ESTIMATES FOR LINEAR SYSTEMS OF OPERATOR EQUATIONS

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1. INTRODUCTION

This is a description of joint work(*) with Alan McIntosh and Werner Ricker of Macquarie University.

Throughout, X and Y denote (complex) Banach spaces. The space of bounded (linear) operators from X to Y, provided with the operator norm, is denoted $L(X,Y)$ and $L(X) = L(X,X)$. The Taylor spectrum of a commuting m-tuple $\underset{\sim}{S} = (S_1,\ldots,S_m)$ in $L(X)^m$ is denoted $Sp(\underset{\sim}{S})$ or $Sp(S_1,\ldots,S_m)$ or $Sp(\underset{\sim}{S},L(X))$ (see Taylor [9]).

We consider the following linear system of equations

$$\sum_{j=1}^{n} A_{ij}QB_{ij} = U_i \quad \text{for} \quad 1 \leqq i \leqq m. \tag{1.1}$$

Here and elsewhere, $\underset{\sim}{A} = (A_{ij}) \in L(X)^{mn}$, $\underset{\sim}{B} = (B_{ij}) \in L(Y)^{mn}$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, and $\underset{\sim}{A}$, $\underset{\sim}{B}$ are commuting mn-tuples. Moreover, $\underset{\sim}{U} = (U_1,\ldots,U_m) \in L(Y,X)$ is given and an operator $Q \in L(Y,X)$ satisfying (1.1) is to be determined. We will order mn-tuples such as $\underset{\sim}{A} = (A_{ij})$ or $x = (x_{ij}) \in \mathbb{C}^{mn}$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, lexicographically from the left. So, $x = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn})$.

For $m > 1$, the system (1.1) is overdetermined and it is readily seen that a necessary condition for the solubility of (1.1) is the following

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compatibility condition

(1.2) $$\sum_{j=1}^{n} A_{\ell j}U_iB_{\ell j} = \sum_{j=1}^{n} A_{ij}U_\ell B_{ij} \quad \text{for} \quad 1 \leqq i, \ell \leqq n.$$

The operators $T_i \in L(L(Y,X))$, defined for $1 \leqq i \leqq m$ by $T_i(Q) = \sum_{j=1}^{n} A_{ij}QB_{ij}$, are sometimes called elementary operators. Spectral properties of (single) elementary operators, especially on Hilbert space, have been studied by a number of authors. See for example Curto [4] and the references cited there. System (1.1) with $m = 1$ is also the subject of McIntosh, Pryde and Ricker [8].

An interesting special case arises when $n = 2$, $A_{i1} = A_i$, $A_{i2} = -I$, $B_{i1} = I$, $B_{i2} = B_i$. Then (1.1) becomes

(1.3) $$A_iQ - QB_i = U_i \quad \text{for} \quad 1 \leqq i \leqq m.$$

In this case T_i is a generalized derivation.

Under the condition that $\mathrm{Sp}(A_1,\ldots,A_m) \cap \mathrm{Sp}(B_1,\ldots,B_m) = \emptyset$, McIntosh and Pryde [5], [6] have shown that the compatibility condition (1.2) is necessary and sufficient for the solvability of (1.3). Moreover, let $\delta = \mathrm{dist}(\mathrm{Sp}(A_1,\ldots,A_m), \mathrm{Sp}(B_1,\ldots,B_m))$ be positive and suppose $\underset{\sim}{A}$ and $\underset{\sim}{B}$ consist of generalized scalar operators with real spectra. Recall that an operator $S \in L(X)$ is *generalized scalar with real spectrum* if and only if there exist $s \geqq 0$ and $M \geqq 1$ such that $\|\exp(i\lambda S)\| \leqq M(1+|\lambda|)^s$ for all $\lambda \in \mathbb{R}$ (Colojoară and Foiaş, [3]). So there exist constants $s, t \geqq 0$ and $M, N \geqq 1$ such that $\|\exp(i\sum_{\ell=1}^{m}\xi_\ell A_\ell)\| \leqq M(1+|\xi|)^s$, $\|\exp(i\sum_{\ell=1}^{m}\xi_\ell B_\ell)\| \leqq N(1+|\xi|)^t$ for all $(\xi_1,\ldots,\xi_\ell) \in \mathbb{R}^m$. It is proved in [6] that there exists a constant $c = c(m,s+t)$ such that any solution Q of (1.3) satisfies

(1.4) $$\|Q\| \leqq cMN\delta^{-1}\max(1,\delta^{-s})\max(1,\delta^{-t})\,\|\underset{\sim}{U}\|$$

where $\|\underset{\sim}{U}\| = (\sum_{i=1}^{m} \|U_i\|^2)^{\frac{1}{2}}$.

Our original motivation for studying system (1.3) was that it arises in the study of perturbation of spectral subspaces of commuting m-tuples of, say, normal operators on a Hilbert space. For these applications, see [5].

