RINGS OF QUOTIENTS OF ULTRAPRIME BANACH ALGEBRAS. WITH APPLICATIONS TO ELEMENTARY OPERATORS

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1. INTRODUCTION

From its beginning, the investigation of non-commutative prime rings has been connected with the concept of (right) rings of quotients. The former begun by McCoy in [22] and continued by Johnson [15] strongly influenced the development of the latter, whose early stages are related with the names of Johnson [16], Utumi [29], Findlay and Lambek [7], and others. Both ideas have found important applications in many areas of algebra, for instance in the theory of GPI-rings. A comprehensive account of the state of art of rings of quotients was given in Stenström's book [25], where the more sophisticated approach via Gabriel topologies is used.

In the setting of *commutative* Banach algebras, Suciu studied algebras of quotients which can be normed ([26], [27]). For non-commutative prime normed algebras A over $\mathbb{C}$ there are, however, obstructions to endow the appropriate 'algebra of quotients', viz. Utumi's maximal ring of quotients, $Q_a(A)$, with a norm. The center C of $Q_a(A)$, the so-called *extended centroid of* A, is a field containing the center of A and, endowed with an algebra norm, would therefore coincide with $\mathbb{C}$ by Mazur's theorem. As often in Banach algebra theory, the truly non-commutative algebras thus play a distinguished role among general Banach algebras.

In the present paper we will introduce a topological version of primeness in the following sense. We call a normed algebra A *ultraprime* if its ultrapower $\hat{A}_{\mathcal{U}}$ with respect to some countably incomplete ultrafilter $\mathcal{U}$ is a prime algebra. Together with the corresponding notion of an ultraprime ideal and an intrinsic characterization (Lemma 3.1), some basic properties and examples of ultraprime normed algebras will be presented in Section 3. In particular, we prove that the center of every ultraprime normed algebra is trivial and that completion yields an ultraprime Banach algebra (both assertions fail for arbitrary prime normed algebras). In Section 4 we will construct, for each ultraprime algebra A, an ultraprime normed algebra of quotients, $Q(A)$, which contains a homeomorphic image of A and whose center coincides with the extended centroid of A (Theorem 4.1). This construction uses in a natural way two automatic continuity results for right A-module homomorphisms (Lemmas 4.3 and 4.5). *A posteriori*, this is quite understandable; for example, Kaplansky's well-known result that the centroid of a primitive Banach algebra A is trivial ([17], Lemma 9) boils down to the fact that every A-bimodule homomorphism on A is continuous. The behaviour of $Q(A)$ is somewhat similar to that of $Q_a(A)$; e.g., for every (topologically) simple unital Banach algebra A we have that $Q_a(A) = Q(A) = A$.

Since we are using the notion of an ultrapower of a normed algebra both as a conceptual means as well as a technical device, we have collected several mainly known basic results on ultrapowers of normed algebras and (multi-)linear mappings in Section 2. We have however refrained from making use of the language of non-standard analysis.

Originally, our study of ultraprime Banach algebras which was begun in [20] was motivated by questions concerning properties of elementary operators on Banach algebras. We therefore will apply some of the results on ultraprime algebras, in particular Theorem 4.1, to determine the structure of the algebra $\mathcal{E}\ell(A)$ which is generated by all left and right multiplications on A. Denoting by A^{op} the opposite algebra of a normed algebra A we have in the case of a prime algebra that $A \otimes_C A^{\mathrm{op}} \cong \mathcal{E}\ell(A)$, and thus, if A is ultraprime, $A \otimes A^{\mathrm{op}} \cong \mathcal{E}\ell(A)$ (Theorem 5.1). Some information on the spectra of elementary operators will conclude this paper. We remark that many of the results in Section 5 were previously obtained in the special case of prime C*-algebras in [21]; therefore, we merely provide the additional arguments, where necessary.

The idea of ultraprimeness as a tool in the spectral theory of elementary operators is reminiscent in many of the existing contributions. It was used for the first time by Lumer and Rosenblum in the formulation of condition (a) in Lemma 3.1 (see below) for the algebra $L(E)$ of all bounded operators on some Banach space E in their paper [18], and afterwards taken up by a number of their successors. It appeared explicitly in [19], § 4.

Although primitive Banach algebras are the prototypes of non-commutative prime Banach algebras and are *centrally closed*, i.e. the extended centroid C is trivial, the interrelations between them and ultraprime Banach algebras remain yet undetermined. This problem may be connected with Kaplansky's (still unsolved) question whether every prime C*-algebra is primitive.

