* $T = \{(x,\varphi(x)) : x \in \partial\Omega\}$, $Tu = \{(x,u(x)) : x \in \partial\Omega\}$. *Then* $\Lambda = \operatorname{graph}(u) + Q$ *is the unique area* *minimizing integral current whose boundary is the given* T *and whose support lies in* $\bar{\Omega} \times \mathbb{R}$.

COROLLARY 2 $\Lambda \sim T$ *is a* $C^{1,1}$ *surface with the* $C^{1,1}$ *character depending only on the* $L_\infty(\partial\Omega)$ *norm of* φ , *the* $C^{1,1}$ *character of* $\partial\Omega$, *and the distance to* T .

LEMMA 2 *Let* φ , u , Ω *be as before, we defined* φ^* *as follows*
$\varphi^* = \begin{cases} \varphi & \text{on } \partial\Omega \setminus B_{r_0}(x_0) \\ \varphi + 2r_0 & \text{on } \partial\Omega \cap B_{r_0}(x_0) \end{cases}$. *Let* u^* *be the corresponding solution of* (0.2) . *Then* $u \equiv u^*$.

PROOF The conclusion follows from Corollary 2 and the generalized maximum principle of R. Finn, see for example [GT, Thm. 13.10].

1.3 From now on, we will restrict ourselves to a neighbourhood of each point in $\gamma_* = \{(x,u(x)) : x \in \gamma_+ \cap B_{\gamma_{0/2}}(x_0)\}$ and to study the local behavior of Λ near such points. Notice that $\mathrm{dist}(\gamma_*,\partial\Lambda) \geq \gamma_{0/2}$ by the virtue of Lemma 2. For convenience, we fix a point $p \in \gamma_*$, and introduce a new coordinate system $(y,y_3) = (y_1,y_2,y_3)$ such that

(i) $p = (0,0,0) \in \mathbb{R}^3$,

(ii) the y_1-axis is the same as x_3-axis ,

(iii) the (y_2,y_3) plane is the tangent plane of $\partial\Omega \times \mathbb{R}$ at p ,

(iv) $e_3 = (0,0,1)$ is the unit normal of $\partial\Omega \times \mathbb{R}$ at p , hence normal to Λ at p by Corollary 2 .

By Lemma 2, Corollary 2, Λ locally, say over $D_\delta = \{y \in \mathbb{R}^2 : |y| < \delta\}$, $\delta = \delta(\gamma_0, \partial\Omega) > 0$, can be represented as the graph of a $C^{1,1}$ function $u = u(y)$ (we still use u for the convenience). $\partial\Omega \times \mathbb{R}$ over D_δ can be represented as the graph of $C^{2,\alpha}$ function $w(y) \equiv w(y_2)$. Let us assume that $\delta = 1$ (This can be achieved by a suitable scaling in $\mathbb{R}^3$ around the point p), and let

$$K = \{ v \in C^{0,1}(D_1) : v \geq w \text{ in } B_1 \text{ and } v = u \text{ on } \partial B_1\} .$$

Then one easily verify that $u \in K$ solves the variational inequality

$$\int_{D_1} a_j(Du)(v-u)_j dy \geq 0 \quad \text{for all} \quad v \in K , \tag{1.3}$$

where $a_j(Du) = u_j/\sqrt{1+|Du|^2}$, $u_j = u_{y_j}$.

Let $h = u - w$ so that $h \in C^{1,1}(B_1)$. Let

$$\begin{aligned} \Lambda(h) &= \{y \in B_1 : h(y) = 0\} , \\ \Omega(h) &= \{y \in B_1 : h(y) > 0 \} , \text{ and} \\ F(h) &= \partial(\Omega(h)) \cup \partial(\Lambda(h)) . \end{aligned} \tag{1.4}$$

One sees immediately that $u(F(h)) = \gamma_*$ in $B_1 \times \mathbb{R}$.

The remaining section will devoted to show γ_* is the graph of a $C^{1,\alpha}$ - function over $\gamma_+ \cap B_{\gamma_{0/2}}(x_0)$. This is divided into two steps. The first step is to show $F(h)$, hence γ_* is a $C^{1,\alpha}$ - curve. The second step is to show γ_* is a C^1 - graph over $\gamma_+ \cap B_{\gamma_{0/2}}(x_0)$.

