THE BOUNDARY ASSOCIATED WITH A PROBABILITY MEASURE ON A GROUP

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Let G be a locally compact group. Each probability measure, μ, on G determines a random walk on G with transition probabilities given by μ. Associated with the random walk are several different notions of 'boundary'. A discussion of the boundary is given in section 0 of [12] and some of these different notions are defined there. The Martin boundary, which is described in this volume in [13], is one of these. The boundary which is most appropriate for this paper however is the Poisson boundary which is usually much smaller than the Martin boundary (see section 0 of [12]).

The definitions of random walks and their various boundaries are not given here because we wish to describe another, more algebraic, way to associate a boundary to each probability measure μ. This boundary is Borel isomorphic to the Poisson boundary but its algebraic description provides an alternate approach to the study of boundaries. It also turns out that the boundary may be used to prove some results about $L^1(G)$. These points will be illustrated below. Details will appear in [15].

For each probability measure μ on a locally compact group G, let $J_\mu = [L^1(G) * (\delta_e - \mu)]^-$, where δ_e denotes the point mass at e, the unit element of G. Then J_μ is a closed, left ideal in $L^1(G)$ with a right, bounded approximate identity. The quotient space, $L^1(G)/J_\mu$, is therefore a left $L^1(G)$-module. It may be shown that, when it is equipped with a certain partial ordering, $L^1(G)/J_\mu$ is an abstract L^1-space and so there is a measure space (Ω, σ) such that $L^1(G)/J_\mu$ is isometric to $L^1(\Omega, \sigma)$. A G-action may be defined on Ω for which σ is quasi-invariant and so that $L^1(G)/J_\mu$ and $L^1(\Omega, \sigma)$ are isomorphic as $L^1(G)$-modules. There is also a μ-stationary probability measure, ν, which may be defined on Ω in a natural way, where

"ν is μ-stationary" means that $\mu * \nu = \nu$. Then the measurable G-space (Ω, σ) and the probability measure ν constitute what we shall call the *boundary* associated with μ.

Some examples will illustrate these ideas.

Example 1. Let G be any locally compact group and $\mu = \delta_e$. Then $J_\mu = \{0\}$, $(\Omega, \sigma) = (G, m_G)$, where m_G denotes Haar measure on G, and $\nu = \delta_e$. This example is so trivial that it is not usually regarded as being a random walk but it will be useful for us to allow all possibilities.

Example 2. Let G be a finite group and μ be a probability measure such that supp $(\mu) = G$. Then it may be shown that $\mu^k \to m_G$ as $k \to \infty$. It follows that $J_\mu = \{f \in L^1(G) | \sum_{x \in G} f(x) = 0\}$, which has codimension one in $L^1(G)$. More generally, for each G define $L_0^1(G) = \{f \in L^1(G) | \int_G f \, dm_G = 0\}$. Then it may be shown that, if G is abelian (see [3]) or compact (see [10]), and if μ is a probability measure on G such that supp (μ) generates G, then $J_\mu = L_0^1(G)$. In these cases Ω is a single point.

Example 3. Let G be $\mathbb{F}_2$, the free group on two generators a and b, and let $\mu = \frac{1}{4}(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}})$. Then it may be shown (see [5]) that $\Omega = \{\underset{\sim}{\omega} = (\omega_i)_{i=1}^\infty | \omega_i = a, b, a^{-1}$ or b^{-1}, and $\omega_i \omega_{i+1} \neq e, \; i = 1, 2, 3, \ldots\}$, i.e. Ω is the set of all semi-infinite reduced words in the generators and their inverses. The μ-stationary measure, ν, is described in [6] and in section 4.1 of [8]. In this case ν is quasi-invariant and σ may be chosen to be equal to ν. This example is closely related to those discussed in this volume in [13].

Example 4. For the final example we refer to [7]. It should be noted though that the term "boundary " has a different meaning in [7] and what we are calling a (Poisson) boundary is called a "Poisson space" in that paper. Now let G be a semisimple Lie group with finite centre and let μ be an absolutely continuous probability measure on G. Then (theorem 5.2 of [7]) the boundary corresponding to μ is one of a finite number of possibilities, each of which is a homogeneous space of G. These possibilities are all of the covering spaces of G/H, where H is a certain maximal amenable subgroup of G. The G-space G/H is denoted by $B(G)$ in [7]. Since Ω, the boundary corresponding to μ, is always a homogeneous space $[\sigma]$ is the unique invariant measure class on Ω.

Let Ω_1 and Ω_2 be two G-spaces. Then, following definition I.6 in [1], Ω_1 is said to be *greater than* Ω_2 if there is a surjection from Ω_1 to Ω_2 which commutes with the G-actions. This defines a partial ordering of the set of possible boundaries of G. Since each boundary is a covering space of $B(G)$, $B(G)$ is the smallest boundary with respect to this partial order.