2. PREREQUISITES ON ULTRAPOWERS OF NORMED ALGEBRAS

In this section we record all the basic properties of ultrapowers which will be needed in the sequel. Some of this material is taken from [3], [11] and [24], and for some others we have no reference. We will always assume that $\mathcal{U}$ is a *countably incomplete* ultrafilter on an infinite set $\mathbf{I}$, i.e. there is a sequence of elements $U_k \in \mathcal{U}$ satisfying

$$U_1 \supseteq U_2 \supseteq \ldots \supseteq U_k \supseteq \ldots \quad \text{and} \quad \bigcap_{k=1}^{\infty} U_k = \emptyset.$$

For example, every free ultrafilter on $\mathbf{N}$ is countably incomplete.

Let E be a normed space, and denote by $\ell^\infty(\mathbf{I}, E)$ the space of all bounded functions from $\mathbf{I}$ into E. The set $n_{\mathcal{U}} = \{(x_i)_{i \in \mathbf{I}} \in \ell^\infty(\mathbf{I}, E) \mid \lim_{\mathcal{U}} \|x_i\| = 0\}$ is a closed subspace of $\ell^\infty(\mathbf{I}, E)$ (equipped with the sup-norm), and the quotient $\ell^\infty(\mathbf{I}, E)/n_{\mathcal{U}}$ is called *the ultrapower of* E *(with respect to the ultrafilter* $\mathcal{U}$*)*; we will denote it by $\hat{E}_{\mathcal{U}}$. If we write (x_i) for the elements in $\ell^\infty(\mathbf{I}, E)$ and $(x_i)_{\mathcal{U}}$ for the cosets in $\hat{E}_{\mathcal{U}}$, we have that $\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|$ (cf., e.g., [24]). We will identify E with an isometric subspace of $\hat{E}_{\mathcal{U}}$ via $x \mapsto (x, x, \ldots)_{\mathcal{U}}$.

The following is an immediate consequence of the ultrafilter property.

PROPOSITION 2.1. *If* $\hat{x} \in \hat{E}_{\mathcal{U}}$ *has norm one, then there is* $(x_i) \in \ell^\infty(\mathbf{I}, E)$ *with* $(x_i)_{\mathcal{U}} = \hat{x}$ *and* $\|x_i\| = 1$ *for all* $i \in \mathbf{I}$.

If A is a normed algebra (over $\mathbf{C}$), the 'coordinatewise' operations in $\ell^\infty(\mathbf{I}, A)$ will induce algebra operations in $\hat{A}_{\mathcal{U}}$, i.e. $(x_i)_{\mathcal{U}}(y_i)_{\mathcal{U}} := (x_i y_i)_{\mathcal{U}}$. Endowed with these, $\hat{A}_{\mathcal{U}}$ becomes a Banach algebra, and a C*-algebra if A is a C*-algebra.

Suppose now that E, $E^{(1)}, \ldots, E^{(n)}$ are normed spaces and $T : E^{(1)} \times \ldots \times E^{(n)} \to E$ is a bounded multilinear mapping. Putting

$$\hat{T}_{\mathcal{U}}(x_i^{(1)}, \ldots, x_i^{(n)})_{\mathcal{U}} := (T(x_i^{(1)}, \ldots, x_i^{(n)}))_{\mathcal{U}}$$

we obtain a well-defined mapping into $\hat{E}_{\mathcal{U}}$ which is bounded and multilinear if we identify $(E^{(1)} \oplus \ldots \oplus E^{(n)})\hat{}_{\mathcal{U}}$ with $\hat{E}_{\mathcal{U}}^{(1)} \oplus \ldots \oplus \hat{E}_{\mathcal{U}}^{(n)}$ canonically using the ℓ^1-norm on both spaces. We have $\|T\| = \|\hat{T}_{\mathcal{U}}\|$. (The arguments for the linear case, cf. [24], carry over verbatim.)

Applying this to bilinear mappings yields the following.

PROPOSITION 2.2. *Let A be a (unital) normed algebra and M be a (unit linked) normed A-module. Then $\hat{M}_{\mathcal{U}}$ is a (unit linked) Banach $\hat{A}_{\mathcal{U}}$-module. In particular, if I is an ideal of A, then $\hat{I}_{\mathcal{U}}$ is a closed ideal of $\hat{A}_{\mathcal{U}}$.*

PROPOSITION 2.3. *Let I be a closed ideal of the normed algebra A. Then the ultrapower $(A/I)\hat{}_{\mathcal{U}}$ of the quotient algebra A/I and the quotient algebra $\hat{A}_{\mathcal{U}}/\hat{I}_{\mathcal{U}}$ are isometrically isomorphic.*