We recall the following result due to L. Caffarelli and N. Riviere (see [C-R]) : $\underline{0} \in F(h)$ be as above, then <u>either</u>

(i) $\underline{0}$ is the isolated point of $F(h)$ and

$$h(y_1,y_2) = ay_1^2 + by_2^2 + O(y_1^2+y_2^2) \ , \ \text{for} \quad (y_1,y_2) \in \Omega(h)$$

and $y_1^2 + y_2^2 \leq \eta^2$

<u>or</u>

(ii) $\underline{0}$ is a point of positive density with respect to $\Lambda(h)$, and $\underline{0}$ is a regular point of $F(h)$, i.e. $F(h) \cap B_\eta(0)$ is a C^1 - curve , and $h(y) = ay_2^2 + o(y_1^2 + y_2^2)$ for $(y_1,y_2) \in \Omega(h)$ $y_1^2 + y_2^2 \leq \eta^2$;

<u>or</u>

(iii) $\underline{0}$ is a point of zero density with respect to $\Lambda(h)$, and in $B_\eta(0)$, $\Lambda(h) \cap B_\eta(0) \subset \mathbb{C} = \{y, \theta(y,(\pm 1,0)) \leq \Phi^{-1}(h)\}$, and $h(y) = ay_2^2 + o(y_1^2+y_2^2)$, for $(y_1,y_2) \in \Omega(h) \sim C$ and $(y_1,y_2) \in B_\eta(0)$.

Here a, b, η are positive constants for the problem (1.3), (1.4).

We notice, in particular, that $\Lambda(h) \cap B_\eta(0) \subset \mathbb{C}_1$ in the case (iii) where $C_1 = \{(y_1,y_2) \in B_\eta(0) : y_1 \geq |y_2|\}$.

1.4 Since $u(F(h)) = \gamma_*$ is the trace of a $BV(\Omega)$ function in our problem, we see that first case (i) never occurs.

Therefore to show the C^1 - regularity of $F(h)$, it suffice to show (iii) cannot occur. To do so, we consider the Jocabi-field equation on the minimal surfaces:

$$\Delta_M V + |A_M|^2 V = 0 , \tag{1.5}$$

where $|A_M|^2$ = square of the length of the second fundamental form of M, Δ_M = the Laplace-Beltrami operator on M.

In our coordinate system, we have $V = -u_{y_1} / \sqrt{1+|du|^2}$ which is a Lip function on $\overline{\Omega(h)}$, and such that

$$\begin{cases} \Delta_M V + |A_M|^2 V = 0 \quad \text{in} \quad \Omega(h) , \ V > 0 \quad \text{in } \Omega(h) \\ V|_{F(h)} = 0 . \end{cases} \tag{1.6}$$

On $\mathbb{C}_1$ we have $U_0(y) = r^\alpha \cos \alpha\,\theta$, where $\alpha = 2/3$, $\theta(y_1,y_2)$ is the angle between vectors (y_1,y_2) and $(-1,0)$ such that

$$\begin{cases} \Delta U_0 = 0 \quad , \quad U_0 > 0 \quad \text{in} \quad \mathbb{C}_1 \\ U_0|_{\partial\mathbb{C}_1} = 0 . \end{cases} \tag{1.7}$$

By the maximum-principle, we have

(1.8) $\qquad V \geq \epsilon U_0$ in $\mathbb{C}_1 \cap B_\eta(0)$, for some $\epsilon > 0$.

Then it is clear that V cannot Lipschitz continuous at $\underline{0}$, thus we obtain a contradiction and hence we have C^1 - regularity of $F(h)$. By [K,N] , we have $\gamma_* = u(F(h))$ is a $C^{1,\alpha}$ - curve.

1.5 Next we want to show $\gamma_* = u(F(h))$ is the $C^{1,\alpha}$ graph of a $C^{1,\alpha}$ - function defined on γ_* .

To show this we argue by contradiction. Since $V = -u_{y_1} / \sqrt{1+|Du|^2} > 0$ in $\Omega(h)$, and $V|_{F(h)} = 0$. Moreover u is uniformly C^2 in $\overline{\Omega(h)} \cap B_{1/2}$, by [C] . If γ_* is not a uniformly C^1 - graph over γ_+ we would obtain a point, say $\underline{0}$, on $F(h) \cap D_{1/2}$ such that, the second order "blow up" of h at $\underline{0}$ is of form:

(1.9) $$h_0(y) = ay_2^2 \ , \ 2a = -H_{\partial\Omega}(\underline{0}) > 0 \ .$$

That is, $h(y) = ay_2^2 + o(y_1^2+y_2^2)$.