In this paper we attempt to obtain estimates similar to (1.4) for the more general system (1.1). To do this it will, at times, be necessary to assume that $\underset{\sim}{A} = (A_{ij})$ and $\underset{\sim}{B} = (B_{ij})$ are commuting mn-tuples of generalized scalar operators with real spectra. So, there exist $s_{ij}, t_{ij} \geqq 0$ and $M_{ij}, N_{ij} \geqq 1$ for $1 \leqq i \leqq m$, $1 \leqq j \leqq n$ such that

(1.5) $\quad \|\exp(i\lambda A_{\ell j})\| \leqq M_{\ell j}(1+|\lambda|)^{s_{\ell j}}, \ \|\exp(i\lambda B_{\ell j})\| \leqq N_{\ell j}(1+|\lambda|)^{t_{\ell j}}$

for all $\lambda \in \mathbb{R}$, $1 \leq \ell \leq m$ and $1 \leq j \leq n$.

It follows from (1.5) that $\underset{\sim}{T} = (T_1,\ldots,T_m)$ is also a commuting tuple of generalized scalar operators with real spectra ; that is, there exist $u \geqq 0$, $P \geqq 1$ such that

(1.6) $\quad \|\exp(i\langle\xi,\underset{\sim}{T}\rangle)\| \leqq P(1+|\xi|)^u$ for all $\xi \in \mathbb{R}^m$ where $\langle\xi,\underset{\sim}{T}\rangle = \sum_{j=1}^{m} \xi_j T_j$.

By McIntosh and Pryde [6, Theorem 11.1] any solution Q of (1.1) satisfies

(1.7) $\quad \|Q\| \leqq cP\delta^{-1}\max(1,\delta^{-u})\|\underset{\sim}{U}\|$

where $c = c(m,u)$ and $\delta = \mathrm{dist}(0,\mathrm{Sp}(\underset{\sim}{T})) > 0$.

However, we are in general unable to find a relationship between (u,P) and $(s_{ij},t_{ij},M_{ij},N_{ij})$. In McIntosh, Pryde and Ricker [8] it is shown that we can take $u = \sum_{i,j}(s_{ij}+t_{ij})$ when X, Y are finite dimensional. In the infinite dimensional case, if X = Y, it follows from Albrecht [1] that $u \leqq \sum_{i,j}(s_{ij}+t_{ij}+2)$. In a private communication, M. Hladnik has given an explicit example $(X = Y = \ell_2)$ where $u > \sum_{i,j} (s_{ij}+t_{ij})$.

In this paper we seek estimates for solutions Q of (1.1) in terms of the parameters $(s_{ij}, t_{ij}, M_{ij}, N_{ij})$ and not in terms of (u,P).

Note that, given (1.5), $\underset{\sim}{A}$ and $\underset{\sim}{B}$ satisfy

(1.8) $\quad \|\exp(i\langle\xi,\underset{\sim}{A}\rangle)\| \leqq M(1+|\xi|)^s, \quad \|\exp(i\langle\xi,\underset{\sim}{B}\rangle)\| \leqq N(1+|\xi|)^t$

for certain constants $s, t \geqq 0$ and $M,N \geqq 1$ and all $\xi \in \mathbb{R}^{mn}$.

In fact, since $\exp(i\langle\xi,\underset{\sim}{A}\rangle) = \prod_{\ell,j} \exp(i\xi_{\ell j}A_{\ell j})$, with a similar expression for $\underset{\sim}{B}$, it follows that we can take $s = \sum_{i,j} s_{ij}$, $t = \sum_{i,j} t_{ij}$, $M = \prod_{i,j} M_{ij}$, $N = \prod_{i,j} N_{ij}$.

2. EXISTENCE, UNIQUENESS THEOREM

Let $L_{ij}, R_{ij} \in L(L(Y,X))$ for $1 \leqq i \leqq m$, $1 \leqq j \leqq n$ be defined by $L_{ij}(Q) = A_{ij}Q$ and $R_{ij}(Q) = QB_{ij}$. Let $\underset{\sim}{L} = (L_{ij})$ and $\underset{\sim}{R} = (R_{ij})$ so that $(\underset{\sim}{L},\underset{\sim}{R})$ is a commuting 2mn-tuple.

Define $\psi : \mathbb{C}^{2mn} \to \mathbb{C}^m$ by $\psi = (\psi_1,\ldots,\psi_m)$ where $\psi_i(x,y) = \sum_{j=1}^{n} x_{ij}y_{ij}$ for $x,y \in \mathbb{C}^{mn}$ and we make the identification $\mathbb{C}^{2mn} = \mathbb{C}^{mn} \times \mathbb{C}^{mn}$. If $\underset{\sim}{T} = (T_1,\ldots,T_m)$ then

(2.1) $\quad \underset{\sim}{T} = \psi(\underset{\sim}{L},\underset{\sim}{R})$.

In the next proposition, and in section 3, we will assume that $\underset{\sim}{A} = (A_{\ell j})$, $\underset{\sim}{B} = (B_{\ell j})$ are of the form

(2.2) $\quad A_{\ell j} = A_{\ell j1} + iA_{\ell j2}\,, \quad B_{\ell j} = B_{\ell j1} + iB_{\ell j2}$

where $(A_{\ell jk})$, $(B_{\ell jk})$ for $1 \leqq \ell \leqq m$, $1 \leqq j \leqq n$, $1 \leqq k \leqq 2$ are commuting 2mn-tuples in $L(X)^{2mn}$, $L(Y)^{2mn}$ respectively and all $A_{\ell jk}$, $B_{\ell jk}$ have real spectra.