Outline of proof. Since the canonical epimorphism $\pi : A \to A/I$ is open, its extension $\hat{\pi}_{\mathcal{U}} : \hat{A}_{\mathcal{U}} \to (A/I)\hat{}_{\mathcal{U}}$ is open too. If $\hat{x} = (x_i)_{\mathcal{U}} \in \hat{A}_{\mathcal{U}}$ then, for each $\varepsilon > 0$, there is $(y_i) \in \ell^\infty(\mathbf{I}, I)$ such that $\|\pi(x_i)\| \geq \|x_i + y_i\| - \varepsilon$ whence

$$\|\hat{\pi}_{\mathcal{U}}(\hat{x})\| = \lim_{\mathcal{U}} \|\pi(x_i)\| \geq \lim_{\mathcal{U}} \|x_i + y_i\| - \varepsilon.$$

Thus, $\|\hat{\pi}_{\mathcal{U}}(\hat{x})\| \geq \|\rho(\hat{x})\|$ where $\rho : \hat{A}_{\mathcal{U}} \to \hat{A}_{\mathcal{U}}/\hat{I}_{\mathcal{U}}$ is the canonical epimorphism. In particular, $\ker \hat{\pi}_{\mathcal{U}} \subseteq \ker \rho$ and we obtain a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\pi"'] & \hat{A}_{\mathcal{U}} \arrow[r, "\rho"] \arrow[d, "\hat{\pi}_{\mathcal{U}}"] & \hat{A}_{\mathcal{U}}/\hat{I}_{\mathcal{U}} \\ A/I \arrow[r] & (A/I)\hat{}_{\mathcal{U}} \arrow[ur, "\sigma"'] & \end{tikzcd}$$

where $\sigma : (A/I)\hat{}_{\mathcal{U}} \to \hat{A}_{\mathcal{U}}/\hat{I}_{\mathcal{U}}$ is defined by $\sigma \circ \hat{\pi}_{\mathcal{U}} = \rho$ and is an open contractive algebra homomorphism. A simple argument shows that σ is indeed an isometry.

For a complex normed algebra A without identity denote by $A+\mathbf{C}$ its *unitization*. Its ultrapower has the following simple description.

PROPOSITION 2.4. *The Banach algebras $(A+\mathbf{C})\hat{}_{\mathcal{U}}$ and $\hat{A}_{\mathcal{U}}+\mathbf{C}$ are isometrically isomorphic.*

If A is a unital normed algebra, we consider the algebra $M_n(A)$, $n \in \mathbf{N}$, of $n \times n$-matrices over A as a subalgebra of $L(A^n)$ where A^n is endowed with the ℓ^1-norm.

PROPOSITION 2.5. *The Banach algebras $M_n(A)\hat{}_{\mathcal{U}}$ and $M_n(\hat{A}_{\mathcal{U}})$ are isometrically isomorphic.*

The isomorphism is, of course, given by $(a_i)_{\mathcal{U}} \mapsto [(a_i(\mu,\nu))_{\mathcal{U}}]_{1\le\mu,\nu\le n}$ where $(a_i) \in \ell^\infty(\mathbf{I}, M_n(A))$, $a_i = [a_i(\mu,\nu)]_{1\le\mu,\nu\le n}$.

The last result in this preliminary section will be used to show that the class of ultraprime normed algebras is stable under ultrapowers. A proof of it can be found in [24; § 13]. If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on $\mathbf{I}$ and $\mathbf{J}$ respectively, their *product* $\mathcal{U}\times\mathcal{V}$ consists of all sets Y satisfying

$$\{j \in \mathbf{J} \mid \{i \in \mathbf{I} \mid (i,j) \in Y\} \in \mathcal{U}\} \in \mathcal{V}$$

and defines an ultrafilter on $\mathbf{I}\times\mathbf{J}$ which is countably incomplete if either $\mathcal{U}$ or $\mathcal{V}$ is countably incomplete.

PROPOSITION 2.6. *Let A be a normed algebra. Then there is a natural isometric algebra isomorphism from $(\hat{A}_{\mathcal{U}})\hat{}_{\mathcal{V}}$ onto $\hat{A}_{\mathcal{U}\times\mathcal{V}}$.*

3. THE CONCEPT OF AN ULTRAPRIME NORMED ALGEBRA

Since the initiating paper by McCoy [22], the idea of a *non-commutative* prime ring has been elaborated to an important concept in algebra and has found a vast variety of applications. While closed prime ideals in topological rings have also been studied (see e.g. [30]) and used in analysis, for instance in automatic continuity theory [2], they were mostly replaced by primitive ideals because of the better behaviour of the latter under topological manipulations. In this section we introduce a class of 'well-behaved' closed prime ideals of a normed algebra and study some of their basic properties. The more subtle construction of a ring of quotients (which is fundamental to the theory of prime rings) is deferred to the next section.

