

## 6. SCALAR OPERATORS

The most important applications of integration with respect to Banach space valued measures undoubtedly arise in the theory of spectral operators. To describe its central notion, let  $E$  be a complex Banach space,  $BL(E)$  the algebra of all bounded linear operators on  $E$  and  $I$  the identity operator. A spectral measure is an additive and multiplicative map  $P: \mathcal{Q} \rightarrow BL(E)$ , whose domain,  $\mathcal{Q}$ , is an algebra of sets in a space  $\Omega$ , such that  $P(\Omega) = I$ . An operator  $T \in BL(E)$  is said to be of scalar type if there exists a  $\sigma$ -additive (in the strong operator topology) spectral measure,  $P$ , whose domain is a  $\sigma$ -algebra and a  $P$ -integrable function  $f$  such that

$$(*) \quad T = \int_{\Omega} f dP.$$

This notion, due to N. Dunford, extends to arbitrary Banach space the idea of an operator with diagonalizable matrix on a finite-dimensional space. It proved to be very fruitful as shows the exposition in Part III of the monograph [14]. Many powerful techniques in which scalar operators play a role are based on the requirements that  $\mathcal{Q}$  be a  $\sigma$ -algebra and that  $P$  be  $\sigma$ -additive. But precisely these requirements are responsible for excluding many operators of prime interest from the class of scalar-type operators.

In this chapter, we present a suggestion for extending this class, [35]. It is based on the fact that the integral  $(*)$  exists if and only if there exist  $\mathcal{Q}$ -simple functions  $f_j$ ,  $j = 1, 2, \dots$ , such that

$$\sum_{j=1}^{\infty} \left\| \int_{\Omega} f_j dP \right\| < \infty$$

and the equality

$$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

holds for every  $\omega \in \Omega$  for which

$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty.$$

In that case,

$$\int_{\Omega} f dP = \sum_{j=1}^{\infty} \int_{\Omega} f_j dP.$$

So, the integral with respect to  $P$  can be characterized purely in terms of the operator-norm convergence. Moreover, to use this characterization as a definition of the integral with respect to  $P$ , it is not necessary to assume that the set function  $P$  be bounded, let alone  $\sigma$ -additive, nor that  $\mathcal{Q}$  be a  $\sigma$ -algebra. It suffices to assume that the seminorm

$$f \mapsto \left\| \int_{\Omega} f dP \right\|,$$

on  $\mathcal{Q}$ -simple functions, be integrating. (See Section 2D.)

Thus, as scalar operators in a wider sense, we propose operators which can be expressed in the form (\*) assuming that  $P$  is a spectral measure such that the mentioned seminorm is indeed integrating. Such operators can also be characterized intrinsically, that is, without the reference to any particular definition of integral. Namely, an operator  $T \in \text{BL}(E)$  turns out to be scalar in this sense if and only if there exists a (not necessarily bounded) Boolean algebra of projections belonging to  $\text{BL}(E)$  such that the Banach algebra of operators it generates is semisimple and contains  $T$ . However, in contrast with the classical theory, the Gelfand representations of such a Banach algebra is not necessarily the algebra of all continuous functions on its structure space but only a dense subalgebra.

A. Let  $E$  be a complex Banach space. Let  $\text{BL}(E)$  be the algebra of all bounded linear operators on  $E$ . Then  $\text{BL}(E)$  is a Banach algebra with respect to the operator (uniform) norm, defined by  $\|T\| = \sup\{|Tx| : |x| \leq 1, x \in E\}$ , for every  $T \in \text{BL}(E)$ . The identity operator is denoted by  $I$ .

Let  $\mathcal{Q}$  be a quasialgebra of sets in the space  $\Omega$ . (See Section 1D.) A map  $P : \mathcal{Q} \rightarrow \text{BL}(E)$  is said to be multiplicative if  $P(fg) = P(f)P(g)$  for every  $f \in \text{sim}(\mathcal{Q})$  and  $g \in \text{sim}(\mathcal{Q})$ . For an additive (see Section 1E) map,  $P$ , to be multiplicative it suffices that  $P(X \cap Y) = P(X)P(Y)$  for every  $X \in \mathcal{Q}$  and  $Y \in \mathcal{Q}$ .

An additive and multiplicative map  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  such that  $P(\Omega) = I$  will be called a  $\text{BL}(E)$ -valued spectral set function on  $\mathcal{Q}$ . If  $\mathcal{Q}$  happens to be an algebra of sets, then a spectral set function  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  is called a spectral measure; see [14], Definition XV.2.1.

The generality of the theory presented in this chapter is not substantially increased by the admission of arbitrary spectral set functions instead of spectral measures only. This admission is dictated mainly by convenience in considering the families of sets which classically occur in integration and spectral theories but are merely quasialgebras and not algebras. It also allows for the possibility of distinguishing certain nuances in the presented theory. However, with the exception of a single remark in the last section, this possibility will not be pursued here.

A spectral set function  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  is said to be  $\sigma$ -additive if, for every  $x \in E$ , the  $E$ -valued set function  $X \mapsto P(X)x$ ,  $X \in \mathcal{Q}$ , is  $\sigma$ -additive. (See Section 1F.) That is to say,  $\sigma$ -additivity of spectral set functions is understood in the strong operator topology of  $\text{BL}(E)$ .

In virtue of the Stone representation theorem, a set  $W \subset \text{BL}(E)$  is a Boolean algebra of projection operators if and only if there exist an algebra of sets,  $\mathcal{R}$ , in a space  $\Omega$  and a spectral measure,  $P: \mathcal{R} \rightarrow \text{BL}(E)$ , such that  $W = \{P(X) : X \in \mathcal{R}\}$ . Accordingly, a set of operators  $W \subset \text{BL}(E)$  is called a Boolean quasialgebra of projection operators if it is the range of a  $\text{BL}(E)$ -valued spectral set function, that is, if there exist a quasialgebra of sets,  $\mathcal{Q}$ , in a space  $\Omega$  and a spectral set function,  $P: \mathcal{Q} \rightarrow \text{BL}(E)$ , such that  $W = \{P(X) : X \in \mathcal{Q}\}$ .

If  $W \subset \text{BL}(E)$ , then by  $A(W)$  is denoted the least uniformly closed algebra of operators which contains  $W$ . If  $W = \{P(X) : X \in \mathcal{Q}\}$  is the range of a spectral set function  $P: \mathcal{Q} \rightarrow \text{BL}(E)$ , we write  $A(W) = A(P)$ . Clearly,  $A(P)$  is then the closure of the family of operators  $\{P(f) : f \in \text{sim}(\mathcal{Q})\}$  in the space  $\text{BL}(E)$ .