If $\underset{\sim}{A}, \underset{\sim}{B}$ satisfy (2.2) they are called *strongly commuting*, and the tuples $\pi(\underset{\sim}{A}) = (A_{\ell jk})$, $\pi(\underset{\sim}{B}) = (B_{\ell jk})$ are referred to as *partitions* of $\underset{\sim}{A}, \underset{\sim}{B}$. If X, Y are finite dimensional, then any commuting tuples are strongly commuting. If X, Y are Hilbert and $\underset{\sim}{A}, \underset{\sim}{B}$ are commuting tuples of normal operators, then $\underset{\sim}{A}, \underset{\sim}{B}$ are strongly commuting. Other examples may be found in McIntosh, Pryde and Ricker [7].

PROPOSITION 2.3 *Suppose one of the following conditions is satisfied*

a) $m = n = 1$,

b) $X = Y$,

c) X, Y *are Hilbert spaces, or*

d) $\underset{\sim}{A}, \underset{\sim}{B}$ *are strongly commuting.*

Then $\mathrm{Sp}(\underset{\sim}{L}) \subset \mathrm{Sp}(\underset{\sim}{A})$ *and* $\mathrm{Sp}(\underset{\sim}{R}) \subset \mathrm{Sp}(\underset{\sim}{B})$.

Proof. Define $\ell : L(X) \to L(L(Y,X))$ and $r : L(Y) \to L(L(Y,X))$ by $\ell(A)(Q) = AQ$ and $r(B)(Q) = QB$. It is easy to check that $\mathrm{Sp}(\ell(A)) \subset \mathrm{Sp}(A)$ and $\mathrm{Sp}(r(B)) \subset \mathrm{Sp}(B)$, proving the result for a).

If $X = Y$ or if X, Y are Hilbert spaces, then ℓ and r are isometries onto (closed) unital subalgebras of $L(L(Y,X))$. Further ℓ is a homomorphism and r an order-reversing-homomorphism. Hence $\mathrm{Sp}(\underset{\sim}{A}) \subset \mathrm{Sp}(\ell(\underset{\sim}{A}), L(L(Y,X))) = \mathrm{Sp}(\underset{\sim}{L})$ and $\mathrm{Sp}(\underset{\sim}{B}) \subset \mathrm{Sp}(r(\underset{\sim}{B}), L(L(Y,X))) = \mathrm{Sp}(\underset{\sim}{R})$, proving the result for b), c).

Suppose $\underset{\sim}{A}, \underset{\sim}{B}$ are strongly commuting with partitions $\pi(\underset{\sim}{A}), \pi(\underset{\sim}{B})$. Since $\mathrm{Sp}(\pi(\underset{\sim}{A})) \subset \mathbb{R}^{2mn}$, by [7, Theorem 1] $\mathrm{Sp}(\pi(\underset{\sim}{A})) = \gamma(\pi(\underset{\sim}{A})) = \{\lambda \in \mathbb{R}^{2mn} : 0 \in \mathrm{Sp}(\Sigma(A_{ijk}-\lambda_{ijk})^2)\}$. Define $p : \mathbb{C}^{2mn} \to \mathbb{C}^{mn}$ by $p(x) = y$, where $x = (x_{\ell jk})$, $y = (y_{\ell j})$, $y_{\ell j} = x_{\ell j1} + ix_{\ell j2}$. Then

$$\mathrm{Sp}(\underset{\sim}{A}) = p(\mathrm{Sp}(\pi(\underset{\sim}{A})))$$

(by Taylor's spectral mapping theorem [10])

$$= p(\gamma(\pi(\underset{\sim}{A})))$$

$$\supset p(\gamma(\ell(\pi(\underset{\sim}{A}))))$$

(by the result proved above for a))

$$= p(Sp(\ell(\pi(\underset{\sim}{A}))))$$

(since $Sp(\ell(\pi(\underset{\sim}{A}))) \subset \mathbb{R}^{2mn}$)

$$= Sp(\underset{\sim}{L}).$$

Similarly, $Sp(\underset{\sim}{B}) \supset Sp(\underset{\sim}{R})$. ▮

PROPOSITION 2.4 *Suppose one of the conditions* 2.3a) - d) *is satisfied. Then* $Sp(\underset{\sim}{T}) \subset \psi(Sp(\underset{\sim}{A}) \times Sp(\underset{\sim}{B}))$.

Proof. By (2.1), Taylor's spectral mapping theorem and Proposition 2.3,

$$\begin{aligned} Sp(\underset{\sim}{T}) &= Sp(\psi(\underset{\sim}{L},\underset{\sim}{R})) \\ &= \psi(Sp(\underset{\sim}{L},\underset{\sim}{R})) \\ &\subset \psi(Sp(\underset{\sim}{L}) \times Sp(\underset{\sim}{R})) \\ &\subset \psi(Sp(\underset{\sim}{A}) \times Sp(\underset{\sim}{B})). \end{aligned}$$ ▮

THEOREM 2.5 *Suppose one of the conditions* 2.3a) - d) *is satisfied and* $0 \notin \psi(Sp(\underset{\sim}{A}) \times Sp(\underset{\sim}{B}))$. *Then system (1.1) has a solution* $Q \in L(Y,X)$ *if and only if the compatibility conditions (1.2) are satisfied. Moreover, when a solution exists it is unique.*