We saw in example 2 that, if G is an abelian or compact group, then there is a probability measure, μ, on G such that $J_\mu = L_0^1(G)$. It was conjectured by Furstenberg [9] that, if G is any amenable group, then there is a probability measure, μ, on G such that $J_\mu = L_0^1(G)$. (It is not difficult to show that amenability is a necessary condition, see [14]). This conjecture was proved by Rosenblatt [14] and independently by Kaimanovich and Versik [12]. In [15] this result is put into a new context by proving it as part of a more general theorem. For this, let $\mathcal{J}(G) = \{J_\mu | \mu$ is a probability measure on $G\}$. Then $\mathcal{J}(G)$ is partially ordered by inclusion. It follows from the discussion in example 2 that, if G is compact or abelian, then $\mathcal{J}(G)$ may be identified with the partially ordered set of closed subgroups of G. The following result is proved in [15].

THEOREM 1. *Let G be a σ-compact group. Then:*

(i) each J_μ in $\mathcal{J}(G)$ is contained in a maximal element of $\mathcal{J}(G)$; and

(ii) $\mathcal{J}(G)$ has a unique maximal element if and only if G is amenable, in which case the unique maximal element is $L^1_0(G)$.

Part (ii) of the theorem implies Furstenberg's conjecture. This theorem raises the problem of the significance of the order structure on $\mathcal{J}(G)$ and of the maximal elements in $\mathcal{J}(G)$.

Some examples of maximal elements in $\mathcal{J}(G)$, when G is not amenable, are given in section 4 of [15]. One of these is provided by the probability measure on $\mathbb{F}_2$ described in example 3. If μ is this measure, then J_μ is maximal, as may be shown by appealing to a result of Furstenberg, proved in section 4.1 of [8]. Another example is provided by the measure δ_b, where b is one of the generators of $\mathbb{F}_2$. A calculation made in [15] shows directly that J_{δ_b} is maximal. Another class of examples is on Lie groups. We have seen that in the case when G is a semisimple Lie group there is an order relation on the possible boundaries of G. This partial ordering of the boundaries reflects the partial ordering at the level of ideals, as it may be shown that, if μ_1 and μ_2 are absolutely continuous probability measures with boundaries Ω_1 and Ω_2 respectively and if $J_{\mu_1} \subseteq J_{\mu_2}$, then Ω_1 is greater than Ω_2. In particular, it may be shown that, if μ is an absolutely continuous probability measure on a semisimple Lie group, G, then $B(G)$ is the boundary of μ if and only if J_μ is maximal in $\mathcal{J}(G)$.

In the semisimple Lie group case, the order relation on boundaries may also be described more directly in terms of μ, as shown in [1]. There, an open semigroup, S_μ, contained in G is associated with the support of each absolutely continuous probability measure, μ, on G. It is shown that, for each pair of absolutely continuous probability measures μ_1 and μ_2, if $S_{\mu_1} \subseteq S_{\mu_2}$, then Ω_1 is greater than Ω_2, where Ω_1 and Ω_2 are the boundaries associated with μ_1 and μ_2. When G is not a semi-simple Lie group there does not appear to be such a close connection between the order relation on $\mathcal{J}(G)$, the possible boundaries of G and the supports of the probability measures on G. An obvious problem is to determine just what connection, if any, there is in general.

Another problem is suggested by the theorem. Part(i) hints at the possibility of a structure theory for group algebras analogous to that for rings or Banach algebras (see [21], sections 24-26). Part (ii) of the theorem implies that the group algebras of amenable groups would play the part of radical algebras under this analogy. However, to establish such a theory it would be nessesary to prove some results analogous to the lemmas 26.1 and 26.2 in [2] and this appears to be difficult.

Part (ii) of theorem 1 shows that, if G is a σ-compact, amenable group, then there is a probability measure, μ, on G such that $L_0^1(G) = [L^1(G) * (\delta_e - \mu)]^-$. It follows in particular that $L_0^1(G)$ has a right bounded approximate identity. There is a two-sided generalization of this fact for non-amenable groups which is proved in section 3 of [15]. To state it we will need to introduce the notion of a non-degenerate probability measure. An absolutely continuous probability measure, μ, on G is said to be *non-degenerate* if the smallest closed semigroup containing the support of μ is G. The generalization then is contained in the following

THEOREM 2. *Let G be a σ-compact group and μ be a non-degenerate probability measure on G. Then*

$$L_0^1(G) = [L^1(G) * (\delta_e - \mu)]^- + [(\delta_e - \mu) * L^1(G)]^-.$$

It follows that $L_0^1(G)$ is the algebraic sum of a closed, left ideal with a right bounded approximate identity and a closed, right ideal with a left bounded approximate identity.