Recall that, if  $A$  is a commutative Banach algebra with unit, then the structure space,  $\Delta$ , of  $A$  is the set of all homomorphisms of  $A$  onto the field of complex numbers. For an element  $T$  of  $A$ , by  $\hat{T}$  is denoted the Gelfand transform

of  $T$ ; it is the function on  $\Delta$  defined by  $\hat{T}(h) = h(T)$ , for every  $h \in \Delta$ . It is well-known (see e.g. [46], 23B) that  $\sup\{|\hat{T}(h)| : h \in \Delta\} \leq \|T\|$  and that the coarsest topology on  $\Delta$  which makes all the functions  $\hat{T}$ ,  $T \in A$ , continuous turns  $\Delta$  into a compact Hausdorff space. Hence the Gelfand transform is a norm-decreasing homomorphism of the algebra  $A$  into the algebra,  $\mathcal{C}(\Delta)$ , of all complex continuous functions on  $\Delta$ . If the Gelfand transform is injective, then the algebra  $A$  is called semisimple.

Recall that an operator  $T \in \text{BL}(E)$  is called nonsingular if it is invertible in  $\text{BL}(E)$ , that is, if there exists an operator  $S \in \text{BL}(E)$  such that  $ST = TS = I$ . Then of course  $S = T^{-1}$  is the inverse of each of  $T$ . A full algebra of operators is uniformly closed algebra of operators which contains the inverse of each of its nonsingular elements; see [14], Definition XVII.1.1.

**LEMMA 6.1.** *Let  $\mathcal{Q}$  be a quasialgebra of sets in a space  $\Omega$  and let  $P : \mathcal{Q} \rightarrow \text{BL}(E)$  be a spectral set function.*

(i) *If  $f \in \text{sim}(\mathcal{Q})$ , then the operator  $P(f)$  is nonsingular if and only if the function  $f$  can be represented in the form*

$$(A.1) \quad f = \sum_{j=1}^n c_j X_j,$$

*where the  $n$  is a natural number, the  $c_j$  are non-zero complex numbers and the  $X_j$  are pair-wise disjoint sets from  $\mathcal{Q}$ ,  $j = 1, 2, \dots, n$ , such that*

$$\sum_{j=1}^n P(X_j) = I.$$

*In that case,  $(P(f))^{-1} = P(g)$ , where*

$$g = \sum_{j=1}^n c_j^{-1} X_j.$$

(ii) *Let  $f \in \text{sim}(\mathcal{Q})$  be a function expressed in the form (A.1) where  $X_j \in \mathcal{Q}$  are pair-wise disjoint sets such that  $P(X_j) \neq 0$ , for every  $j = 1, 2, \dots, n$ , and let*

$$c = \sup\{|c_j| : j = 1, 2, \dots, n\} \text{ and}$$

$$d = \sup\{\|\sum_{j \in J} P(X_j)\| : J \subset \{1, 2, \dots, n\}\}.$$

Then  $c \leq \|P(f)\| \leq 4cd$ .

(iii)  $A(P)$  is a full algebra of operators.

**Proof.** Let  $n \geq 1$  be an integer. Let  $X_j \in \mathcal{Q}$  be pair-wise disjoint sets, such that  $P(X_j) \neq 0$ , for every  $j = 1, 2, \dots, n$  and the sum of the operators  $P(X_j)$ ,  $j = 1, 2, \dots, n$ , is equal to  $I$ . Then the family of operators

$$\sum_{j=1}^n c_j P(X_j),$$

with arbitrary complex  $c_j$ ,  $j = 1, 2, \dots, n$ , is a closed algebra of operators generated by the Boolean algebra of projections

$$\sum_{j \in J} P(X_j),$$

where  $J$  varies over all subsets of  $\{1, 2, \dots, n\}$ . Then (i) holds by Lemma XVII.2.1 and (ii) by Lemma XVII.2.2 in [14].

To show that  $A(P)$  is a full algebra of operators let  $T$  be a non-singular element of  $A(P)$ . Let  $f_n \in \text{sim}(\mathcal{Q})$ ,  $n = 1, 2, \dots$ , be functions such that  $\|T - P(f_n)\| \rightarrow 0$ , as  $n \rightarrow \infty$ . Then for all sufficiently large  $n$ , the operator  $P(f_n)$  is nonsingular and  $\|T^{-1} - (P(f_n))^{-1}\| \rightarrow 0$ . But, by (i), for each such  $n$ , there exists a function  $g_n \in \text{sim}(\mathcal{Q})$  such that  $(P(f_n))^{-1} = P(g_n)$ . Therefore,  $T^{-1} \in A(P)$ .

**B.** With a spectral set function  $P: \mathcal{Q} \rightarrow \text{BL}(E)$ , we shall associate the seminorm  $\rho_P$  on  $\text{sim}(\mathcal{Q})$  defined by

$$(B.1) \quad \rho_P(f) = \|P(f)\|$$

for every  $f \in \text{sim}(\mathcal{Q})$ .

**PROPOSITION 6.2.** A set  $Y \subset \Omega$  is  $\rho_P$ -null if and only if there exist sets  $X_j \in \mathcal{Q}$  such that  $P(X_j) = 0$ , for every  $j = 1, 2, \dots$ , and

$$(B.2) \quad Y \subset \bigcup_{j=1}^{\infty} X_j.$$

**Proof.** Let  $X_j \in \mathcal{Q}$  be sets such that  $P(X_j) = 0$  for every  $j = 1, 2, \dots$  and (B.2) holds. Let us repeat each set countably many times, arrange the resulting family of sets into a single sequence and call their characteristic functions  $f_j$ ,  $j = 1, 2, \dots$ . Then  $f_j \in \text{sim}(\mathcal{Q})$ , for every  $j = 1, 2, \dots$ ,

$$(B.3) \quad \sum_{j=1}^{\infty} \rho_P(f_j) < \infty$$

and

$$(B.4) \quad \sum_{j=1}^{\infty} |f_j(\omega)| = \infty$$

for every  $\omega \in Y$ . So, by Proposition 2.2, the set  $Y$  is  $\rho_P$ -null.

Conversely, assume that  $f_j \in \text{sim}(\mathcal{Q})$ ,  $j = 1, 2, \dots$ , are functions, satisfying (B.3), such that (B.4) holds for every  $\omega \in Y$ . Let

$$f_j = \sum_{k=1}^{n_j} c_{jk} X_{jk},$$

with some integer  $n_j \geq 1$ , numbers  $c_{jk}$  and pair-wise disjoint sets  $X_{jk} \in \mathcal{Q}$ ,  $k = 1, 2, \dots, n_j$ , for every  $j = 1, 2, \dots$ . By Lemma 6.1,  $\|P(f_j)\| \geq |c_{jk}|$ , whenever  $P(X_{jk}) \neq 0$ . Therefore if we modify each function  $f_j$  by omitting those sets  $X_{jk}$ , together with the corresponding numbers  $c_{jk}$ , for which  $P(X_{jk}) \neq 0$ , then (B.4) will remain satisfied for every  $\omega \in Y$ . But then,  $Y$  is covered by the remaining sets  $X_{jk}$ ,  $k = 1, 2, \dots, n_j$ ,  $j = 1, 2, \dots$ .

In view of this proposition,  $\rho_P$ -null sets will be called simply  $P$ -null.