Proof. We have observed already that the compatibility conditions are necessary for solubility of (1.1). Conversely, if $0 \notin \psi(Sp(\underset{\sim}{A}) \times Sp(\underset{\sim}{B}))$ then by Proposition 2.4 and the definition of the Taylor spectrum, the Koszul complex for $\underset{\sim}{T}$ is exact. In particular, $Q \mapsto (T_1(Q),\dots,T_m(Q))$ is an injection from $L(X,Y)$ into $L(Y,X)^m$ whose range is precisely those $\underset{\sim}{U}$ satisfying (1.2). ▮

3. ESTIMATES FOR THE SOLUTION : REAL SPECTRA

In order to prove estimates for the solution of (1.1) we must place restrictions on $\underset{\sim}{A}$, $\underset{\sim}{B}$. Throughout this section, we assume $0 \notin \psi(\mathrm{Sp}(\underset{\sim}{A}) \times \mathrm{Sp}(\underset{\sim}{B}))$ and moreover that $\underset{\sim}{A}$, $\underset{\sim}{B}$ are commuting mn-tuples of generalized scalar operators with real spectra. In particular we assume that condition (1.8) is satisfied.

It follows that $(\underset{\sim}{L},\underset{\sim}{R})$ is a commuting 2mn-tuple of generalized scalar operators with real spectra. In particular, if $K = MN$ and $r = s + t$, then

$$(3.1) \qquad \|\exp(i<(\xi,\eta),\ (\underset{\sim}{L},\underset{\sim}{R})>\| \leqq K(1+|(\xi,\eta)|)^r$$
$$\text{for all } \xi,\ \eta \in \mathbb{R}^{mn}.$$

Let k be a positive integer and r any non-negative real. We denote by $L_1^{\vee}(r,\mathbb{R}^k)$ the space of inverse Fourier transforms g of complex-valued functions h for which $(1+|\xi|)^r h \in L_1(\mathbb{R}^k)$. In particular, $g(x) = h^{\vee}(x) = (2\pi)^{-k} \int_{\mathbb{R}^k} \exp(i<\xi,x>)h(\xi)d\xi$. The norm $\|g\| = (2\pi)^{-k} \int_{\mathbb{R}^k} (1+|\xi|)^r |h(\xi)|d\xi$ makes $L_1^{\vee}(r,\mathbb{R}^k)$ a Banach algebra with respect to pointwise multiplication. For the details, see McIntosh and Pryde [6].

In view of condition (3.1), it follows that $(\underset{\sim}{L},\underset{\sim}{R})$ has a functional calculus based on $L_1^{\vee}(r,\mathbb{R}^{2mn})$. In fact there is a continuous homomorphism

$$(3.2) \qquad \Phi : L_1^{\vee}(r,\mathbb{R}^{2mn}) \to L(L(Y,X))$$

defined by

$$\Phi(g) = (2\pi)^{-2mn} \int_{\mathbb{R}^{2mn}} \exp(i<(\xi,\eta),\ (\underset{\sim}{L},\underset{\sim}{R})>)\hat{g}(\xi,\eta)d\xi d\eta.$$

If $p : \mathbb{R}^{2mn} \to \mathbb{C}$ is a polynomial and $\theta \in C_c^{\infty}(\mathbb{R}^{2mn})$ is 1 on a neighbourhood of $\mathrm{Sp}(\underset{\sim}{L},\underset{\sim}{R})$ then $\theta p \in L_1^v(R, \mathbb{R}^{2mn})$ and

(3.3) $\quad \Phi(\theta p) = p(\underset{\sim}{L},\underset{\sim}{R})$.

From condition (3.1) it follows readily that

(3.4) $\quad \|\Phi(g)\| \leqq K\|g\|$ for all $g \in L_1^v(r, \mathbb{R}^{2mn})$.

Since $0 \notin \psi(\mathrm{Sp}(\underset{\sim}{A}) \times \mathrm{Sp}(\underset{\sim}{B}))$ and $\mathrm{Sp}(\underset{\sim}{A}) \times \mathrm{Sp}(\underset{\sim}{B})$ is compact, $|\psi|^{-2}\psi_i$ is C^∞ on a neighbourhood of $\mathrm{Sp}(\underset{\sim}{A}) \times \mathrm{Sp}(\underset{\sim}{B})$ for $1 \leqq i \leqq m$. So there exists $g = (g_1,\ldots,g_m)$ such that

(3.5) $\quad g \in L_1^v(r, \mathbb{R}^{2mn})^m$ and $g = |\psi|^{-2}\psi$ on a neighbourhood of $\mathrm{Sp}(\underset{\sim}{A}) \times \mathrm{Sp}(\underset{\sim}{B})$.

With $\|g\| = (\sum_{i=1}^{m} \|g_i\|^2)^{\frac{1}{2}}$ define

(3.6) $\quad c(m,n,r,\mathrm{Sp}(\underset{\sim}{A}),\ \mathrm{Sp}(\underset{\sim}{B})) = \inf\{\|g\| : g$ satisfies (3.5)$\}$.

