ON REMOVABLE ISOLATED SINGULARITIES OF SOLUTIONS TO A CLASS OF QUASI-LINEAR ELLIPTIC EQUATIONS

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1. INTRODUCTION

Let Ω be some open subset of $\mathbb{R}^N$ containing 0 and $\Omega' = \Omega \sim \{0\}$. Let u be a solution of

$$-\Delta u + u|u|^{q-1} = 0 \qquad \text{in } \Omega' . \tag{1.1}$$

Brezis and Véron [2] proved that u can be extended to be a solution of (1.1) in all of Ω if $q \geq N/(N-2)$, $N \geq 3$. Hence isolated singularities of (1.1) are "removable". Véron [8] showed that the exponent $N/(N-2)$ is the best possible because there exist singular solutions when $1 < q < N/(N-2)$. Aviles [1] generalized the result in [2] by replacing the Laplacian by some linear operators in divergence form. Vazquez and Véron showed that we can also replace the Laplacian by the quasi-linear p-Laplacian $\operatorname{div}(|Du|^{p-2}Du)$, $N > p > 1$. Here $Du = (D_1u, \ldots, D_Nu)$ denotes the gradient of the function of u.

A natural question is to ask whether the Laplacian can be replaced by a more general class of quasi-linear elliptic operators which include the above mentioned examples.

In this paper, we shall show that the Brezis-Veron result is indeed true for a wide class of quasi-linear operators satisfying certain growth and ellipticity conditions. Specific examples are given in section 4.

2. PRELIMINARIES

For simplicity we assume that $\Omega = \{x \in \mathbb{R}^N : |x| < 2\}$, $N \geq 2$, and we set $\Omega' = \Omega \sim \{0\}$. We consider the following equation

$$-\mathrm{div}\, A(x, Du) + B(x, u) = 0 , \tag{2.1}$$

where $A(x, p) = (A_1(x, p), \ldots, A_N(x, p))$ is a vector-valued function belonging to $C^0(\Omega \times \mathbb{R}^N) \cap C^1(\Omega \times (\mathbb{R}^N \sim \{0\}))$ for $(x, p) \in \Omega \times \mathbb{R}^N$ and $B(x, u) \in C^0(\Omega \times \mathbb{R})$ for $(x, u) \in \Omega \times \mathbb{R}$. Denote

$$E(x, p) = D_{p_j} A_i(x, p) p_i p_j = a_{ij}(x, p) p_i p_j , \tag{2.2}$$

$$T(x, p) = \sum_{i=1}^{N} a_{ii}(x, p) . \tag{2.3}$$

From now on, we shall use the convention that repeated indices represent summation from 1 to N . Furthermore, we assume the following ellipticity and growth conditions: for some constants $c_1 > 0$, $c_2 > 0$, $1 < m < N$,

$$|p|(|A(x, p)| + |D_{x_i} A_i(x, p)|) + N|p|^2 \sum_{i,j=1}^{N} |D_{p_j} A_i(x, p)| \leq c_1 |p|^m ,$$

for all $x \in \Omega$, $|p| \geq c_2$,

$$p_i A_i(x, p) \geq |p|^m - c_1 , \text{ for all } (x, p) \in \Omega \times \mathbb{R}^N ,$$

$A(x, 0) = 0$, for all $x \in \Omega$,

$|A(x, p)| \le c_1$, for all $x \in \Omega$, $|p| \le c_2$, (A1)

$D_{p_j} A_i(x, p)\xi_i\xi_j \ge c_1^{-1}(\kappa + |p|)^{m-2}|\xi|^2$,

for all $x \in \Omega$, $0 \ne p \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$, for some $\kappa \in [0, 1]$, (A2a)

$(A(x, p) - A(x, q))(p - q) \ge 0$,

for all $(x, p, q) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N$, (A2b)

$$\liminf_{t \to +\infty} \frac{B(x, t)}{t^{\frac{N(m-1)}{N-m}}} > 0 , \quad \limsup_{t \to -\infty} \frac{B(x, t)}{|t|^{\frac{N(m-1)}{N-m}}} < 0 ,$$

uniformly on Ω . (B1)

Definition 2.1. A function $u \in W^{1,m}_{loc}(\Omega') \cap L^{\infty}_{loc}(\Omega')$ is said to be a (weak) solution (resp. sub-solution) of (2.1) in Ω' if

$$\int_\Omega A_i(x, Du)D_i\varphi + B(x, u)\varphi \, dx = 0 \text{ (resp. } \le 0) \tag{2.4}$$

for all $\varphi \in C^1_o(\Omega')$ (resp. $0 \le \varphi \in C^1_o(\Omega')$).