For a function  $f$  on  $\Omega$ , let

$$\|f\|_{\infty} = \inf\{\sup\{|f(\omega)| : \omega \in \Omega \setminus Y\} : Y \in \mathcal{N}\},$$

where  $\mathcal{N}$  is the family of all  $P$ -null sets. Then  $0 \leq \|f\|_\infty \leq \infty$ . The function  $f$  is said to be  $P$ -essentially bounded if  $\|f\|_\infty < \infty$ . In that case, the infimum is actually a minimum because any subset of the union of countably many  $P$ -null sets is  $P$ -null. That is to say, for any  $P$ -essentially bounded function  $f$ , there exists a  $P$ -null set,  $Y$ , such that

$$\|f\|_\infty = \sup\{|f(\omega)| : \omega \in \Omega \setminus Y\}.$$

Following the custom, we shall call  $P$ -null any function  $f$  on  $\Omega$  such that  $\|f\|_\infty = 0$ . The  $P$ -equivalence class of a function  $f$  will be denoted by  $[f]$ , or by  $[f]_P$  if the spectral set function  $P$  needs to be indicated. To be sure,  $[f]$  is the set of all functions  $g$  on  $\Omega$  such that  $\|f-g\|_\infty = 0$ .

Let  $\mathcal{L}^\infty(P)$  be the family of all functions  $f$  on  $\Omega$  such that, for every  $\epsilon > 0$ , there exists a function  $g \in \text{sim}(\mathcal{Q})$  for which  $\|f-g\|_\infty < \epsilon$ . Then  $\mathcal{L}^\infty(P)$  is an algebra under the point-wise operations.

Let  $L^\infty(P) = \{[f] : f \in \mathcal{L}^\infty(P)\}$ . Then  $L^\infty(P)$  is a Banach algebra with respect to the operations induced by the operations in the algebra  $\mathcal{L}^\infty(P)$  and the norm,  $\|\cdot\|_\infty$ , induced by the seminorm  $f \mapsto \|f\|_\infty$ ,  $f \in \mathcal{L}^\infty(P)$ .

The Banach algebra  $L^\infty(P)$  is semisimple (see e.g. [46], Theorem 24C). Actually, if  $\Delta$  is the structure space of  $L^\infty(P)$ , then the Gelfand transform is an isometric isomorphism of  $L^\infty(P)$  onto the whole of  $C(\Delta)$ . Moreover, for any function  $f \in \mathcal{L}^\infty(P)$ , the equality

$$(B.5) \quad \{[f]^\wedge(h) : h \in \Delta\} = \bigcap_{Y \in \mathcal{N}} \{f(\omega) : \omega \in \Omega \setminus Y\}^-$$

holds, where  $\mathcal{N}$  is the family of all  $P$ -null sets and the bar indicates the closure in the complex plane. The set (B.5) is called the  $P$ -essential range of the function  $f$ .

C. A spectral set function  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  will be called closable if the associated seminorm,  $\rho_P$ , defined by (B.1) on  $\text{sim}(\mathcal{Q})$  is integrating. Obviously, in that case,  $\rho_P$  integrates for  $P$ . Because  $\rho_P$  is determined by  $P$ , we shall write

$$\mathcal{L}(P) = \mathcal{L}(\rho_P, \text{sim}(\mathcal{Q})) , \quad L(P) = L(\rho_P, \text{sim}(\mathcal{Q})) , \quad q_{\rho_P} = \rho_P \quad \text{and}$$

$$P(f) = \int_{\Omega} f(\omega) P(d\omega) = \int_{\Omega} f dP = \int_{\Omega} f d\rho_P^P ,$$

for every  $f \in \mathcal{L}(P)$  , omitting the subscript.

**PROPOSITION 6.3.** *Let  $P : \mathcal{Q} \rightarrow \text{BL}(E)$  be a closable spectral set function.*

*The equality  $\|f\|_{\infty} = 0$  holds for a function  $f$  on  $\Omega$  if and only if  $f \in \mathcal{L}(P)$  and  $(f) = 0$  . Furthermore,  $\mathcal{L}(P) \subset \mathcal{L}^{\infty}(P)$  and  $\|f\|_{\infty} \leq \|P(f)\|$  , for every function  $f \in \mathcal{L}(P)$  .*

*If  $f \in \mathcal{L}(P)$  and  $g \in \mathcal{L}(P)$  then  $fg \in \mathcal{L}(P)$  and  $P(fg) = P(f)P(g)$  . So,  $\mathcal{L}(P)$  is an algebra of functions.*

*The range of the integration map  $P : \mathcal{L}(P) \rightarrow \text{BL}(E)$  is equal to  $A(P)$  . The Banach algebra  $A(P)$  is semisimple. The integration map  $P : L(P) \rightarrow A(P)$  is an isomorphism of the algebra  $L(P)$  onto the algebra  $A(P)$  .*

*If  $f \in \mathcal{L}(P)$  , then the spectrum of the operator  $T = P(f)$  is equal to the  $P$ -essential range of the function  $f$  .*

**Proof.** If  $f$  is a function on  $\Omega$  such that  $\|f\|_{\infty} = 0$  , then by the definitions of the  $P$ -null sets,  $P$ -null functions and integral,  $f \in \mathcal{L}(P)$  and  $P(f) = 0$  .

Let  $f \in \mathcal{L}(P)$  . Let  $f_j \in \text{sim}(\mathcal{Q})$  ,  $j = 1, 2, \dots$ , be functions, satisfying condition (B.3), such that

$$(C.1) \quad f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

for every  $\omega \in \Omega$  for which

$$(C.2) \quad \sum_{j=1}^{\infty} |f_j(\omega)| < \infty .$$

Then, by Lemma 6.1,

$$(C.30) \quad \sum_{j=1}^{\infty} \|f_j\|_{\infty} < \infty .$$

By the completeness of the space  $L^{\infty}(P)$  , there exists a function  $g \in \mathcal{L}^{\infty}(P)$  such that



$$(C.4) \quad [g] = \sum_{j=1}^{\infty} [f_j]$$

in  $L^{\infty}(P)$ . Since, by Proposition 6.2, the set of points  $\omega \in \Omega$ , for which the equality (C.1) does not hold, is  $P$ -null, we have  $\|f-g\|_{\infty} = 0$ , and so,  $f \in L^{\infty}(P)$ . Moreover, by Lemma 6.1,

$$\left\| \sum_{j=1}^n f_j \right\|_{\infty} \leq \left\| P \left[ \sum_{j=1}^n f_j \right] \right\|,$$

for every  $n = 1, 2, \dots$ . Therefore, by Proposition 2.1 and the continuity of norms,  $\|f\|_{\infty} \leq \|P(f)\|$ .

If, moreover,  $g \in \text{sim}(\mathcal{Q})$ , then, by Lemma 6.1,  $P(f_j g) = P(f_j)P(g)$ , for every  $j = 1, 2, \dots$ , and, by (B.3),

$$(C.5) \quad \sum_{j=1}^{\infty} \|P(f_j g)\| \leq \sum_{j=1}^{\infty} \|P(f_j)\| \|P(g)\| < \infty.$$

Hence,  $fg \in \mathcal{L}(P)$  and  $P(fg) = P(f)P(g)$ . But then, we can write (C.5) for any function  $g \in \mathcal{L}(P)$ . Consequently, by Proposition 2.1,  $fg \in \mathcal{L}(P)$  and  $P(fg) = P(f)P(g)$  for any  $f \in \mathcal{L}(P)$  and  $g \in \mathcal{L}(P)$ .