THEOREM 3.7 *Let* $\underset{\sim}{A}$, $\underset{\sim}{B}$ *be commuting* mn*-tuples of generalized scalar operators with real spectra such that* $0 \notin \psi(\mathrm{Sp}(\underset{\sim}{A}) \times \mathrm{Sp}(\underset{\sim}{B}))$. *In particular, suppose condition (3.1) is satisfied. If* Q *is a solution of system (1.1) in* $L(Y,X)$ *then*

$$\|Q\| \leqq K\, c(m,n,r,\mathrm{Sp}(\underset{\sim}{A}),\ \mathrm{Sp}(\underset{\sim}{B}))\ \|\underset{\sim}{U}\|.$$

Proof. Let Φ be the functional calculus homomorphism (3.2) and g any function satisfying (3.5). Let $P = \sum_{\ell=1}^{m} \Phi(g_\ell)U_\ell$. If $\theta \in C_c^\infty(\mathbb{R}^{2mn})$ is 1 on a neighbourhood of $\mathrm{Sp}(\underset{\sim}{L},\underset{\sim}{R})$, then for $1 \leqq i \leqq m$,

$$\begin{aligned} T_i(P) &= T_i\Big(\sum_{\ell=1}^{m} \Phi(g_\ell)U_\ell\Big) \\ &= \sum_{\ell=1}^{m} \Phi(g_\ell)T_i(U_\ell) \end{aligned}$$

$$= \sum_{\ell=1}^{m} \Phi(g_\ell) T_\ell (U_i)$$

(using the compatibility condition (1.2))

$$= \sum_{\ell=1}^{m} \Phi(g_\ell) \Phi(\theta\psi_\ell) U_i$$

(by (2.1) and (3.3))

$$= \Phi\left(\sum_{\ell=1}^{m} g_\ell \theta \psi_\ell \right) U_i$$

$$= \Phi(\theta) U_i$$

(by (3.5) and proposition 2.2)

$$= U_i$$

(by (3.3)). Hence $P = Q$ and by (3.4),

$$\begin{aligned} \|Q\| &= \left\| \sum_{\ell=1}^{m} \Phi(g_\ell) U_i \right\| \\ &\leqq K \sum_{\ell=1}^{m} \|g_\ell\| \, \|U_\ell\| \\ &\leqq K \, \|g\| \, \|\underset{\sim}{U}\| \end{aligned}$$

from which the result follows. ∎

4. ESTIMATES FOR THE SOLUTION : COMPLEX SPECTRA

A more general result for operators with complex spectra can also be obtained. Again we assume that $0 \notin \psi(Sp(\underset{\sim}{A}) \times Sp(\underset{\sim}{B}))$. In addition we assume that $\underset{\sim}{A}$, $\underset{\sim}{B}$ are strongly commuting mn-tuples whose partitions $\pi(\underset{\sim}{A}) = (A_{\ell jk})$ and $\pi(\underset{\sim}{B}) = (B_{\ell jk})$ consist of generalized scalar operators (with real spectra).

We define operators $L_{\ell jk}$, $R_{\ell jk} \in L(L(Y,X))$ by $L_{\ell jk}(Q) = A_{\ell jk}Q$,

$R_{\ell jk}(Q) = QB_{\ell jk}$ and set $\underset{\sim}{L}^{(k)} = (L_{\ell jk})$, $\underset{\sim}{R}^{(k)} = (R_{\ell jk})$, $1 \leqq \ell \leqq m$, $1 \leqq j \leqq n$, $1 \leqq k \leqq 2$. Then $(\underset{\sim}{L}^{(1)}, \underset{\sim}{L}^{(2)}, \underset{\sim}{R}^{(1)}, \underset{\sim}{R}^{(2)})$ is a commuting 4mn-tuple of generalized scalar operators with real spectra. Hence there exist $r \geqq 0$, $K \geqq 1$ such that

$$(4.1) \qquad \|\exp(i \sum_{\ell,j,k} (\xi_{\ell jk} L_{\ell jk} + \eta_{\ell jk} R_{\ell jk}))\| \leqq K(1+|(\xi,\eta)|)^r$$

for all $\xi = (\xi_{\ell jk})$, $\eta = (\eta_{\ell jk}) \in \mathbb{R}^{2mn}$.

Moreover,

$$T_\ell = \sum_j L_{\ell j} R_{\ell j}$$

$$= \sum_j (L_{\ell j1} R_{\ell j1} - L_{\ell j2} R_{\ell j2}) + i(L_{\ell j2} R_{\ell j1} + L_{\ell j1} R_{\ell j2}).$$

Hence

$$(4.2) \qquad T_\ell = \phi_\ell(\underset{\sim}{L}^{(1)}, \underset{\sim}{L}^{(2)}, \underset{\sim}{R}^{(1)}, \underset{\sim}{R}^{(2)})$$

where $\phi_\ell : \mathbb{R}^{4mn} \to \mathbb{C}$ is defined by

$$\phi_\ell(x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)})$$

$$= \sum (x_{\ell j1} y_{\ell j1} - x_{\ell j2} y_{\ell j2}) + i(x_{\ell j2} y_{\ell j1} + x_{\ell j1} y_{\ell j2})$$

$$= \psi_\ell(x^{(1)} + ix^{(2)}, y^{(1)} + iy^{(2)})$$

for $x^{(k)} = (x_{\ell jk})$, $y^{(k)} = (y_{\ell jk}) \in \mathbb{R}^{mn}$.

Let $\phi = (\phi_\ell) : \mathbb{R}^{4mn} \to \mathbb{C}^m$, let $\phi_\ell^{\#} = \bar{\phi}_\ell$ the complex conjugate of ϕ_ℓ, and define $T_\ell^{\#} = \phi_\ell^{\#}(\underset{\sim}{L}^{(1)}, \underset{\sim}{L}^{(2)}, \underset{\sim}{R}^{(1)}, \underset{\sim}{R}^{(2)})$.