Throughout, A will denote a normed algebra over $\mathbf{C}$ if not further specified. An ideal I of A is called *prime* if $I_1 I_2 \subseteq I$, I_1, I_2 ideals of A, implies $I_1 \subseteq I$ or $I_2 \subseteq I$. The algebra A is called *prime* if the zero ideal 0 is a prime ideal. Hence, I is a prime ideal if and only if the quotient algebra A/I is prime. The property of being a prime ideal is easily expressed in terms of operators. Let $M_{a,b} \colon A \to A$, $x \mapsto axb$, where $a, b \in A$, denote a (*two-sided*) *multiplication on* A. Then, I is a prime ideal if and only if $M_{a,b}A \subseteq I$ implies $a \in I$ or $b \in I$ (cf. e.g. [13], Prop. VIII.2.2).

We prepare our definition by the following lemma.

LEMMA 3.1. *The following conditions are equivalent:*

(a) *For each pair* $((x_k)_{k\in\mathbf{N}}, (y_k)_{k\in\mathbf{N}})$ *of sequences in* A *with* $\|x_k\| = \|y_k\| = 1$ *for all* $k \in \mathbf{N}$ *there exists a bounded sequence* $(z_k)_{k\in\mathbf{N}}$ *in* A *such that the sequence* $(x_k z_k y_k)_{k\in\mathbf{N}}$ *does not converge to zero.*

(b) *There exists a constant* $\kappa > 0$ *such that for all* $a, b \in A$

$$\|M_{a,b}\| \geq \kappa\, \|a\|\, \|b\|.$$

(c) *Every ultrapower* $\hat{A}_{\mathcal{U}}$ *of* A *is prime.*

(d) *There exists an ultrapower* $\hat{A}_{\mathcal{U}}$ *of* A *which is prime.*

Proof. (a) $\Rightarrow$ (b) Put $\kappa = \inf\{\|M_{a,b}\| \mid \|a\| = \|b\| = 1\}$. If $\kappa = 0$, then there is a pair of sequences $(x_k)_{k\in\mathbf{N}}$, $(y_k)_{k\in\mathbf{N}}$ in the unit sphere of A such that $\lim\limits_{k\to\infty} \|M_{x_k,y_k}\| = 0$. Hence, for each $(z_k)_{k\in\mathbf{N}} \in \ell^\infty(\mathbf{N}, A)$ the sequence $(x_k z_k y_k)_{k\in\mathbf{N}}$ tends to zero.

(b) $\Rightarrow$ (c) Let $\hat{a}, \hat{b} \in \hat{A}_{\mathcal{U}}$ with $\|\hat{a}\| = \|\hat{b}\| = 1$. If $(a_i) \in \hat{a}$, $(b_i) \in \hat{b}$ are such that $\|a_i\| = \|b_i\| = 1$ for all i (cf. Proposition 2.1), then $\|M_{a_i,b_i}\| \geq \kappa > 0$ whence $\|M_{\hat{a},\hat{b}}\| = \lim_{\mathcal{U}} \|M_{a_i,b_i}\| > 0$. Thus, $\hat{A}_{\mathcal{U}}$ is prime.

(c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (a) Let $(U_k)_{k\in\mathbf{N}}$ be a decreasing sequence of elements $U_k \in \mathcal{U}$ such that $\bigcap_{k\in\mathbf{N}} U_k = \emptyset$. We may assume that $U_1 = \mathbf{I}$ and that $U_k \setminus U_{k+1} \neq \emptyset$. Then, for each $i \in \mathbf{I}$, there is precisely one $k \in \mathbf{N}$ such that $i \in U_k \setminus U_{k+1}$. By means of this, we define a mapping $\ell^\infty(\mathbf{N}, A) \to \ell^\infty(\mathbf{I}, A)$, $(x_k) \mapsto (x_i)$ by $x_i := x_k$ if $i \in U_k \setminus U_{k+1}$. (Clearly, this is an isometry.) If (x_k), $(y_k) \in \ell^\infty(\mathbf{N}, A)$ are such that $\|x_k\| = \|y_k\| = 1$ for all $k \in \mathbf{N}$, we thus obtain $\hat{x} := (x_i)_\mathcal{U}$, $\hat{y} := (y_i)_\mathcal{U} \in \hat{A}_\mathcal{U}$ with $\|\hat{x}\| = \|\hat{y}\| = 1$. Since $\hat{A}_\mathcal{U}$ is prime, there is $\hat{z} = (z_i)_\mathcal{U} \in \hat{A}_\mathcal{U}$ with $\hat{x}\hat{z}\hat{y} \neq 0$. Therefore, for some $\varepsilon > 0$, the set $V = \{i \in \mathbf{I} \mid \|x_i z_i y_i\| \geq \varepsilon\} \in \mathcal{U}$. Putting $V_k = V \cap U_k$ we obtain a decreasing sequence $(V_k)_{k\in\mathbf{N}} \subseteq \mathcal{U}$ with empty intersection. Hence, there are infinitely many $k \in \mathbf{N}$ with $V_k \setminus V_{k+1} \neq \emptyset$. For each $k \in \mathbf{N}$ we let $z_k = z_i$ where $i \in V_k \setminus V_{k+1} \neq \emptyset$ is arbitrary and, if $V_k \setminus V_{k+1} = \emptyset$, i is arbitrarily choosen from $U_k \setminus U_{k+1}$. Thus, we obtain $(z_k) \in \ell^\infty(\mathbf{N}, A)$ satisfying $\|x_k z_k y_k\| \geq \varepsilon$ for infinitely many k, i.e. $x_k z_k y_k \not\to 0$.