On the other hand, we have, by (1.6) and Hopt-boundary point Lemma, that $\frac{\partial V}{\partial y_2} \neq 0$ at $\underline{0}$. Hence $\frac{\partial^2 h}{\partial y_1 , \partial y_2}(\underline{0}) \neq 0$. This contradicts to (1.9). Hence we have complete the proof of Theorem 1.

§2 Surface with prescribed mean curvature

2.1 Let Ω be a bounded C^2 domain in R^2 , and let H be a Lipschitz function on $\bar{\Omega}$. Consider the unique solution u (up to constants) of

(0.4) with H satisfying (0.5), (0.6). Let $\Omega_1 \subsetneq \Omega$ with $\operatorname{per}(\Omega_1) < \infty$, then the set $U = \{(x_1,x_2,x_3) \in \Omega_1 \times \mathbb{R} : x_3 < u(x)\}$ is a minimizer for the functional

$$(2.1) \qquad \int |DX_U| dx + \int HX_U dx dx_3 \quad \text{in} \quad \Omega_1 \times \mathbb{R} .$$

That means that for every set $V \subset \Omega_1 \times \mathbb{R}$, coinciding with U outside some compact set $K \subset \Omega_1 \times \mathbb{R}$, we have

$$(2.2) \qquad \begin{aligned} &\int_K |DX_U| d\underline{\bar{x}} + \int_K HX_U dx dx_3 \\ \le &\int_K |DX_U| dz + \int_K HX_U dx dx_3 \end{aligned}$$

see [G2] for further discussions.

Moreover, see [G2], if in a neighbourhood of $x_0 \in \partial\Omega$ we have $H_{\partial\Omega}(x) < H(x)$, then $u(x)$ is bounded above there. Notice that u is always bounded below in this case, see also [G2]. Thus we can assume that $u \geq 0$. Suppose there are positive numbers C_0, r_0 such that

$$(2.3) \qquad H_{\partial\Omega}(x) \le H(x) - C_0 \ , \quad \text{for all} \quad x \in B_{r_0}(x_0) \cap \partial\Omega \ ,$$

for a point $x_0 \in \partial\Omega$. We want to show the restriction of u on $B_{r_{0/2}}(x_0) \cap \partial\Omega$ is smooth in the following sense.

THEOREM 2 *Let* Ω *be a bounded* $C^{2,\alpha}$ *domain in* $\mathbb{R}^2$, *and let* H *be a Lipschitz function on* $\bar{\Omega}$ *which satisfies* (0.5), (0.6). *Suppose*

(2.3), *then the restriction of* u *to* $B_{r_{0/2}}(x_0) \cap \partial\Omega$ *is uniformly* $C^{1,\alpha}$, *and* u *is* $C^{1/2}$ *in* $\bar{\Omega} \cap B_{r_{0/2}}(x_0)$.

COROLLARY 3 *In Theorem 2, (i) if* Ω *is* $C^{k,\alpha}$, *and if* H *is* $C^{k-2,\alpha}$, *then the restriction of* u *to* $B_{r_{0/2}}(x_0) \cap \partial\Omega$ *is* $C^{k-1,\alpha}$, *for* $k = 3,4,\ldots$, $0 < \alpha < 1$; *(ii) if* Ω *and* H *are analytic, then* u *is also analytic on* $\partial\Omega \cap B_{r_{0/2}}(x_0)$.

Corollary 3 again follows from Theorem 2, see [K,N]. To show Theorem 2, we follow the same arguments as in Section 1.

Let Q be the multiplicity one integral current in $\mathbb{R}^3$ with $\mathrm{spt}(Q) \subseteq (\partial\Omega \cap B_{r_0}(x_0)) \times \mathbb{R}$ and $\partial Q = T - Tu$, where $T = \{(x,u^*(x)) : x \in \partial\Omega\}$, $Tu = \{(x,u(x)) : x \in \partial\Omega\}$,

$$u^*(x) = \begin{cases} u(x) & \text{if } x \in \partial\Omega \setminus B_{r_0}(x_0) \\ \\ u(x) + \eta(x) & \text{if } x \in B_{r_0}(x_0) \cap \partial\,\Omega \,. \end{cases}$$

Here $\eta(x)$ is a smooth function such that $0 \le \eta \le M_0$, and $\eta(x) \equiv M_0 = \sup\{u(x) : x \in \partial\Omega \cap B_{r_0}(r_0)\}$ for $x \in B_{r_{0/2}}(x_0) \cap \partial\Omega$. Then we have that, see [L,4] for the proof.