It is clear, from the definition of the integral, that for any  $f \in \mathcal{L}(P)$ , the operator  $P(f)$  belongs to  $A(P)$ , the closure of the set  $\{P(h) : h \in \text{sim}(\mathcal{Q})\}$ . Hence, to show that  $\{P(h) : h \in \mathcal{L}(P)\} = A(P)$ , it suffices to show that the set  $\{P(h) : h \in \mathcal{L}(P)\}$  is closed in  $\text{BL}(E)$ . So, let the operator  $T$  be in the closure of this set. Let  $h_j \in \mathcal{L}(P)$  be functions such that  $\|T - P(h_j)\| < 2^{-j}$  for every  $j = 1, 2, \dots$ . Let  $f_1 = h_1$  and  $f_j = h_j - h_{j-1}$ , for every  $j = 2, 3, \dots$ . Then the condition (B.3) is satisfied, and, so by Proposition 2.1, if  $f$  is a function such that (C.1) holds for every  $\omega \in \Omega$  for which (C.2) does, then  $f \in \mathcal{L}(P)$  and  $T = P(f)$ .

It is now obvious that the integration map  $P : L(P) \rightarrow A(P)$  is an isomorphism of the algebras  $L(P)$  and  $A(P)$ . Because the algebra  $L(P)$  is semisimple, being a dense subalgebra of  $L^{\infty}(P)$ , the algebra  $A(P)$  too is semisimple.

By Lemma 6.1, the algebra  $A(P)$  is full. Therefore, the spectrum of an operator  $T$  belonging to  $A(P)$  coincides with its spectrum as an element of this

algebra. Because of the isomorphism of  $A(P)$  and  $L(P)$ , this spectrum coincides with the spectrum of the element,  $[f]$ , of the algebra  $L(P)$  such that  $T = P(f)$ , which is equal to the essential range of the function  $f$ .

**D.** If  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  is a closable spectral set function, then by Proposition 6.3,  $L(P) \subset L^\infty(P)$ . Clearly, if  $P$  is not bounded on the algebra generated by  $\mathcal{Q}$ , then the integration map is not continuous in the norm of the space  $L^\infty(P)$  and its domain,  $L(P)$ , is not equal to the whole of  $L^\infty(P)$ . This domain is of course dense in  $L^\infty(P)$  and the following proposition implies that the integration map is closed.

**PROPOSITION 6.4.** *A spectral set function  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  is closable if and only if there exists an injective map  $\Phi: A(P) \rightarrow L^\infty(P)$  such that  $\|\Phi(T)\|_\infty \leq \|T\|$ , for every  $T \in A(P)$ , and  $\Phi(P(f)) = [f]$ , for every  $f \in \text{sim}(\mathcal{Q})$ .*

*If the spectral set function  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  is indeed closable then such a map  $\Phi$  is unique, its range is equal to  $L(P)$  and the map  $\Phi$  is equal to the inverse of the integration map.*

**Proof.** If such a map  $\Phi: A(P) \rightarrow L^\infty(P)$  exists, then it is unique and linear because  $\{P(f): f \in \text{sim}(\mathcal{Q})\}$  is a dense subspace of  $A(P)$ . Let then  $f_j \in \text{sim}(\mathcal{Q})$ ,  $j = 1, 2, \dots$ , be functions satisfying condition (B.3) and let

$$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every  $\omega \in \Omega$  for which (C.2) holds. Let  $T \in \text{BL}(E)$  be the operator such that

$$\lim_{n \rightarrow \infty} \|T - \sum_{j=1}^n P(f_j)\| = 0.$$

Then of course  $T \in A(P)$ . Because the map  $\Phi$  is norm-decreasing, condition (B.3) implies that (C.3) holds and, if  $[g] = \Phi(T)$ , then (C.4) does. Now, by Proposition 6.2, the set of the points  $\omega \in \Omega$  for which (B.4) holds is  $P$ -null, and so,  $[g] = 0$ .

Consequently,  $T = 0$  because the map  $\Phi$  is injective. That is

$$\sum_{j=1}^{\infty} P(f_j) = 0 ,$$

and, by Proposition 2.8, the set function  $P$  is closable.

If the set function  $P$  is closable, then by Proposition 6.2, such a map  $\Phi : A(P) \rightarrow L^{\infty}(P)$  exists: it is the inverse of the integration map.

Let us now mention a sufficient condition for a spectral set function to be closable. But first a definition:

A spectral set function  $P : \mathcal{Q} \rightarrow \text{BL}(E)$  is said to be stable if  $P(Y) = 0$  for every  $P$ -null set  $Y$  which belongs to  $\mathcal{Q}$ .

**PROPOSITION 6.5.** *If  $\mathcal{Q}$  is an algebra of sets and  $P : \mathcal{Q} \rightarrow \text{BL}(E)$  a bounded and stable spectral set function, then  $P$  is closable.*

**Proof.** Let  $[\text{sim}(\mathcal{Q})] = \{[f] : f \in \text{sim}(\mathcal{Q})\}$ . Because  $P$  is stable, there is a map  $\tilde{P} : [\text{sim}(\mathcal{Q})] \rightarrow \text{BL}(E)$ , unambiguously defined by  $\tilde{P}([f]) = P(f)$ , for every  $f \in \text{sim}(\mathcal{Q})$ . Because  $P$  is bounded and  $\mathcal{Q}$  is an algebra, by Lemma 1, the map  $\tilde{P}$  is bounded. Then  $\tilde{P}$  has a unique continuous extension onto the whole of  $L^{\infty}(P)$ . By Lemma 1,  $\tilde{P}$  and its extension are norm-increasing. Therefore,  $\tilde{P}$  so extended has an inverse,  $\Phi$ , which is norm-decreasing. Because both maps,  $\tilde{P}$  and  $\Phi$ , are bounded, the domain of  $\Phi$  is closed and, hence, equal to  $A(P)$ . So, by Proposition 6.4, the set function  $P$  is closable.

**COROLLARY 6.6.** *Let  $P : \mathcal{Q} \rightarrow \text{BL}(E)$  be a spectral set function such that, for every  $x \in E$  and  $x' \in E'$ , the set function  $X \mapsto x' P(X)x$ ,  $X \in \mathcal{Q}$ , generates a  $\sigma$ -additive measure of finite variation. Then the set function  $P$  is closable.*

**Proof.** The assumption implies that the additive extension of  $P$  onto the algebra of sets generated by  $\mathcal{Q}$  is bounded and stable.

Corollary 6.6 implies, in particular, that a  $\sigma$ -additive spectral measure whose domain is a  $\sigma$ -algebra of sets is closable.

E. Let us call a Boolean quasialgebra of projections  $W \subset \text{BL}(E)$  semisimple if the Banach algebra,  $A(W)$ , it generates is semisimple.