REMARK. Condition (a) is included in the above lemma mainly for historical reasons (see the Introduction and Section 5).

Let I be a closed ideal of A. By Proposition 2.3, the ultrapower $(A/I)\hat{}_\mathcal{U}$ is a prime algebra if and only if $\hat{I}_\mathcal{U}$ is a prime ideal of $\hat{A}_\mathcal{U}$. This leads us to the following definition.

DEFINITION 3.2. A closed ideal I of A is called an *ultraprime ideal* if $\hat{I}_\mathcal{U}$ is a prime ideal in some ultrapower $\hat{A}_\mathcal{U}$ of A. The normed algebra A is called *ultraprime* if some ultrapower $\hat{A}_\mathcal{U}$ is a prime Banach algebra. By the above, I is an ultraprime ideal if and only if A/I is an ultraprime algebra.

Before giving examples we list some of the basic properties of ultraprime ideals and algebras. The first is immediate from the definition and Lemma 3.1; the question of reversing its statement will be discussed below.

PROPOSITION 3.3. *Every ultraprime ideal is a closed prime ideal.*

PROPOSITION 3.4. *The center of every ultraprime normed algebra is trivial.*

Proof. If the dimension of the center is at least two, then, by [31], 14.4, its completion contains a non-zero topological divisor of zero. By a 3ε-argument there exist sequences $(x_k)_{k\in\mathbf{N}}$, $(y_k)_{k\in\mathbf{N}}$ of unit vectors in the center such that $\lim\limits_{k\to\infty} x_k y_k = 0$. Thus, $\lim\limits_{k\to\infty} x_k z_k y_k = 0$ for every bounded sequence $(z_k)_{k\in\mathbf{N}}$, which is impossible if the algebra is ultraprime (Lemma 3.1).

Since a prime Banach algebra may well be commutative, e.g. the convolution algebra $L^1(\mathbf{R}_+)$, this result indicates that the ultraprime algebras form a distinguished class among all prime Banach algebras. This is also supported by the next proposition which shows in particular that the completion of an ultraprime normed algebra is an ultraprime Banach algebra whereas an analogous result for prime algebras fails.

PROPOSITION 3.5. *Let B be a subalgebra of the normed algebra A. Then B is ultraprime if and only if its closure $\overline{B}$ is ultraprime.*

Proof. This follows immediately from the fact that $\hat{B}_{\mathcal{U}}$ and $(\overline{B})\hat{}_{\mathcal{U}}$ are isometrically isomorphic.

The next three results are consequences of corresponding results which hold in the purely algebraic setting of prime ideals.

PROPOSITION 3.6. *Every ideal of an ultraprime normed algebra is an ultraprime algebra.*

Proof. If I is an ideal of the ultraprime algebra A, then $\hat{I}_{\mathcal{U}}$ is an ideal in the prime algebra $\hat{A}_{\mathcal{U}}$ (Proposition 2.2), and thus a prime algebra by [22], Lemma 2.

PROPOSITION 3.7. *A normed algebra A without identity is ultraprime if and only if its unitization $A + \mathbf{C}$ is ultraprime.*

Proof. Since A is an ideal in $A + \mathbf{C}$, the "if"-part follows from the preceeding proposition. By Proposition 2.4, $(A+\mathbf{C})\hat{}_{\mathcal{U}} = \hat{A}_{\mathcal{U}} + \mathbf{C}$ is the unitization of $\hat{A}_{\mathcal{U}}$ whence the "only if"-part is a consequence of the fact that the unitization of a prime algebra is still prime.