Remark 2.2. By an approximation argument, we can take the test function φ in (2.4) to be in $\varphi \in W^{1,m}_o(\Omega')$.

Remark 2.3. By the regularity result of [4], any weak solution of (2.1) in Ω' has to be in $C^{1,\alpha}(\Omega')$ for some $0 < \alpha < 1$. So without loss of generality, we can always assume that $u \in C^{1,\alpha}(\Omega')$.

3. MAIN RESULT

Theorem 3.1. Suppose $1 < m < N$ and (A1), (A2), (B1) (as stated in section 2) hold. Let $u \in C^{1,\alpha}(\Omega')$, $0 < \alpha < 1$, be a solution of

$$-\operatorname{div} A(x, Du) + B(x, u) = 0 \qquad \text{in } \Omega' . \tag{3.1}$$

Then u can be extended to all of Ω so that the resulting function $\bar{u}$ is a (weak) solution of (2.1) in Ω. Hence by [4], $\bar{u} \in C^{1,\alpha}(\Omega)$ for some (may be different) $\alpha \in (0, 1)$.

To prove Theorem 3.1, we need the following two lemmata.

Lemma 3.2. Assume (A1), (A2), (B1). Suppose that $1 < m < N$, $q = \frac{N(m-1)}{N-m}$, $u \in W^{1,m}_{\ell oc} \cap L^{\infty}_{\ell oc}(\Omega')$ satisfies (in the weak sense)

$$-\operatorname{div} A(x, Du) + a\,u^q - C \le 0 \tag{3.2}$$

on $\{x \in \Omega' : u(x) > 0\}$, for some positive constants a and C. Assume that (A1), (A2) hold. Then

$$u(x) \le \frac{c_3}{|x|^{\frac{m}{q+1-m}}} \qquad \text{a.e. on } \{x : 0 < |x| < 1\}, \tag{3.3}$$

where c_3 is a constant depending on N, m, q, a, C and $\max\limits_{|x|=1} u(x)$.

Lemma 3.3. Under the hypotheses of Lemma 3.2, we have $u^+ \in L^{\infty}_{\ell oc}(\Omega)$.

Proof of Lemma 3.2. We shall use the convention that $c(m, q, \ldots)$ denotes some constant depending on $m, q, \ldots$.

Let $r_o > 0$ be given such that $4r_o < 1$. Consider the function

$$v(x) = L(r(x)^2 - r_o^2)^{-t} + M \tag{3.4}$$

defined on the annulus $D = \{x : r_o \le r(x) = (x_1^2 + \ldots + x_N^2)^{\frac{1}{2}} \le 4r_o\}$. L, M, t are some positive constants to be chosen. A routine computation shows:

$$D_i v(x) = -2Lt(r^2 - r_o^2)^{-t-1} x_i ,$$

$$|Dv(x)| = 2Lt(r^2 - r_o^2)^{-t-1} r ,$$

$$D_{ij} v(x) = -2Lt(r^2 - r_o^2)^{-t-1} \delta_{ij} + 4Lt(t+1)(r^2 - r_o^2)^{-t-2} x_i x_j ,$$

$$\begin{aligned}
\operatorname{Div} A(x, Dv) &= D_{p_j} A_i(x, Dv) D_{ij} v + D_{x_i} A_i(x, Dv) \\
&= \frac{(t+1)}{Lt} (r^2 - r_o^2)^t \, E(x, Dv) - 2Lt(r^2 - r_o^2)^{-t-1} \, T(x, Dv) \\
&\quad + D_{x_i} A_i(x, Dv) \\
&\le \frac{(t+1)}{Lt} (r^2 - r_o^2)^t \, c_1 \{2Lt(r^2 - r_o^2)^{-t-1} r\}^m \\
&\quad + c_1 \{2Lt(r^2 - r_o^2)^{-t-1} r\}^{m-1} ,
\end{aligned}$$

by (A1). We shall check that $|Dv| \ge c_2$ after L, t, M have been chosen.