LEMMA 4.3 *If the compatibility conditions (1.2) are satisfied for* $\underset{\sim}{U}$ *then the solution* Q *of (1.1) is*

$$Q = (\sum_\ell T_\ell^{\#} T_\ell)^{-1} (\sum_\ell T_\ell^{\#} U_\ell).$$

Proof. By (1.2), assuming that $\sum_\ell T_\ell^\# T_\ell$ is invertible,

$$\begin{aligned} T_i(Q) &= (\sum_\ell T_\ell^\# T_\ell)^{-1}(\sum_\ell T_\ell^\# T_i U_\ell) \\ &= (\sum_\ell T_\ell^\# T_\ell)^{-1}(\sum_\ell T_\ell^\# T_\ell U_i) \\ &= U_i. \end{aligned}$$

To prove that $\sum_\ell T_\ell^\# T_\ell$ is invertible, we note that $\sum_\ell T_\ell^\# T_\ell = \sum_\ell (\psi_\ell^\# \psi_\ell)(\underset{\sim}{L}^{(1)}, \underset{\sim}{L}^{(2)}, \underset{\sim}{R}^{(1)}, \underset{\sim}{R}^{(2)})$. Since $\sum_\ell \phi_\ell^\# \phi_\ell : \mathbb{R}^{4mn} \to \mathbb{R}$ is a polynomial, it follows from Taylor's spectral mapping theorem [10] and Proposition 2.3 that

$$\begin{aligned} Sp(\sum_\ell T_\ell^\# T_\ell) &= \sum_\ell (\phi_\ell^\# \phi_\ell)(Sp(\underset{\sim}{L}^{(1)}, \underset{\sim}{L}^{(2)}, \underset{\sim}{R}^{(1)}, \underset{\sim}{R}^{(2)})) \\ &= \sum_\ell |\psi_\ell|^2 (Sp(\underset{\sim}{L}^{(1)} + i\underset{\sim}{L}^{(2)}, \underset{\sim}{R}^{(1)} + i\underset{\sim}{R}^{(2)})) \\ &= |\psi|^2 (Sp(\underset{\sim}{L}, \underset{\sim}{R})) \\ &\subset \{|\psi(x,y)|^2 : x \in Sp(\underset{\sim}{A}),\ y \in Sp(\underset{\sim}{B})\}. \end{aligned}$$

Since $0 \notin \psi(Sp(\underset{\sim}{A}) \times Sp(\underset{\sim}{B}))$, $\sum_\ell T_\ell^\# T_\ell$ is invertible. ∎

Now $|\phi|^{-2}\phi_i^\#$ is C^∞ in a neighbourhood of $Sp(\underset{\sim}{L}^{(1)}, \underset{\sim}{L}^{(2)}, \underset{\sim}{R}^{(1)}, \underset{\sim}{R}^{(2)})$. So there exists a function g such that

(4.4) $g \in \check{L}_1(r, \mathbb{R}^{4mn})^m$ and $g = |\phi|^{-2}\phi_i^\#$ on a neighbourhood of $Sp(\underset{\sim}{L}^{(1)}, \underset{\sim}{L}^{(2)}, \underset{\sim}{R}^{(1)}, \underset{\sim}{R}^{(2)})$.

Analogously to (3.6) we define

(4.5) $c(m,n,r,Sp(\underset{\sim}{A}),Sp(\underset{\sim}{B})) = \inf\{\|g\| : g$ satisfies (4.4)$\}$.

For g satisfying (4.4) and Q any solution of (1.1), we conclude from Lemma 4.3 that $Q = \sum_\ell \Phi(g_\ell)U_\ell$. Hence :

THEOREM 4.6 *Let* $\underset{\sim}{A}$, $\underset{\sim}{B}$ *be strongly commuting mn-tuples of generalized scalar operators such that* $0 \notin \psi(Sp(\underset{\sim}{A}) \times Sp(\underset{\sim}{B}))$ *and condition (4.1) is satisfied. If* Q *is a solution of system (1.1) in* $L(Y,X)$ *then*

$$\|Q\| \leqq K\, c(m,n,r,Sp(\underset{\sim}{A}),Sp(\underset{\sim}{B}))\ \|\underset{\sim}{U}\|.$$

5. UNIVERSAL ESTIMATES

The estimate (1.7) for system (1.3) reduces to

$$(5.1) \qquad \|Q\| \leqq c(m)\,\delta^{-1}\,\|\underset{\sim}{U}\|$$

in the case where $\underset{\sim}{A}$, $\underset{\sim}{B}$ are, say, commuting m-tuples of self-adjoint operators on Hilbert spaces, $c(m)$ being a universal constant with respect to such tuples.

In this section we attempt to improve the estimate of Theorem 3.7 by obtaining a more general constant.