PROPOSITION 3.8. *Let A be a unital ultraprime normed algebra. Then the $n \times n$-matrix algebra $M_n(A)$ is also ultraprime.*

Proof. From Proposition 2.5 and [22], Thm. 8 we conclude that $M_n(A)\hat{_{\mathcal{U}}} = M_n(\hat{A}_{\mathcal{U}})$ is prime, if $\hat{A}_{\mathcal{U}}$ is prime.

The next result shows that the class of ultraprime normed algebras is stable under ultrapowers.

PROPOSITION 3.9. *Every ultrapower of an ultraprime normed algebra is ultraprime.*

Proof. This follows at once from Proposition 2.6 and Lemma 3.1. Alternatively, one may analyze the proof of Lemma 3.1 in order to realize that condition (b) is inherited by any ultrapower.

We conclude this section with some examples of ultraprime algebras. Observe at first that each finite dimensional prime algebra A is ultraprime (with respect to any norm), for the algebras A and $\hat{A}_{\mathcal{U}}$ are canonically isomorphic. The algebra $L(E)$ of all bounded linear operators on a normed space E is ultraprime since $\|M_{a,b}\| = \|a\|\,\|b\|$ for all $a, b \in L(E)$. More generally, every subalgebra of $L(E)$ which contains all finite rank operators is ultraprime. It was proved in [21; I] that every closed prime ideal of a C*-algebra is an ultraprime ideal. Even more is true in the von Neumann algebra case. Recall that a *factor* is a von Neumann algebra with one-dimensional center.

EXAMPLE. Every closed ideal of a factor is an ultraprime ideal.

If I is a closed ideal of the factor A, it suffices to show that I is a prime ideal. Let $\pi\colon A \to A/I$ denote the canonical quotient map and take $a, b \in A \setminus I$. Without loss of generality we may assume $a, b \geq 0$. If $a = \int_0^\infty \lambda \, de_\lambda$ is the spectral decomposition of a, then, for each $\varepsilon > 0$, we put $p_\varepsilon = 1 - e_{[0,\varepsilon)}$; then, $ap_\varepsilon \geq \varepsilon p_\varepsilon$ and $\|a - ap_\varepsilon\| \leq \varepsilon$. Since $a \notin I$, there exists $p_\varepsilon \notin I$. Similarly, there is $q_\varepsilon \notin I$ such that $bq_\varepsilon \geq \varepsilon q_\varepsilon$ and $\|b - bq_\varepsilon\| \leq \varepsilon$ (there is, of course, a common ε). Using the comparability theorem we may write $p_\varepsilon \geq uu^*$ and $u^*u = q_\varepsilon$ for some $u \in A$, say. Then,

$$\begin{aligned}\|\pi(a)\pi(u)\pi(b)\|^2 &= \|\pi(ap_\varepsilon uq_\varepsilon b)\|^2 \\ &= \|\pi(bq_\varepsilon u^* p_\varepsilon a^2 p_\varepsilon uq_\varepsilon b)\| \\ &\geq \varepsilon^2 \|\pi(bq_\varepsilon u^* p_\varepsilon uq_\varepsilon b)\| \\ &= \varepsilon^2 \|\pi(uq_\varepsilon b)\|^2 \\ &= \varepsilon^2 \|\pi(uq_\varepsilon b^2 q_\varepsilon u^*)\| \\ &\geq \varepsilon^4 \|\pi(uq_\varepsilon u^*)\| > 0,\end{aligned}$$

whence $aub \notin I$. Hence, $\pi(A)$ is a prime algebra and I is a prime ideal.

4. A CONSTRUCTION OF A RING OF QUOTIENTS

Throughout this section, A will denote a non-zero ultraprime normed algebra. We will construct an 'algebra of quotients' of A, closely related to Utumi's ring of quotients, which can be normed in such a way that it becomes an ultraprime algebra containing a homeomorphic image of A. For a thorough discussion of the algebraic theory of rings of quotients we refer to [25].

If I is a (not necessarily closed) ideal of A, by $\mathrm{Hom}_A(I, A)$ we will denote the set of all continuous right A-module homomorphisms from I into A. Using the primeness of A, it is easily seen that each $f \in \mathrm{Hom}_A(I, A)$ is in fact linear; hence, if endowed with the operator norm, $\mathrm{Hom}_A(I, A)$ becomes a normed left A-module. If I_1, I_2 are non-zero ideals of A, a coherent family of (continuous) linear mappings

$$\mathrm{Hom}_A(I_1, A) \longrightarrow \mathrm{Hom}_A(I_2, A), \quad \text{whenever } I_2 \subseteq I_1,$$

is defined by restricting each $f \in \mathrm{Hom}_A(I_1, A)$ to I_2. As a direct limit of complex vector spaces, $Q(A) = \varinjlim \mathrm{Hom}_A(I, A)$; in order to endow $Q(A)$, the 'algebra of quotients' of A, with the structure of a normed algebra we will construct $Q(A)$ in the following alternative way.