**PROPOSITION 6.7.** *A Boolean quasialgebra of projection operators,  $W \subset \text{BL}(E)$ , is semisimple if and only if there exists a quasialgebra of sets,  $\mathcal{Q}$ , in a space  $\Omega$ , and a closable spectral set function,  $P: \mathcal{Q} \rightarrow \text{BL}(E)$ , such that  $A(W) = A(P)$ .*

**Proof.** Let  $W$  be semisimple. Let  $\Omega$  be the structure space of the Banach algebra  $A(W)$ . Let us denote by  $\Phi$  the Gelfand transform and put  $\mathcal{Q} = \{\Phi(S) : S \in W\}$ . Because we identify sets with their characteristic functions,  $\mathcal{Q}$  is a quasialgebra of sets in the space  $\Omega$ . Let  $P(\Phi(S)) = S$ , for every  $S \in W$ . This defines a spectral set function  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  such that the empty set is the only  $P$ -null set. Therefore,  $L^\infty(P) = C(\Omega)$  and the Gelfand transform is clearly a norm-decreasing injective map from  $A(P) = A(W)$  into  $L^\infty(P)$  such that  $\Phi(P(f)) = [f]$  for every  $f \in \text{sim}(\mathcal{Q})$ . So, by Proposition 6.4, the spectral set function  $P$  is closable.

Conversely, if a closable spectral set function  $P$  such that  $A(W) = A(P)$  exists, then, by Proposition 6.3, the Banach algebra  $A(W)$  is semisimple.

**COROLLARY 6.8.** *Any bounded Boolean algebra of projections is semisimple.*

**Proof.** By the Stone representation theorem, for any Boolean algebra of operators,  $W$ , there exists an algebra of sets,  $\mathcal{Q}$ , and a spectral set function  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  such that  $\emptyset$  is the only  $P$ -null set and  $\{P(X) : X \in \mathcal{Q}\} = W$ . By Proposition 4.5, the set function  $P$  is closable.

Let us call an operator  $T \in \text{BL}(E)$  scalar in the wider sense if there exists a semisimple Boolean quasialgebra of operators  $W \subset \text{BL}(E)$  such that  $T \in A(W)$ . By Proposition 4.7, and Proposition 4.3, an operator  $T$  is scalar in the wider sense if and

only if there exist a quasialgebra of sets,  $\mathcal{Q}$ , in a space  $\Omega$ , a closable spectral set function  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  and a  $P$ -integrable function  $f$  such that  $T = P(f)$ .

An operator is said to be scalar in the sense of N. Dunford if there exist a  $\sigma$ -algebra of sets,  $\mathcal{Q}$ , in a space  $\Omega$ , a  $\sigma$ -additive spectral measure  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  and a function  $f \in \mathcal{L}(P)$  such that  $T = P(f)$ . We may also call such operators  $\sigma$ -scalar. By Corollary 6.6, operators which are scalar in the sense of Dunford are scalar in the wider sense. Moreover, these operators can be characterized in terms introduced here.

By a Boolean  $\sigma$ -algebra of projection operators is understood a Boolean algebra of projection operators which contains the strong limit of every monotonic sequence of its elements.

**PROPOSITION 6.9.** *An operator  $T \in \text{BL}(E)$  is scalar in the sense of Dunford if and only if there exists a Boolean  $\sigma$ -algebra of projection operators,  $W \subset \text{BL}(E)$ , such that  $T \in A(W)$  and every element of  $W$  commutes with every operator from  $\text{BL}(E)$  which commutes with  $T$ .*

**Proof.** If the operator  $T \in \text{BL}(E)$  is scalar in the sense of Dunford, then there exist a  $\sigma$ -algebra of sets,  $\mathcal{Q}$ , in a space  $\Omega$ , a  $\sigma$ -additive spectral measure  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  and a function  $f \in \mathcal{L}(P)$  such that  $T = P(f)$ . Let  $\mathcal{Q}_f$  be the minimal  $\sigma$ -algebra of sets such that  $\mathcal{Q}_f \subset \mathcal{Q}$  and, if  $P_f$  is the restriction of  $P$  to  $\mathcal{Q}_f$ , then  $f \in \mathcal{L}(P_f)$ . The range,  $W = \{P(X) : X \in \mathcal{Q}_f\}$ , of the spectral measure  $P_f$  is then a Boolean  $\sigma$ -algebra of projections such that  $T \in A(W)$  and every element of  $W$  commutes with every operator commuting with  $T$ .

Conversely, let  $W \subset \text{BL}(E)$  be a Boolean  $\sigma$ -algebra of projections such that  $T \in A(W)$ . By the Stone representation theorem there exist a compact space  $\Omega$ , an algebra  $\mathcal{R}$  consisting of its compact and open subsets and a spectral set function  $P: \mathcal{R} \rightarrow \text{BL}(E)$  such that  $W = \{P(X) : X \in \mathcal{R}\}$ . Let  $\mathcal{Q}$  be the  $\sigma$ -algebra of sets generated by  $\mathcal{R}$ . Because  $P$  is in fact  $\sigma$ -additive and  $W$  is a  $\sigma$ -algebra of operators, the set function  $P$  has a strongly  $\sigma$ -additive extension onto  $\mathcal{Q}$ , still

denoted by  $P$ , whose range remains equal to  $W$ ; see, for example, [30]. Then  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  is a spectral measure such that, by Proposition 4.3,  $T = P(f)$ , for some function  $f \in \mathcal{L}(P)$ .

Operators which are scalar in the wider sense but not scalar in the sense of Dunford abound. A way of producing a wealth of such operators is indicated by the following

**EXAMPLE 6.10.** Let  $\Omega = (0,1]$ ,  $\mathcal{Q} = \{(s,t] : 0 \leq s \leq t \leq 1\}$ . Let  $p > 1$  and  $\rho(X) = (\iota(X))^{1/p}$ , for every  $X \in \mathcal{Q}$ , where  $\iota$  is the one-dimensional Lebesgue measure. By Proposition 2.13 and Proposition 2.26,  $\rho$  is an integrating gauge on  $\mathcal{Q}$ . Let  $E = L(\rho, \mathcal{Q})$ .