LEMMA 3 $\Delta = [\mathrm{graph}\ (u) + Q] \cap (B_{r_0}(x_0) \times \mathbb{R})$ *is the unique integral current which minimizes*

$$M(\$) + \int_{B_{r_0}(x_0) \cap \Omega} H(x)\, \eta_{\$}(x)\, dx$$

where $\eta_{\$}(x)$ *is the slice* $\langle \$, P, x\rangle\, (x_3)$, *see* [F, §4.2] , $P(x_1, x_3) = x$, *and* $\partial\$ = \partial\Lambda$.

One follows exactly the same procedure as in section 1 to obtain the conclusion of Theorem 2. The only difference is the Jacobi-field equation, see [L, §5], here we have

$$\text{(2.4)} \qquad \Delta_M V + (|A_M|^2 - H^2)\, V = \delta_3 H ,$$

where $V = \dfrac{1}{\sqrt{1+|Du|^2}}$ and $|\delta_3 H| \le |DH| V$.

However, estimates similar to (1.6) and (1.8) remain true.

2.2 If $H(x) \equiv H_{\partial\Omega}(x)$ in an open arc $\ell \subset \partial\Omega$, then $\lim_{x\to x_0} u(x) = +\infty$ uniformly in $x_0 \in K \subset\subset \ell$. This was shown by E. Giusti [G2]. This can be also deduced from Lemma 3 and the regularity of obstacle problems, see [L]. For if there is a sequence $x_i \in K$, with $\lim_{x_i\to x_0} u(x_i) = u(x_0) < \infty$, then $(x_0, u(x_0))$ will be a point of the support of the minimizer. One chooses a ball $\partial\Omega \cap B_{\gamma_0}(x_0, t)$ moving up-wards from $t < 0$ towards $(x_0, u(x_0))$. (Here γ_0 is small enough so that $B_{\gamma_0}(x, t) \subset K \times \mathbb{R})$. It must touch the free boundary at some point p which may different from $(x_0, u(x_0))$. Then locally at p we view vertical cylinder $\partial\Omega \times \mathbb{R}$ and the graph (u) as graphs over the tangent plane of $\partial\Omega \times \mathbb{R}$ at p . Applying Hopf-Boundary point Lemma at p to obtain a contradiction.

2.3 Let us consider the Dirichlet problem

$$(2.5)\qquad \begin{cases} \sum_{i=1}^{2} \dfrac{\partial}{\partial x_i}\left[\dfrac{\partial u}{\partial x_i} \Big/ \sqrt{1+|Du|^2}\right] = H(x) & \text{in } \Omega \\ U\big|_{\partial\Omega} = \varphi & \text{on } \partial\Omega \end{cases}$$

Suppose that $\varphi \in C^0(\partial\Omega)$, and H is a Lipschitz function on $\bar{\Omega}$ which satisfies

$$(2.6)\qquad \left|\iint_A H(x)dx\right| < \text{per}(A)\ , \text{ for all } \varphi \neq A \subset \Omega\ ,$$

and

$$(2.7)\qquad H_{\partial\Omega}(x) \geq H(x)\ , \text{ for all } x \in \partial\Omega\ .$$

Then (2.5) has a unique solution $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$. As in the case of (0.1), one asks the question that if (2.7) can be omitted when φ and $\partial\Omega$ are smooth, and when φ is almost linear in some sense. The answer turns out to be N_0 . One can easily construct, by using the generalized maximum - principle [GT, Thm 13.10], a smooth positive function H on a closed unit ball $\subseteq \mathbb{R}^n$ which verifies (2.6) $(\Omega \equiv B)$,and $\varphi \equiv$ constant, such that (2.5) is not solvable in the classical sense. One can also construct an example for which $H \equiv$ constant > 0 , $\varphi \equiv$ constant, and Ω smooth such that (2.6) valid, but (2.5) is not solvable in the classical sense. This can be achieved by a suitable perturbation of spherical cap which contains a closed hemisphere, and the generalized maximum - principle of R. Finn. We leave the details to reader.