Let Ω be the unit sphere $\{x \in \mathbb{R}^{mn} : |x| = 1\}$. If K_1, K_2 are compact subsets of Ω we define

$$(5.2) \qquad \delta(K_1,K_2) = \inf\{|\psi(x,y)| : x \in K_1,\ y \in K_2\}.$$

If $\alpha \geqq 0$ and V is any subset of Ω we define

$$(5.3) \qquad \Gamma_\alpha(V) = \{tx : t \in \mathbb{R},\ |t| \geqq \alpha,\ x \in V\}.$$

As in previous sections, we will consider mn-tuples $\underset{\sim}{A}$, $\underset{\sim}{B}$ of operators with real spectra, such that $0 \notin \psi(Sp(\underset{\sim}{A}) \times Sp(\underset{\sim}{B}))$. In addition we will take compact subsets K_1, K_2 of Ω such that

(5.4) $\quad Sp(A) \subset \Gamma_0(K_1)$, $Sp(\underset{\sim}{B}) \subset \Gamma_0(K_2)$ and $\delta(K_1,K_2) > 0$.

For example, we could take $K_1 = \{|x|^{-1}x : x \in Sp(\underset{\sim}{A})\}$ and $K_2 = \{|x|^{-1}x : x \in Sp(\underset{\sim}{B})\}$, in which case $\delta(K_1,K_2) > 0$ follows from the condition $0 \notin \psi(Sp(\underset{\sim}{A}) \times Sp(\underset{\sim}{B}))$.

LEMMA 5.5 *If* K_1,K_2 *are compact subsets of* Ω *with* $\delta(K_1,K_2) > 0$, *there exists* $g \in C(\mathbb{R}^{2mn})^m$ *such that* $g \in L_1^{\vee}(r, \mathbb{R}^{2mn})$ *for all* $r \geqq 0$ *and* $g = |\psi|^{-2}\psi$ *in a neighbourhood of* $\Gamma_1(K_1) \times \Gamma_1(K_2)$.

Proof. Let $\delta = \delta(K_1,K_2)$. Since ψ is continuous, there exist open neighbourhoods U_1, U_2 in Ω of K_1, K_2 respectively, such that $|\psi(x,y)| > \frac{1}{2}\delta$ on $U_1 \times U_2$. Choose open neighbourhoods V_1, V_2 in Ω of K_1, K_2 whose closures are contained in U_1, U_2 respectively.

Let $p \in C_c^{\infty}(\mathbb{R})$ and $q_h \in C^{\infty}(\Omega)$ for $h = 1, 2$ be even functions satisfying $p(t) = 1$ for $|t| \leqq \frac{1}{2}$, $p(t) = 0$ for $|t| \geqq 1$; $q_h(\omega) = 1$ for $\omega \in V_h$, $q_h(\omega) = 0$ for $\omega \notin U_h$; and $p(t)$, $q_h(\omega) \in [0,1]$ for all $t \in \mathbb{R}$, $\omega \in \Omega$.

For integers k and $h = 1, 2$ let $\phi_k \in C_c^{\infty}(\mathbb{R}^{mn})$ and $\eta_h \in C^{\infty}(\mathbb{R}^{mn} \setminus \{0\})$ be defined by $\phi_k(x) = p(2^{-k}|x|)$ and $\eta_h(x) = q_h(|x|^{-1}x)$. For integers k, ℓ let $\mu_{k,\ell} \in C_c^{\infty}(\mathbb{R}^{2mn})$ be defined by $\mu_{k,\ell}(x,y) = [\phi_k(x) - \phi_{k-1}(x)][\phi_\ell(y) - \phi_{\ell-1}(y)]\eta_1(x)\eta_2(y)$.

Then $|\mu_{k,\ell}(x,y)| \leqq 1$ for all $x, y \in \mathbb{R}^{mn}$ and $\mu_{k,\ell}$ has support in the set $\{(x,y) \in \Gamma_0(U_1) \times \Gamma_0(U_2) : 2^{k-2} \leqq |x| \leqq 2^k,\ 2^{\ell-2} \leqq |y| \leqq 2^\ell\}$. Moreover, for K, L positive integers,

$$\sum_{k=0}^{K} \sum_{\ell=0}^{L} \mu_{k,\ell}(x,y) = (\phi_K - \phi_{-1})(x)\,(\phi_L - \phi_{-1})(y)\,\eta_1(x)\,\eta_2(y)$$

which is identically 1 on the set

$$\{(x,y) \in \Gamma_0(V_1) \times \Gamma_0(V_2) : \tfrac{1}{2} \leqq |x| \leqq 2^{K-1},\ \tfrac{1}{2} \leqq |y| \leqq 2^{L-1}\}.$$

For $1 \leqq j \leqq m$ and k,ℓ integers, let $G_{k,\ell,j} \in C_c^\infty(\mathbb{R}^{2mn})$ be defined by

$$G_{k,\ell,j}(x,y) = |\psi(x,y)|^{-2}\psi_j(x,y)\ \mu_{k,\ell}(x,y) = 2^{-k-\ell}G_j(2^{-k}x, 2^{-\ell}y)$$

where $G_j = G_{0,0,j}$. Then $|G_{k,\ell,j}(x,y)| \leqq 2^{5-k-\ell}\delta^{-1}$ because $|\psi(x,y)| = |x|\ |y|\ |\psi(|x|^{-1}x, |y|^{-1}y)| \geqq 2^{-5}\delta$ on the support of $\mu_{0,0}$. Hence $\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} G_{k,\ell,j}(x,y)$ converges uniformly on $\mathbb{R}^{2mn}$. If $g_j(x,y)$ denotes the limit, then $g = (g_1,\ldots,g_m) \in C(\mathbb{R}^{2mn})^m$ and $g = |\psi|^{-2}\psi$ on $\Gamma_{\frac{1}{2}}(V_1) \times \Gamma_{\frac{1}{2}}(V_2)$ a neighbourhood of $\Gamma_1(K_1) \times \Gamma_1(K_2)$. Further, $\sum_{k,\ell} G_{k,\ell,j}$ converges to g_j in $\mathcal{S}'(\mathbb{R}^{2mn})$ the Schwartz space of tempered distributions. Taking Fourier transforms we conclude that $\sum_{k,\ell}\hat{G}_{k,\ell,j}$ converges to $\hat{g}_j$ in $\mathcal{S}'(\mathbb{R}^{2mn})$. Now