For every  $X \in \mathcal{Q}$ , let  $P(X)$  be the operator of point-wise multiplication by the characteristic function of the set  $X$ . That is,  $P(X)[u]_\rho = [Xu]_\rho$ , for every  $u \in \mathcal{L}(\rho, \mathcal{Q})$ . Because  $\mathcal{L}(\rho, \mathcal{Q}) \neq \mathcal{L}^p(\iota)$  (see Example 4.16(ii) in Section 4C) the so-defined spectral set function  $P: \mathcal{Q} \rightarrow \text{BL}(E)$  is surely not  $\sigma$ -additive; indeed, its additive extension on the algebra of sets generated by  $\mathcal{Q}$  is not bounded. Nevertheless,  $P$  is closable. Moreover, if  $n \geq 1$  is an integer and a set  $X$  is equal to the union of  $n$  pair-wise disjoint sets,  $X_k$ ,  $k = 1, 2, \dots, n$ , belonging to  $\mathcal{Q}$ , then  $\|P(X)\| \leq n^{(p-1)/p}$ . In fact, let  $u$  be a function belonging to  $\mathcal{L}(\rho, \mathcal{Q})$ . Let  $c_j$  be numbers and  $Y_j \in \mathcal{Q}$  sets,  $j = 1, 2, \dots$ , such that

$$\sum_{j=1}^{\infty} |c_j| \rho(Y_j) < \infty$$

and

$$u(\omega) = \sum_{j=1}^{\infty} c_j Y_j(\omega)$$

for every  $\omega \in \Omega$  for which

$$\sum_{j=1}^{\infty} |c_j| Y_j(\omega) < \infty.$$

Then

$$q_\rho(Y_j \cap X) \leq \sum_{k=1}^n \rho(Y_j \cap X_k) \leq n^{(p-1)/p} \rho(Y_j),$$

for every  $j = 1, 2, \dots$ , and

$$(Xu)(\omega) = \sum_{j=1}^{\infty} c_j Y_j(\omega) X(\omega)$$

for every  $\omega \in \Omega$  for which

$$\sum_{j=1}^{\infty} |c_j| (Y_j \cap X)(\omega) < \infty.$$

Therefore,  $Xu \in \mathcal{L}(\rho, \mathcal{Q})$  and  $q_{\rho}(Xu) \leq n^{(p-1)/p} q_{\rho}(u)$ .

Now, let  $Z$  be the function on  $\mathbb{R}$  which is periodic with period 1 and its restriction to  $\Omega$  is equal to the characteristic function of the interval  $(\frac{1}{2}, 1]$ . For every  $j = 1, 2, \dots$ , let  $X_j$  be the function  $\omega \mapsto Z(2^{j-1}\omega)$ ,  $\omega \in \Omega$ . Hence,  $X_j \in \text{sim}(\mathcal{Q})$  and  $\|P(X_j)\| \leq 2^{(j-1)(p-1)/p}$ , for every  $j = 1, 2, \dots$ . Also, if  $f(\omega) = \omega$ , then

$$f(\omega) = \sum_{j=1}^{\infty} 2^{-j} X_j(\omega),$$

for every  $\omega \in \Omega$ . Therefore,  $f \in \mathcal{L}(P)$  and

$$\|P(f)\| \leq \sum_{j=1}^{\infty} 2^{-j} 2^{(j-1)(p-1)/p} = \frac{1}{2} \frac{2^{1/p}}{2^{1/p} - p}.$$

F. This and the next sections are devoted to an example, or, rather, a class of examples, which is sufficiently rich to display all the features of the presented theory.

Let  $G$  be a locally compact Abelian group and  $\Gamma$  its dual group. The value of a character  $\xi \in \Gamma$  on an element  $x \in G$  is denoted by  $\langle x, \xi \rangle$ .

Let  $1 < p < \infty$  and let  $E = L^p(G)$ , with respect to a fixed Haar measure on the group  $G$ .

Let  $\mathcal{M}^p(\Gamma)$  be the family of all individual functions on  $\Gamma$  which determine multiplier operators on  $E$ . That is,  $f \in \mathcal{M}^p(\Gamma)$  if and only if there exists an operator  $T_f \in \text{BL}(E)$  such that  $(T_f \varphi)^{\wedge} = f^{\wedge} \hat{\varphi}$ , for every  $\varphi \in L^2 \cap L^p(G)$ . Here, of course,  $\hat{\varphi}$  denotes the Fourier-Plancherel transform of an element  $\varphi$  of  $L^2(G)$ .

Functions belonging to  $\mathcal{M}^p(\Gamma)$  are essentially bounded. In fact,  $\|f\|_\infty \leq \|T_f\|$ , for every  $f \in \mathcal{M}^p(\Gamma)$ , where  $\|f\|_\infty$  is the essential supremum norm of  $f$  with respect to the Haar measure. The operator  $T_f$  depends only on the equivalence class of the function  $f$ . That is, if  $f \in \mathcal{M}^p(\Gamma)$  and if  $g$  is a function on  $\Gamma$  such that  $g(\xi) = f(\xi)$  for almost every  $\xi \in \Gamma$ , relative to the Haar measure, then  $g \in \mathcal{M}^p(\Gamma)$  and  $T_g = T_f$ .

It is well-known that an operator  $T \in \text{BL}(E)$  commutes with all translations of  $G$  if and only if there exists a function  $f \in \mathcal{M}^p(\Gamma)$  such that  $T = T_f$ . So,  $\{T_f : f \in \mathcal{M}^p(\Gamma)\}$  is a commutative algebra of operators, containing the identity operator, which is closed in  $\text{BL}(E)$ . Clearly,  $\mathcal{M}^p(\Gamma)$  is an algebra of functions and the map  $f \mapsto T_f$ ,  $f \in \mathcal{M}^p(\Gamma)$ , is multiplicative and linear.

Let  $\mathcal{R}^p(\Gamma)$  be the family of all sets  $X \subset \Gamma$  such that  $X \in \mathcal{M}^p(\Gamma)$ . Let  $P_\Gamma^p(X) = T_X$ , for every  $X \in \mathcal{R}^p(\Gamma)$ .

**PROPOSITION 6.11.** *The family  $\mathcal{R}^p(\Gamma)$  is an algebra of sets in  $\Gamma$  and  $P_\Gamma^p : \mathcal{R}^p(\Gamma) \rightarrow B(L^p(G))$  is a closable spectral set function.*

**Proof.** It follows from the mentioned properties of the map  $f \mapsto T_f$ ,  $f \in \mathcal{M}^p(\Gamma)$ , that  $\mathcal{R}^p(\Gamma)$  is an algebra of sets and the set function  $P = P_\Gamma^p$  is spectral. Furthermore, a set  $Y \subset \Gamma$  is  $P$ -null if and only if it is null with respect to the Haar measure on  $\Gamma$ . Consequently the Haar measure equivalence classes of functions on  $\Gamma$  are the same as the  $P$ -equivalence classes and so are their  $\infty$ -norms. Therefore,  $L^\infty(P)$  is a Banach subspace of  $L^\infty(\Gamma)$ . Now,  $A(P)$  is a closed subalgebra of the Banach algebra  $\{T_f : f \in \mathcal{M}^p(\Gamma)\}$ . For every  $T \in A(P)$ , let  $\Phi(T) = [f]$ , where  $f \in \mathcal{M}^p(\Gamma)$  is a function such that  $T = T_f$ . Then  $\Phi$  is an unambiguously defined norm-decreasing map from  $A(P)$  into  $L^\infty(P)$  such that  $\Phi(P(f)) = [f]$ , for every  $f \in \text{sim}(\mathcal{R}^p(\Gamma))$ . Therefore, by Proposition 6.4, the set function  $P$  is closable.

The usefulness of this proposition depends of course on how rich is the algebra of sets  $\mathcal{R}^p(\Gamma)$ . A result of T.A. Gillespie implies that it is rich enough to permit complete spectral analysis of translation operators. Let us introduce the necessary relevant notation.