$$\begin{aligned}
&\le (r^2 - r_o^2)^{-(m-1)(t+1)-1} (r + 2(t+1)) c_1 (2t)^{m-1} r^m L^{m-1} \\
&= (r^2 - r_o^2)^{-(m-1)(t+1)-1} c_4(t, m) r^m L^{m-1} ,
\end{aligned}$$

$$- av(x)^q \le - \frac{a}{2} L^q(r^2 - r_o^2)^{-tq} - \frac{a}{2} M^q . \tag{3.5}$$

Hence $\mathrm{Div}\, A(x, Dv) - av(x)^q + C \le 0$ if we choose

$$t = \frac{m}{q+1-m} ,$$

$$L \ge \left(\frac{2^{2m+1} c_4 \, r_o^m}{a} \right)^{\frac{1}{q+1-m}}$$

and

$$M \ge \max\left\{ \left(\frac{2C}{a} \right)^{\frac{1}{q}} , \max_{|x|=4r_o} u(x) \right\} . \tag{3.6}$$

By these choices, it can be easily checked that for sufficiently small $r_o > 0$, $|Dv| \ge c_2$. We now proceed to show that $u(x) \le v(x)$ in D. Choose $0 \le \varphi_n \in C_o^1(D)$ so that $\varphi_n \equiv 1$ on $\{x : (1+\frac{1}{n})r_o \le r(x) \le (4-\frac{1}{n})r_o\}$ and β a C^1 bounded function vanishing on $(-\infty, 0]$, nondecreasing on $[0, +\infty)$. Then we have

$$\begin{aligned} &\int_D -\mathrm{div}(A(x, Du) - A(x, Dv))\beta(u - v)\varphi_n \, dx \\ &= \int_D (A(x, Du) - A(x, Dv))\beta'(u - v)(Du - Dv)\varphi_n \, dx \\ &\quad + \int_D (A(x, Du) - A(x, Dv))\beta(u - v)D\varphi_n \, dx \quad 0 \end{aligned} \tag{3.7}$$

because $u < v$ near ∂D and (A2). Hence

$$\int_D a(u^q - v^q)\beta(u - v)\varphi_n \, dx \le 0 \quad \text{for all } n , \tag{3.8}$$

which implies $u \le v$ in D. In particular

$$u(x) \le v(2r_o) = c_5(m, q)r_o^{-t} + M \tag{3.9}$$

for x such that $r(x) = 2r_o$. By an iteration argument, we proved our assertion. Q.E.D.

Proof of Lemma 3.3. Let

$$\zeta_n(x) = \begin{cases} 0 & \text{if } |x| < \frac{1}{2n} \text{ or } |x| > 1 , \\ 1 & \text{if } \frac{1}{n} < |x| < \frac{1}{2} , \end{cases} \tag{3.10}$$

and $0 \le \zeta_n \le 1$, $|D\zeta_n| \le c_6 n$, for some constant $c_6 > 0$. Let β be a C^1 bounded function vanishing on $(-\infty, 0)$, nondecreasing on $[0, \infty)$. Denote

$$T_n = \{x : \frac{1}{2n} < |x| < \frac{1}{n}\} . \tag{3.11}$$

Choose $M \ge \max\left\{ \left(\frac{C}{a}\right)^{\frac{1}{q}} , \sup\limits_{\frac{1}{2} \le |x| \le 1} u(x)\right\}$. Then $\varphi = \beta(u - M)\zeta_n$ is a legitimate test function and

$$\begin{aligned} 0 &\le \int_\Omega (au^q - C)\beta(u - M)\zeta_n \, dx \equiv I_n \\ &\le -\int_\Omega A_i(x, Du)\{\beta(u - M)D_i\zeta_n + \zeta_n\beta'(u - M)D_i(u - M)^+\}dx \\ &\le \int_\Omega \beta(u - M)|A(x, Du)| \, |D\zeta_n|dx \qquad \text{by (A2)} , \\ &\le c_7(c_6, N)\|\beta\|_{L^\infty} n^{1 - \frac{N}{m}} \left\{\int_{T_n \cap \{x : u(x) > M\}} |A(x, Du)|^{\frac{m}{m-1}}dx\right\}^{\frac{m-1}{m}} . \end{aligned} \tag{3.12}$$