Let $\mathcal{I}$ denote the lattice of all non-zero ideals of A, and put $\mathcal{M} = \{(I, f) \mid I \in \mathcal{I}, f \in \mathrm{Hom}_A(I, A)\}$. We define an equivalence relation on $\mathcal{M}$ by

$$(I, f) \sim (J, g) \text{ if there is } U \in \mathcal{I}, U \subseteq I \cap J \text{ such that } f_{|U} = g_{|U}.$$

(The primeness of A yields the transitivity of this relation.)

REMARK. If $(I,f) \sim (J,g)$ then $f_{|I\cap J} = g_{|I\cap J}$. In fact, put $h = f - g$ and take $U \in \mathcal{I}$ with $h(U) = 0$. For $x \in I \cap J$, $y \in U$ we have $h(x)y = h(xy) = 0$, i.e. $h(I \cap J)U = 0$. Since $h(I \cap J)$ is a right ideal of A and A is prime, it follows that $h(I \cap J) = 0$. This remark will be used repeatedly in the sequel.

Let $Q(A) := \mathcal{M}/\sim$ be the set of all equivalence classes, and denote the equivalence class of $(I,f) \in \mathcal{M}$ by $[(I,f)]$. Using the primeness of A, it is easily verified that the following operations are well-defined

$$[(I,f)] + [(J,g)] := [(I \cap J, f+g)],$$

$$\lambda\,[(I,f)] := [(I, \lambda f)],$$

$$[(I,f)] \cdot [(J,g)] := [(JI, fg)],$$

for all (I,f), $(J,g) \in \mathcal{M}$ and $\lambda \in \mathbf{C}$. The usual associativity and distributivity laws are satisfied too.

THEOREM 4.1. *Let A be an ultraprime normed algebra.*

(a) *With the operations defined as above, $Q(A)$ becomes a prime unital algebra containing an isomorphic image of A. Its center $Z(Q(A))$ is a field and coincides with the extended centroid of A.*

(b) *By $\|q\| = \inf\{\|f\| \mid (I,f) \in q\}$, $q \in Q(A)$, an algebra norm is defined on $Q(A)$, whose restriction to A is equivalent to the original norm. Endowed with this norm, $Q(A)$ becomes a unital ultraprime normed algebra.*

DEFINITION 4.2. $Q(A)$ is called *the normed algebra of quotients* of the ultraprime algebra A; its completion $\tilde{Q}(A)$ is *the Banach algebra of quotients* of A. By Theorem 4.1 and Proposition 3.5, $\tilde{Q}(A)$ is an ultraprime Banach algebra.

We will divide the proof of Theorem 4.1 into several lemmas. The crucial one is the following automatic continuity result. In the sequel, κ will always denote a positive real number satisfying condition (b) of Lemma 3.1.

LEMMA 4.3. *Every non-zero A-bimodule homomorphism $f\colon I \to A$, $I \in \mathcal{I}$, is a topological isomorphism onto its image.*

Proof. For all $x, y \in I$ we have

$$\|x\|\,\|f(y)\| \geq \|M_{x,f(y)}\| = \|M_{f(x),y}\| \geq \kappa\,\|f(x)\|\,\|y\|.$$

Thus, $\|f\| = \sup\{\|f(x)\| \mid \|x\| \leq 1\} < \infty$ and $\|f(y)\| \geq \kappa\,\|f\|\,\|y\|$ for all $y \in I$.

Utumi's maximal ring of quotients arises from extending the equivalence relation $\sim$ to $\mathcal{M}_a$, the set of all pairs (I, f) with $I \in \mathcal{I}$ and $f\colon I \to A$ an arbitrary right A-module homomorphism; $Q_a(A) := \mathcal{M}_a/\!\sim$ becomes a prime unital algebra whose center $Z(Q_a(A))$ is called the *extended centroid* of A (cf. [12], Chap. 1.3).

COROLLARY 4.4. $Z(Q(A)) = Z(Q_a(A))$ *is a field.*

Proof. Let $f\colon I \to A$ be an A-bimodule homomorphism on $I \in \mathcal{I}$. If $(J, g) \in \mathcal{M}_a$, then, for all $x \in I$, $y, z \in J$, we have

$$fg(yxz) = f(g(y)xz) = g(y)f(xz) = g(yf(xz)) = gf(yxz);$$

hence, $fg_{|JIJ} = gf_{|JIJ}$ and $0 \neq JIJ \subseteq JI \cap IJ$. It follows that $[(I,f)]$ commutes with $[(J,g)]$.