Let  $\mathbb{T}$  be the circle group,  $\{z \in \mathbb{C} : |z| = 1\}$ , with its usual topology of a subset of the complex plane. Connected subsets of  $\mathbb{T}$  will be called arcs. For an element  $x$  of the group  $G$  and an arc  $Z \subset \mathbb{T}$ , let

$$X_{Z,x} = \{\xi \in \Gamma : \langle x, \xi \rangle \in Z\}.$$

Let  $\mathcal{K}_1(\Gamma)$  be the family of all sets  $X_{Z,x}$  corresponding to arcs  $Z \subset \mathbb{T}$  and elements of  $x \in G$ . The classes of sets  $\mathcal{K}_n(\Gamma)$ ,  $n = 2, 3, \dots$ , are then defined recursively by requiring that  $\mathcal{K}_n(\Gamma)$  consist of all sets  $X \cap Y$  such that  $X \in \mathcal{K}_{n-1}(\Gamma)$  and  $Y \in \mathcal{K}_1(\Gamma)$ .

**LEMMA 6.12.** *The inclusion  $\mathcal{K}_n(\Gamma) \subset \mathcal{R}^p(\Gamma)$  is valid for every  $p \in (1, \infty)$  and every  $n = 1, 2, \dots$ . Moreover, for every  $p \in (1, \infty)$ , there exists a constant  $C_p \geq 1$  such that  $\|P_\Gamma^p(X)\| \leq C_p^n$ , for every  $X \in \mathcal{K}_n(\Gamma)$ , every  $n = 1, 2, \dots$ , and every locally compact Abelian group  $\Gamma$ .*

**Proof.** For  $n = 1$ , this is a simple re-formulation of Lemma 6 of [18]. (See also Lemma 20.15 in [12].) By induction, the result follows for every  $n = 2, 3, \dots$ .

Let  $\mathcal{J}_1$  be the family of all subsets of  $\mathbb{R}$  which contains all members of  $\mathcal{K}_1(\mathbb{R})$  and all intervals in  $\mathbb{R}$  and no other sets. The families  $\mathcal{J}_n$ ,  $n = 2, 3, \dots$ , are then defined recursively by requiring that  $\mathcal{J}_n$  consist of all sets  $X \cap Y$  such that  $X \in \mathcal{J}_{n-1}$  and  $Y \in \mathcal{J}_1$ .

If we combine Lemma 6.12 with a classical theorem of M. Riesz (interpreted to the effect that intervals belong to  $\mathcal{M}^p(\mathbb{R})$  and determine a bounded family of multiplier operators; see e.g. [8], Theorem 6.3.3) we obtain the following

**COROLLARY 6.13.** *The inclusion  $\mathcal{J}_n \subset \mathcal{R}^p(\mathbb{R})$  is valid for every  $p \in (1, \infty)$  and every  $n = 1, 2, \dots$ . Moreover, for every  $p \in (1, \infty)$ , there exists a constant  $D_p \geq 1$  such that  $\|P_\mathbb{R}^p(X)\| \leq D_p^n$ , for every  $X \in \mathcal{J}_n$  and  $n = 1, 2, \dots$ .*

**G.** The (total) variation of a function  $f$  of bounded variation on  $\mathbb{R}$  or on  $\mathbb{T}$  will be denoted by  $\text{var}(f)$ . Recall that every function,  $f$ , of bounded variation has a decomposition,  $f = f_1 + f_2 + f_3$ , such that the function  $f_1$  is absolutely

continuous,  $f_2$  is continuous and singular (its derivative vanishes almost everywhere) and  $f_3$  is a jump-function. If the function  $f$  vanishes at a point (or at  $-\infty$ ) then there is only one such decomposition with all the three components,  $f_1$ ,  $f_2$  and  $f_3$ , vanishing at that point. If the continuous singular component,  $f_2$ , is identically equal to zero, then the function  $f$  is called non-singular.

**LEMMA 6.14.** *Let  $\alpha, \beta$  and  $b$  be real numbers such that  $\alpha \leq \beta$ . Let  $u$  be the function on  $\mathbb{R}$  such that  $u(t) = 0$  for  $t < \alpha$ ,  $u(t) = b(t - \alpha)$  for  $\alpha \leq t \leq \beta$ , and  $u(t) = b(\beta - \alpha)$  for  $t \geq \beta$ . Then there exist numbers  $c_j$  and sets  $X_j \in \mathcal{J}_2$ ,  $j = 0, 1, 2, \dots$ , such that*

$$\sum_{j=0}^{\infty} |c_j| \|P_{\mathbb{R}}^p(X_j)\| \leq 2D_p^2 \text{var}(u)$$

and

$$\sum_{j=0}^{\infty} c_j X_j(t) = u(t)$$

for every  $t \in \mathbb{R}$ .

**Proof.** Because  $\text{var}(u) = |b|(\beta - \alpha)$ , by Corollary 6.13, the statement holds with  $c_j = 2^{-j}b(\beta - \alpha)$ ,  $j = 0, 1, 2, \dots$ ,  $X_0 = [\beta, \infty)$  and

$$X_j = \left\{ t \in \mathbb{R} : \exp \left[ \frac{2^j \pi (t - \alpha)}{\beta - \alpha} i \right] \in \{ \exp si : \pi \leq s < 2\pi \} \right\} \cap [\alpha, \beta),$$

$j = 1, 2, \dots$ .

**PROPOSITION 6.15.** *Let  $f$  be a real non-singular function of bounded variation on  $\mathbb{R}$  such that  $f(-\infty) = 0$ . Then  $f \in \mathcal{L}(P_{\mathbb{R}}^p)$  and*

$$(G.1) \quad P_{\mathbb{R}}^p(f) \leq 3D_p^2 \text{var}(f),$$

for every  $p \in (1, \infty)$ .

**Proof.** Let  $f = f_1 + f_3$  for a function  $g$ , integrable on  $\mathbb{R}$ , such that

$$f_1(t) = \int_{-\infty}^t g(s) ds,$$

$t \in \mathbb{R}$ , and a jump-function  $f_3$  vanishing at  $-\infty$ . Then  $\text{var}(f) = \text{var}(f_1) + \text{var}(f_3)$ . Moreover, there exist numbers  $c_j$  and intervals  $X_j$ ,  $j = 1, 2, 3, \dots$ , such that

$$f_3(t) = \sum_{j=1}^{\infty} c_j X_j(t),$$

for every  $t \in \mathbb{R}$ , and

$$\text{var}(f_3) = \sum_{j=1}^{\infty} |c_j|.$$

There also exist numbers  $b_j$  and bounded intervals  $Y_j$ ,  $j = 1, 2, \dots$ , such that, if

$$u_j(t) = \int_{-\infty}^t b_j Y_j(s) ds,$$

for every  $t \in \mathbb{R}$  and  $j = 1, 2, \dots$ , then

$$\sum_{j=1}^{\infty} \text{var}(u_j) = \sum_{j=1}^{\infty} |u_j(\infty)| < \frac{3}{2} \int_{-\infty}^{\infty} |g(s)| ds = \frac{3}{2} \text{var}(f_1)$$

and

$$f_1(t) = \sum_{j=1}^{\infty} u_j(t)$$

for every  $t \in \mathbb{R}$ . Hence, by Lemma 6.14, and Proposition 2.1,  $f \in \mathcal{L}(P_{\mathbb{R}}^p)$  and the inequality (G.1) holds.