Here we use the fact that $p_i A_i(x, p) \ge 0$ which follows from (A2) and $A(x, 0) = 0$. Letting $\beta(t) \to \text{sign}^+ t$ $(= 1$ if $t > 0$, $= \frac{1}{2}$ if $t = 0$, $= 0$ if $t < 0)$, we have

$$0 \le I_n \le c_7 \, n^{1 - \frac{N}{m}} \left\{\int_{T_n \cap \{x : u(x) > M\}} |A(x, Du)|^{\frac{m}{m-1}}dx\right\}^{\frac{m-1}{m}} . \tag{3.13}$$

As in (3.12), taking $\varphi = (u - M)^+\zeta_{2n}^m$, we obtain

$$\int_\Omega \zeta_{2n}^m A_i(x, Du)D_i(u - M)^+ dx$$

$$\le m\int_{T_{2n}} \zeta_{2n}^{m-1}|A(x, Du)|(u - M)^+|D\zeta_{2n}|dx$$

which implies (by (A1))

$$\int_\Omega |D(u - M)^+|^m \zeta_{2n}^m\, dx$$

$$\le c_1\int_\Omega \zeta_{2n}^m\, dx + m\left\{\int_{\Omega\cap\{x : u(x) > M\}} \zeta_{2n}^m |A|^{\frac{m}{m-1}} dx\right\}^{\frac{m-1}{m}} \left\{\int_{T_{2n}} [(u - M)^+|D\zeta_{2n}|]^m dx\right\}^{\frac{1}{m}}$$

$$\le c_1|\Omega| + (m-1)\varepsilon^{\frac{m}{m-1}} \int_{\Omega\cap\{x : u(x) > M\}} \zeta_{2n}^m |A|^{\frac{m}{m-1}}\, dx$$

$$+ \varepsilon^{-m}\int_{T_{2n}} [(u - M)^+|D\zeta_{2n}|]^m\, dx \quad , \tag{3.14}$$

by Young's inequality. $\varepsilon > 0$ is to be chosen sufficiently small. By (A1) and (3.14), we have

$$\int_{\Omega\cap\{x : u(x) > M \text{ and } |Du| \ge c_2\}} \zeta_{2n}^m |A|^{\frac{m}{m-1}}\, dx \le \int_{\Omega\cap\{x : u(x) > M \text{ and } |Du| \ge c_2\}} \zeta_{2n}^m (c_1|Du|^{m-1})^{\frac{m}{m-1}}\, dx$$

$$\le c_1^{\frac{m}{m-1}} \Big\{c_1|\Omega| + (m-1)\varepsilon^{\frac{m}{m-1}} \int_{\Omega\cap\{x : u(x) > M\}} \zeta_{2n}^m |A|^{\frac{m}{m-1}}\, dx$$

$$+ \varepsilon^{-m}\int_{T_{2n}} [(u - M)^+|D\zeta_{2n}|]^m\, dx\Big\} . \tag{3.15}$$

But by (A1),

$$\int_{\Omega\cap\{x : u(x) > M \text{ and } |Du| \le c_2\}} \zeta_{2n}^m |A|^{\frac{m}{m-1}}\, dx \le c_1^{\frac{m}{m-1}}|\Omega| . \tag{3.16}$$

Combining (3.15) and (3.16), we obtain

$$\int_{\Omega\cap\{x:u(x)>M\}} \zeta_{2n}^m |A|^{\frac{m}{m-1}} dx \le c_8(c_1, c_2, m, |\Omega|)$$

$$\{1 + (m-1)\varepsilon^{\frac{m}{m-1}} \int_{\Omega\cap\{x:u(x)>M\}} \zeta_{2n}^m |A|^{\frac{m}{m-1}} dx$$

$$+ \varepsilon^{-m} \int_{T_{2n}} [(u-M)^+ |D\zeta_{2n}|]^m dx\} . \qquad (3.17)$$