The proof of theorem 2 makes essential use of the isomorphism between $L^1(G)/J_\mu$ and $L^1(\Omega, \sigma)$. It thus illustrates the way in which the boundary may be used to obtain information about $L^1(G)$. The information so obtained is an improvement on the main result in [16].

Non-degeneracy is a condition which is often imposed on μ, with some papers on random walks, [12] for example, making a blanket assumption that all probability measures are non-degenerate. The main reason for this is that, if μ is non-degenerate, then ν is quasi-invariant and $[\nu] = [\sigma]$. The measures in examples 2 and 3 are non-degenerate whereas that in example 1 is not unless G has only one element. The absolutely continuous measures on Lie groups discussed in example 4 need not be non-degenerate. Indeed, it is shown in theorem 5.3 of [7] that, if μ is a non-degenerate probability measure on a semi-simple Lie group, then the Poisson boundary of μ is $B(G)$, i.e. J_μ is maximal. Thus the non-degeneracy condition may sometimes exclude some interesting examples. The examples we have seen so far suggest the conjecture that, if μ is non-degenerate, then J_μ is maximal. However, this conjecture is false, as is shown by some examples on soluble groups and locally finite groups, see [11].

A less restrictive condition which is usually imposed upon μ is that it should be absolutely continuous with respect to Haar measure, which is no restriction at all when G is discrete. This less restrictive condition is not sufficient to guarantee that ν is quasi-invariant, but it does imply that ν is absolutely continuous with respect to σ. The following is a more detailed discussion of the relationship between the conditions "μ is absolutely continuous with respect to Haar measure" and "ν is absolutely continuous with respect to σ".

First, there is a weaker condition than absolute continuity which is sometimes imposed on μ. This is that μ should be "spread-out" or "aleatoire", which means that there is an integer, n such that μ^n is not singular with respect to Haar measure. The next proposition shows that, from the algebraic point of view, no greater generality is obtained by imposing this formally weaker condition. Its proof demonstrates the usefulness of algebraic methods.

PROPOSITION 1. *Let μ be a spread-out probability measure on a locally compact* *group G. Then there is an absolutely continuous probability measure, μ', on G such that $J_{\mu'} = J_\mu$.*

Proof. Let n be an integer such that μ^n is not singular. Then $\mu'' = \frac{1}{n}\sum_{k=1}^{n} \mu^k$ is not singular and $J_{\mu''} = J_\mu$.

Since μ'' is not singular, $\mu'' = \mu_1 + \mu_2$, where μ_1 and μ_2 are positive measures and μ_2 is absolutely continuous and not zero. Put $\mu' = (\delta_e - \mu_1)^{-1} * \mu_2$. Then μ' is an absolutely continuous probability measure on G. Since $\delta_e - \mu' = (\delta_e - \mu_1)^{-1} * (\delta_e - \mu_1 - \mu_2) = (\delta_e - \mu_1)^{-1} * (\delta_e - \mu'')$, it follows that $J_{\mu'} = J_{\mu''} = J_\mu$.□

Since, as we have just seen, quite different probability measures may produce the same ideal, it is natural to seek to impose conditions on the ideal J_μ rather than on μ itself. The appropriate condition corresponding absolute continuity of μ is that J_μ should be a modular ideal.

Definition. A left ideal, J, in $L^1(G)$ is said to be *modular* if there is an element, u, in $L^1(G)$ such that $f - f * u$ belongs to J for every f in $L^1(G)$. The element u is said to be a *right modular unit* for J.

If μ is absolutely continuous, then J_μ is modular because we may take μ itself (regarded as belonging to $L^1(G)$) as a right modular unit. Also, it may be shown that J_μ is modular if and only if ν is absolutely continuous with respect to σ. This is proved by showing that, under the isomorphism between $L^1(G)/J_\mu$ and $L^1(\Omega, \sigma)$, $u + J_\mu$ is mapped to ν. (See propositions 2.4 and 2.5 of [15].)

The above remarks leave open the possibility that modularity of J_μ is a strictly weaker condition than absolute continuity of μ. The example outlined below shows that this is indeed the case. It shows that there is a J_μ which is modular but for which there is no absolutely continuous μ' such that $J_{\mu'} = J_\mu$. In other words, there is a probability measure μ such that there is no absolutely continuous μ' with $J_{\mu'} = J_\mu$

and yet the μ-stationary measure ν is absolutely continuous with respect to σ. This example is joint work with W. Moran.