This proposition points at the richness of the space  $\mathcal{L}(P_{\mathbb{R}}^p)$ . To be sure, this space also contains functions of bounded variation which do not vanish at  $-\infty$  and many functions of unbounded variation. In fact, it also contains many functions of unbounded  $r$ -variation, for any  $r > 1$ , because already the characteristic functions of many sets from  $\mathcal{J}_2$  are such. (In this context, see [24].) As  $\mathcal{L}(P_{\mathbb{R}}^p) \subset \mathcal{M}^p(\mathbb{R})$ , we have a large class of multiplier operators which are scalar in the wider sense.

**LEMMA 6.16.** *Let  $r, \alpha, \beta$  and  $b$  be real numbers such that  $r \leq \alpha < \beta \leq r + 2\pi$ . Let  $u$  be the function on  $\mathbb{T}$  such that  $u(\exp it) = 0$  for  $r \leq t < \alpha$ ,  $u(\exp it) = b(t - \alpha)$  for  $\alpha \leq t < \beta$ , and  $u(\exp it) = b(\beta - \alpha)$  for  $\beta \leq t < r + 2\pi$ . Then there exist numbers  $c_j$  and sets  $X_j \in \mathcal{K}_2(\mathbb{T})$ ,  $j = 0, 1, 2, \dots$ , such that*

$$\sum_{j=0}^{\infty} |c_j| \|P_{\mathbb{T}}^p(X_j)\| \leq C_p^2 \text{var}(u)$$

and

$$\sum_{j=1}^{\infty} c_j X_j(z) = u(z)$$

for every  $z \in \mathbb{T}$ .

**Proof.** Let  $m$  be the largest integer such that  $m(\beta - \alpha) \leq 2\pi$ . Let  $\gamma = \alpha + 2\pi m^{-1}$ . Note that  $\text{var}(u) = 2|b|(\beta - \alpha)$ ,  $r \leq \alpha < \beta \leq \gamma \leq r + 2\pi$  and  $m(\gamma - \alpha) = 2\pi$ . Hence, by Lemma 6.12, it suffices to take  $c_0 = b(\beta - \alpha)$ ,  $X_0 = \{\exp ti : \beta \leq t < r + 2\pi\}$ ,  $c_j = 2^{1-j} \pi b m^{-1}$  and

$$X_j = \{\exp(\gamma - t)i : \exp(2^{j-1} m t i) \in \{\exp si : 0 < s \leq \pi\}\} \cap \{\exp ti : \alpha \leq t < \beta\}$$

for  $j = 1, 2, \dots$ .

**PROPOSITION 6.17.** Let  $r \in \mathbb{R}$  and let  $f$  be a real non-singular function of bounded variation on  $\mathbb{T}$  such that  $f(\exp ri) = 0$ . Then  $f \in \mathcal{L}(P_{\mathbb{T}}^p)$  and

$$P_{\mathbb{T}}^p(f) \leq 2C_p^2 \text{var}(f)$$

for every  $p \in (1, \infty)$ .

**Proof.** It is analogous to that of Proposition 6.15 only Lemma 6.16 is used instead of Lemma 6.14.

**COROLLARY 6.18.** Let  $x \in G$ , let  $u$  be a non-singular function of bounded variation on  $\mathbb{T}$  and let  $f(\xi) = u(\langle x, \xi \rangle)$ , for every  $\xi \in \Gamma$ . Then  $f \in \mathcal{L}(P_{\Gamma}^p)$  for every  $p \in (1, \infty)$ .

**Proof.** A power of a character of a group is a character and all characters of  $\mathbb{T}$  are powers of a single one, namely the identity function on  $\mathbb{T}$ . Interpreting  $G$  as the group of characters of  $\Gamma$  we see immediately that, for every  $Y \in \mathcal{K}_n(\mathbb{T})$ , the set  $X = \{\xi \in \Gamma : \langle x, \xi \rangle \in Y\}$  belongs to  $\mathcal{K}_n(\Gamma)$ ,  $n = 1, 2, \dots$ . So, Lemma 6.12 and Proposition 6.17 imply the result.

Now, each element,  $x$ , of the group  $G$  is interpreted as a function on  $\Gamma$  - the character it generates - that is, the function  $\xi \mapsto \langle x, \xi \rangle$ ,  $\xi \in \Gamma$ . Then  $x \in \mathcal{M}^p(\Gamma)$  and  $T_x$  is the operator of translation by  $x$ . By Corollary 4.17,  $x \in \mathcal{L}(P_\Gamma^p)$  and

$$(G.2) \quad T_x = \int_{\Gamma} \langle x, \xi \rangle P_\Gamma^p(d\xi),$$

for every  $x \in G$ . For  $p = 2$ , this is an instance of Stone's theorem (see e.g. [46], 36E).

Some observations about the Stone formula (G.2) could be of interest because they could possibly have somewhat wider implications. Its proof shows that,  $x \in \mathcal{L}(\rho_p, \mathcal{K}_2(\Gamma))$ , for every  $x \in G$ , where  $P = P_\Gamma^p$  with any  $p \in (1, \infty)$ . That is to say, for every  $x \in G$ , there exist numbers  $c_j$  and sets  $X_j \in \mathcal{K}_2(\Gamma)$ ,  $j = 1, 2, \dots$ , which depend of course on  $x$  but not on  $p$ , such that

$$\sum_{j=1}^{\infty} |c_j| \|P_\Gamma^p(X_j)\| < \infty,$$

the equality

$$\langle x, \xi \rangle = \sum_{j=1}^{\infty} c_j X_j(\xi)$$

holds for every  $\xi \in \Gamma$  and

$$T_x = \sum_{j=1}^{\infty} c_j P_\Gamma^p(X_j),$$

for every  $p \in (1, \infty)$ . Hence for each  $p \in (1, \infty)$ , the translation operator,  $T_x$ , is expressed as the sum of the same multiples of the projections  $P_\Gamma^p(X_j)$ ,  $j = 1, 2, \dots$ . These projections too are 'the same' for each  $p$ , only the space,  $E = L^p(G)$ , in which they operate varies with  $p$ .

Also the fact that the sets  $X_j$ ,  $j = 1, 2, \dots$ , belong to the class  $\mathcal{K}_2(\Gamma)$  may possibly be worth noting. The algebra  $\mathcal{R}^p(\Gamma)$  contains of course also sets of much greater complexity than those belonging to  $\mathcal{K}_2(\Gamma)$ . It seems that it would contribute considerably to our understanding of multiplier operators to know what kind of sets, besides those belonging to the classes  $\mathcal{K}_n(\Gamma)$ ,  $n = 1, 2, \dots$ , are in the algebra  $\mathcal{R}^p(\Gamma)$ . The classes  $\mathcal{J}_n$ ,  $n = 1, 2, \dots$ , give us some indication in the case  $\Gamma = \mathbb{R}$ .