$$\begin{aligned}
\|\hat{G}_{k,\ell,j}\|_{L_1(r,\mathbb{R}^{2mn})} &= \int_{\mathbb{R}^{2mn}} (1+|\xi|)^r |\hat{G}_{k,\ell,j}(\xi)|\,d\xi \\
&= 2^{-k-\ell}\int (1+|\xi|)^r |2^{2mn(k+\ell)}\hat{G}_j(2^k\xi', 2^\ell\xi'')|\,d\xi \\
&= 2^{-k-\ell}\int (1+|(2^{-k}\mu', 2^{-\ell}\mu'')|)^r |\hat{G}_j(\mu',\mu'')|\,d\mu \\
&\leqq 2^{-k-\ell}\int (1+|\mu|)^r |\hat{G}_j(\mu)|\,d\mu \\
&= 2^{-k-\ell}\|\hat{G}_j\|_{L_1(r,\mathbb{R}^{2mn})}
\end{aligned}$$

where $\xi = (\xi',\xi'') \in \mathbb{R}^{mn} \times \mathbb{R}^{mn} = \mathbb{R}^{2mn}$ and $k,\ \ell,\ r \geqq 0$. Also $\hat{G}_j \in \mathcal{S}(\mathbb{R}^{2mn}) \subset L_1(r,\mathbb{R}^{2mn})$ and so $\sum_{k,\ell}\|\hat{G}_{k,\ell,j}\|_{L_1(r,\mathbb{R}^{2mn})} < \infty$. Hence $\sum_{k,\ell}\hat{G}_{k,\ell,j}$ converges to $\hat{g}_j$ in $L_1(r,\mathbb{R}^{2mn})$, proving that $g_j \in L_1^\vee(r,\mathbb{R}^{2mn})$. ∎

Analogously to (3.6) and (4.5) we define

(5.6) $\quad c(m,n,r,K_1,K_2) = \inf\{\|g\| : g \in L_1^{\vee}(r, \mathbb{R}^{2mn})^n, \ g \text{ as in Lemma 5.5}\}.$

If $\underset{\sim}{A}$ is a commuting mn-tuple of operators, define

(5.7) $\quad \delta(\underset{\sim}{A}) = \inf\{|x| : x \in Sp(\underset{\sim}{A})\}.$

THEOREM 5.8 *Let* $\underset{\sim}{A}$, $\underset{\sim}{B}$ *be commuting mn-tuples of generalized scalar operators with real spectra such that* $0 \notin \psi(Sp(\underset{\sim}{A}) \times Sp(\underset{\sim}{B}))$. *In particular, suppose condition (1.8) is satisfied. Let* K_1, K_2 *be compact subsets of* Ω *satisfying condition (5.4). If* Q *is a solution of system (1.1) then*

$$\|Q\| \leq cdMN\|\underset{\sim}{U}\|$$

where $\quad c = c(m,n,s+t,K_1,K_2)$

and $\quad d = \delta(\underset{\sim}{A})^{-1}\delta(\underset{\sim}{B})^{-1}\max(1,\delta(\underset{\sim}{A})^{-s})\max(1,\delta(\underset{\sim}{B})^{-t})$.

Proof. If $\delta(\underset{\sim}{A}) = \delta(\underset{\sim}{B}) = 1$, let g be as in Lemma 5.5 with $r = s + t$. Then $g = |\psi|^{-2}\psi$ on a neighbourhood of $Sp(\underset{\sim}{A}) \times Sp(\underset{\sim}{B})$, and so, as in the proof of Theorem 3.7, $Q = \sum_{\ell=1}^{m} \Phi(g_\ell)U_\ell$. Hence $\|Q\| \leq MN\|g\|\,\|\underset{\sim}{U}\|$ from which the required estimate follows.

The result for general $\underset{\sim}{A}$, $\underset{\sim}{B}$ follows by applying the part proved already to the tuples $\underset{\sim}{A}' = \delta(\underset{\sim}{A})^{-1}\underset{\sim}{A}$ and $\underset{\sim}{B}' = \delta(\underset{\sim}{B})^{-1}\underset{\sim}{B}$. Note that $\underset{\sim}{A}$, $\underset{\sim}{B}$ satisfy condition (1.8) with M, N replaced by $M' = M\max(1,\delta(\underset{\sim}{A})^{-s})$, $N' = N\max(1,\delta(\underset{\sim}{B})^{-t})$. ∎

Remark 5.9

a) By the methods of section 4, Theorem 5.8 can be generalized to strongly commuting mn-tuples with partitions consisting of generalized scalar operators.

b) The method for constructing the function g in the proof of

Lemma 5.5, using Littlewood-Paley decompositions, follows a similar construction in Bhatia, Davis and McIntosh [2].

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