Choose $\varepsilon > 0$ small enough so that $c_8(m-1)\varepsilon^{\frac{m}{m-1}} \le \frac{1}{2}$. (3.17) then gives

$$\int_{T_n\cap\{x:u(x)>M\}} |A|^{\frac{m}{m-1}} dx \le c_9(c_1, c_2, m, |\Omega|) \cdot \{1 +$$

$$+ \int_{T_{2n}} [(u-M)^+ |D\zeta_{2n}|]^m dx\} . \qquad (3.18)$$

By Lemma 3.2,

$$\int_{T_{2n}} [(u-M)^+ |D\zeta_{2n}|]^m dx \le c_6 (2n)^m c_3 (4n)^{\frac{m^2}{q+1-m}} \omega_N n^{-N}$$

$$\le c_{10}(m, c_3, c_6, N) n^{m-N+\frac{m^2}{q+1-m}} , \qquad (3.19)$$

where ω_N = the volume of the unit ball in $\mathbb{R}^N$.

Going back to (3.13), in view of (3.18), (3.19), we have

$$0 \le I_n \le c_7 n^{1-\frac{N}{m}} c_9 \left\{1 + c_{10} n^{m-N+\frac{m^2}{q+1-m}}\right\}^{\frac{m-1}{m}} . \qquad (3.20)$$

As $1 - \frac{N}{m} + \left(m - N + \frac{m^2}{q+1-m}\right) \frac{m-1}{m} = 0$, $0 \le I_n \le c_{11}$ for some constant $c_{11} > 0$, independent of n . Letting $n \to \infty$, we conclude that $(au^q - C)\ \mathrm{sign}^+(u-M) \in L^1(\Omega)$. Knowing this fact we can further improve the estimate of I_n .

Case (i): If $q \geq m$ (i.e. $m^2 \geq N$), then

$$n^{1-\frac{N}{m}} \|(u-M)^+ |D\zeta_{2n}|\|^{m-1}_{L^m(T_{2n})}$$

$$\leq n^{1-\frac{N}{m}}(c_6(2n))^{m-1} \|(u-M)^+\|^{m-1}_{L^m(T_{2n})}$$

$$\leq (2c_6)^{m-1}\, n^{m-\frac{N}{m}} |T_n|^{(1-\frac{m}{q})(\frac{m-1}{m})} \|(u-M)^+\|^{m-1}_{L^q(T_{2n})}$$

$$\leq c_{12}(c_6, m, N)\, n^{m-\frac{N}{m}-N(1-\frac{m}{q})(\frac{m-1}{m})} \|(u-M)^+\|^{m-1}_{L^q(T_{2n})}$$

$$= c_{12}\|(u-M)^+\|^{m-1}_{L^q(T_{2n})} \to 0 \quad \text{as} \quad n \to \infty . \tag{3.21}$$

Case (ii): If $q < m$ (i.e. $m^2 < N$), then

$$n^{1-\frac{N}{m}} \|(u-M)^+ |D\zeta_{2n}|\|^{m-1}_{L^m(T_{2n})}$$

$$\leq n^{m-\frac{N}{m}} (2c_6)^{m-1} \|(u-M)^+\|^{m-1}_{L^m(T_{2n})}$$

$$\leq n^{m-\frac{N}{m}} (2c_6)^{m-1} c_3(4n)^{\frac{(m-1)(m-q)}{q+1-m}} \|(u-M)^+\|^{\frac{m-1}{mq}}_{L^q(T_{2n})}$$

$$= c_{13}(c_6, c_3, m, q) \|(u-M)^+\|^{\frac{m-1}{mq}}_{L^q(T_{2n})} \to 0 \quad \text{as} \quad n \to \infty . \tag{3.22}$$

So in any case

$$\int_{|x|\leq 1} (au^q - C)\, \mathrm{sign}^+(u-M)\, dx = 0 .$$

This implies that $u(x) \leq M$ for almost all $|x| \leq 1$. Q.E.D.