For the example, let $G = \mathbb{Z} \times H$, where H is the abelian group

$$H = \{\underset{\sim}{h} = (h_n)_{n=-\infty}^{\infty} | h_n = \pm 1 \text{ and } \exists N \ \ s.t. \ \ h_n = 1, \ \ \forall\, n > N\}$$

with pointwise multiplication and with the product topology. The action of $\mathbb{Z}$ on H is given by the automorphism α defined by $\alpha(\underset{\sim}{h})_n = h_{n-1}, \ \ n \in \mathbb{Z}$. Now H with Haar measure m_H is itself a measurable G-space under the action, '.', given by

$$(m,h).h' = \alpha^m(h+h'), \ \ (m,h) \in G; \ \ h' \in H.$$

The example will be constructed through a series of lemmas.

LEMMA 1. *Let μ be a probabilty measure on G of the form*

$$(*) \qquad \mu = \lambda * \delta_{(1,0)},$$

where λ is a probabilty measure on H which has compact support. Suppose that the support of μ generates G.

Let ν be the probability measure on H given by $\nu = *_{n=1}^{\infty}\lambda^{(n)}$, where $\lambda^{(n)}$ is defined by $\lambda^{(n)}(E) = \lambda(\alpha^n(E))$ for every measurable E in H. Then $L^1(G)/J_\mu$ and $L^1(H, m_H)$ are isomorphic as $L^1(G)$-modules and ν is the unique μ-stationary measure on H. It follows that (H, m_H) and ν are the boundary corresponding to μ.

□

The particular form of μ in $(*)$ is required to facilitate the calculation used to show that $L^1(G)/J_\mu$ is isomorphic to $L^1(H, m_H)$ as well as to determine ν.

LEMMA 2. *Let μ and ν be as in Lemma 1 and suppose that ν is absolutely continuous with respect to m_H (i.e. that J_μ is modular). Then there is an absolutely continuous* *probability measure μ' on G such that $J_{\mu'} = J_\mu$ if and only if $d\nu/dm_H$ is lower semicontinuous almost everywhere.* □

This lemma is very easy to prove in the 'only if' direction which is the direction in which we shall use it. Suppose that $\nu = \mu'.\nu$, where μ' and ν are regarded as integrable functions on G and H respectively and '.' denotes the $L^1(G)$-module action. Then ν is a (possibly infinite) sum of convolution products of positive integrable functions on H and so is lower semicontinuous.

We will need to introduce some more notation in order to describe the measure λ to be chosen in $(*)$. Let $H_0 = \{\underset{\sim}{h} \in H | h_n = 1 \text{ for } n > 0\}$. Then H_0 is a compact open subgroup of H. Let m_0 denote the normalized Haar measure on H_0 and, for $n = 0, 1, 2, \ldots$, let χ_n be the character on H_0 given by $\chi_n(\underset{\sim}{h}) = h_{-n}$. A *Riesz product* is a measure on H_0 of the form

$$\prod_{n=0}^{\infty}(1 + a_n\chi_n)m_0,$$

where $-1 \leq a_n \leq 1$ for each n (see [4]).

All such measures are probability measures on H_0.

Suppose that the measure λ in $(*)$ is a Riesz product for some sequence of a_n's. Then it may be shown that $\nu = *_{n=1}^{\infty}\lambda^{(n)}$ will also be a Riesz product,

$$\nu = \prod_{n=0}^{\infty}(1 + b_n\chi_n)m_0,$$

where $b_n = \prod_{k=0}^{n} a_k$. The final lemma suggests how to choose the a_n's.

LEMMA 3. *Let $\nu = \prod_{n=0}^{\infty}(1 + b_n\chi_n)m_0$.*

(i) If $(b_n)_{n=0}^{\infty}$ is an l_2-sequence, then ν is absolutely continuous with respect to m_0.

(ii) If $(b_n)_{n=0}^{\infty}$ is an l_2-sequence but is not an l_1-sequence then $d\nu/dm_0$ is not lower semicontinuous at any point where it is non-zero.□

Now, let $\lambda = \prod_{n=0}^{\infty}(1 + a_n\chi_n)m_0$, where $a_0 = 1$ and $a_n = n/(n+1)$ for $n > 0$, and put $\mu = \lambda * \delta_{(1,0)}$. Then the μ-stationary measure

$$\nu = *_{n=1}^{\infty}\lambda^{(n)} = \prod_{n=0}^{\infty}(1 + b_n\chi_n)m_0,$$

where $b_n = 1/(n+1)$. By lemma 3(i), ν is absolutely continuous with respect to m_0 and so also with respect to m_H, therefore J_μ is modular. By lemma 3(ii), $d\nu/dm_0$ almost always fails to be lower semicontinuous and so, by Lemma 2, there is no absolutely continuous μ' such that $J_{\mu'} = J_\mu$.

This example raises the question of whether results which are known for absolutely continuous probabilty measures, μ, can be extended to the case when it is supposed only that J_μ is modular. Some further joint work with W. Moran shows that the results of Furstenberg described above in example 4 may be extended in this way.

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