§3 The Parametric elliptic integral case

3.1 Let Φ be an elliptic parametric integral of constant coefficient in $\mathbb{R}^3$, and let $F = \Phi^{\S}$ be the associated nonparametric integral, see [F, §5.1]. We consider the following problem:

$$(3.1) \qquad \begin{cases} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} F_{p_i}(Du) = 0 & \text{in } \Omega \\ u\big|_{\partial\Omega} = \varphi & \text{on } \partial\Omega \end{cases}$$

for a C^2, bounded domain Ω in $\mathbb{R}^2$.

The problem (3.1) is solvable, for each $\varphi \in C^0(\partial\Omega)$, in the classical sense if and only if $\partial\Omega$ is Φ - convex in the sense that, when for each $p \in \partial\Omega \times \mathbb{R}$, one views $\partial\Omega \times \mathbb{R}$ locally as a graph of a function V over the tangent plane in the inward normal direction, the function V becomes a subsolution of the quasi-linear equation of (3.1). See for example [GT, chapter 13]. In general, one can consider the solution of the following variational problem :

$$(3.2) \qquad \min I[u] \text{ , for } u \in BV(\Omega)$$

where $I[u] = \Phi(\Lambda)$, $\Lambda = \text{graph}(u) + Q$, and Q is the part of $\partial\Omega \times \mathbb{R}$ enclosing by two curves $\{(x,\varphi(x)) : x \in \partial\Omega\}$ and $\{(x,u(x) : x \in \partial\Omega\}$. (See section 1 for the related discussions).

Suppose the functional $\int_\Omega F(DV)\,dx$ is regular in the sense that $(D^2F) > 0$. Then, as in [L,§4], one can easily show the following.

LEMMA 4 *There is a unique solution of* (3.2). *Moreover,* Δ *is the unique* Φ *- minimizing integral current whose support lies in* $\bar{\Omega} \times R$ *and whose boundary is* $\{(x,\varphi(x)) : x \in \partial\Omega\}$.

We are here interested in the boundary behavior of u. Let $\partial\_(\Omega)$ denote the part of $\partial\Omega$ which is not Φ - convex, and γ_+, γ_- be defined by (1.2). Then we have the following.

THEOREM 3 Let Ω be a $C^{2,\alpha}$, $0 < \alpha < 1$, bounded domain in $\mathbb{R}^2$, and let φ, γ_+, u (γ_-) be as above. Suppose $x_0 \in \gamma_+$ (γ_-) so that $\partial\Omega \cap B_{\gamma_0}(x_0) \sim \gamma_+$ $(\partial\Omega \cap B_{\gamma_0}(x_0) \sim \gamma_-$, respectively) is a set of H_1 - measure zero, for some $\gamma_0 > 0$. Then u restricted to $\partial\Omega \cap B_{\gamma_{0/2}}(x_0)$ is a $C^{1,\alpha}$ - function provided Φ is of class C^3. Furthermore its $C^{1,\alpha}$ - norm depends only on $L_\infty(\partial\Omega)$ norm of φ, $C^{2,\alpha}$ - norm of $\partial\Omega$, γ_0, and Φ - convexity of γ_+ (γ_-). u is $C^{1/2}$ in $\bar{\Omega} \cap B_{\gamma_{0/2}}(x_0)$.

If, in addition, $\partial\Omega$ is $C^{k,\alpha}$ or analytic, Φ is $C^{k+1,\alpha}$ or analytic, respectively, for $k = 3,4,\dots$, $0 < \alpha < 1$. Then u restricted to $\partial\Omega \cap \beta_{r_{0/2}}(x_0)$ is $C^{k-1,\alpha}$ or analytic respectively.

3.2 For the proof of Theorem 3, Corollary 4 we refer to Section 1. We only need to remark that there is also a similar Jocabi-field equation which can be written as

$$F_{p_i p_j}(Du) u_{x_\ell x_i} u_{x_\ell x_j} - \frac{\partial}{\partial x_i}\left[V F_{p_i p_j}(Du) V_{x_j} \right] = 0 . \tag{3.3}$$

Here $V = \sqrt{1+|Du|^2}$ (See [SL1].)

In the parametric setting, we have the following

$$\mathrm{div}_M\,(\Sigma(v)\nabla_M V) + (A_M^2 \cdot \Sigma(v))V = 0 , \tag{3.4}$$

see [A] for the details.

The $C^{1,1}$ - estimate for the corresponding obstacle problem was proven in [L].

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