Proof of Theorem 3.1. By (B1) we have $B(x, u) \geq au^q - C$ for $u \geq 0$, where a and C are positive constants. So we have

$$-\operatorname{div} A(x, Du) + au^q - C \leq 0 \qquad \text{on } \{x \in \Omega : u(x) > 0\} .$$

By Lemma 3.3, $u^+ \in L^\infty_{loc}(\Omega)$. In the same way $u^- \in L^\infty_{loc}(\Omega)$. So u and $B(x, u)$ are both in $L^\infty_{loc}(\Omega)$.

Let ζ_n be as before and $\eta \in C^1_o(\Omega)$. Substituting the test function $\varphi = u(\zeta_n\eta)^m$ in the equation, we get

$$\int_\Omega A_i(x, Du)D_iu(\zeta_n\eta)^m + u \cdot D_i(\xi_n\eta)m(\zeta_n\eta)^{m-1} + u\,\zeta_n^m\eta^m B(x, u)dx = 0 . \tag{3.23}$$

We proceed as in (3.13) - (3.18) to conclude that $|Du| \in L^m_{loc}(\Omega)$. Then in (2.23) with m replaced by 1, we let $n \to \infty$ and conclude that u is indeed a weak solution of (2.1) in all of Ω. By regularity theory [4], u is almost everywhere equal to a $C^{1,\alpha}(\Omega)$ function for some $0 < \alpha < 1$. Q.E.D.

4. EXAMPLES

Example 4.1. Consider

$$-\Delta u + |u|^{q-1}u = 0 \qquad \text{in } \Omega' ,$$

where $q \geq \frac{N}{N-2}$ and $N > 2$. It is easily checked that (A1), (A2) and (B1) are all satisfied with $m = 2$. This was Brezis and Véron's result [2].

Example 4.2. Consider

$$- \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left\{ a_{ij}(x) \frac{\partial u}{\partial x_j} \right\} + |u|^{q-1} u = 0 \qquad \text{in } \Omega' ,$$

where $q \geq \frac{N}{N-2}$ and $N > 2$. We assume that $a_{ij}(x)$'s are Lipschitz functions in Ω. Again (A1), (A2) and (B1) hold with $m = 2$.

Example 4.3. Consider

$$- \mathrm{Div}(|Du|^{m-2} Du) + |u|^{q-1} u = 0 \qquad \text{in } \Omega' ,$$

where $q \geq \frac{N(m-1)}{N-m}$, $1 < m < N$. Then (A1), (A2) and (B1) hold and we obtain Vàzquez and Véron's result [5].

Example 4.4. Consider

$$- \mathrm{Div}((1 + |Du|^2)^{\frac{m}{2} - 1} Du) + |u|^{q-1} u = 0 \qquad \text{in } \Omega' ,$$

where $1 < m < N$, $q \geq \frac{N(m-1)}{N-M}$. Then (A1), (A2) and (B1) hold and we can apply Theorem 3.1.

Example 4.5. More generally, we can consider the Euler-Lagrange equation of the following functional

$$I(u) = \int_{\Omega} F(x, Du)\, dx + \int_{\Omega} \left(\int_0^{u(x)} B(x, z) dz \right) dx \tag{3.1}$$

where F is a C^2 function in $\Omega \times \mathbb{R}^N$.

The Euler-Lagrange equation has the form

$$- \operatorname{div} F_p(x, Du) + B(x, u) = 0 .$$

Then $A_i(x, p) = F_{p_i}(x, p) = D_{p_i} F(x, p)$. Condition (A1) simply says that $F(x, p)$ grows like $|p|^m$ when $|p|$ is large. The condition (A2a) is a natural assumption for minimizing problems. Notice that if (A2a) holds for all $p \in \mathbb{R}^N$, then (A2b) is automatically satisfied.

Remark. 4.6. In the cases of the uniformly elliptic or m-Laplacian operator, the singular set can be taken to be larger by appropriately increasing the value of q ; cf. [1,7]. Our proof fails to work in this more general case since we are only assuming the various growth conditions when $|p|$ is large. Notice that in Example 4.4, $A(x, p)$ behaves like $|p|^{m-1}$ or $|p|^1$ depending on whether $|p|$ is near ∞ or 0 .

Remark 4.7. The exponent in (B1) is sharp as shown in [8] in the Laplacian case.

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