On the other hand, if $(I,f) \in q \in Z(Q_a(A))$ then, for every $a \in A$, $[(I,f)] \cdot [(A, L_a)] = [(A, L_a)] \cdot [(I,f)]$. Hence $f \circ L_a = L_a \circ f$ on $AI \cap IA$; in particular, for all $x, y \in I$, $b \in A$, we have

$$f(ax)by = f \circ L_a(xby) = L_a \circ f(xby) = af(x)by,$$

and the primeness of A yields $f(ax) = af(x)$, $x \in I$, $a \in A$. Thus, f is an A-bimodule homomorphism.

Lemma 4.3 now shows that $Z(Q_a(A)) = Z(Q(A))$. Moreover, if $q = [(I,f)] \in Z(Q(A))$ is non-zero, then, by Lemma 4.3, $f(I) \in \mathcal{I}$ and $f^{-1}\colon f(I) \to A$ is a continuous A-bimodule homomorphism, whence $q^{-1} = [(f(I), f^{-1})]$.

Proof of 4.1 (a). The axioms of a complex algebra are easily verified. The identity in $Q(A)$ is the equivalence class $[(A, \mathrm{id})]$ of the identical homomorphism on A. From the primeness of A we conclude that $a \mapsto [(A, L_a)]$ embeds A into $Q(A)$, and we will henceforth consider A as a subalgebra of $Q(A)$.

Let $q_1 = [(I_1, f_1)]$, $q_2 = [(I_2, f_2)] \in Q(A) \setminus \{0\}$, and take $x_j \in I_j$ such that $q_j x_j = f_j(x_j) \neq 0$, $j = 1, 2$. Since A is prime, there is $z \in A$ with $q_1 x_1 z q_2 x_2 \neq 0$ whence $M_{q_1, q_2} \neq 0$. It follows that $Q(A)$ is a prime algebra.

REMARKS. 1. Suppose that A is an ultraprime Banach algebra. Then, instead of $\mathcal{M}$, we could start with $\mathcal{M}_c = \{(I, f) \in \mathcal{M} \mid I \text{ is closed}\}$, and by defining the multiplication on $\mathcal{M}_c/\!\sim$ by $[(I, f)] \cdot [(J, g)] = [(\overline{JI}, fg)]$ obtain another algebra $Q_c(A)$. However, since $(I, f) \sim (\overline{I}, f)$ for every $(I, f) \in \mathcal{M}$, we actually have $Q_c(A) = Q(A)$. In the case when A is a prime C*-algebra, an argument similar to [23], 3.12.2 shows that every right A-module homomorphism $f \colon I \to A$ defined on a *closed* ideal of A is automatically continuous, and hence corresponds uniquely to a left multiplier of I. Denoting by $LM(I)$ the space of all left multipliers of $I \in \mathcal{I}$, it thus follows that $Q(A) = \varinjlim LM(I)$.

2. In view of Lemma 4.3, we note that every A-bimodule homomorphism from a *closed* ideal of a prime *Banach* algebra into itself is automatically continuous (cf. [14], Thm 14).

3. The relation between $Q(A)$ and $Q_a(A)$ is clarified by the following observation:

If $q \in Q_a(A)$ has a representative $(I, f) \in \mathcal{M}$, then $q \in Q(A)$.

This is immediate from the next lemma.

LEMMA 4.5. *If $(J, g) \sim (I, f) \in \mathcal{M}$ then $\kappa \|g\| \leq \|f\|$.*

Proof. Since $g_{|I \cap J} = f_{|I \cap J}$, for each $x \in I$, $y \in J$, $z \in A$, we have

$$M_{g(y), x} z = g(y) z x = g(y z x) = f(y z x),$$

whence $\|M_{g(y), x}\| \leq \|f\| \, \|M_{y, x}\|$. From this we conclude that

$$\kappa \, \|g(y)\| \, \|x\| \leq \|M_{g(y), x}\| \leq \|f\| \, \|y\| \, \|x\|,$$

which implies $\kappa\,\|g\| \leq \|f\|$.

COROLLARY 4.6. *The expression* $\|q\| = \inf\{\|f\| \mid (I,f) \in q\}$, $q \in Q(A)$, *defines an algebra norm on* $Q(A)$.

Proof. Let $q, q_1, q_2 \in Q(A)$ and $\lambda \in \mathbf{C}$. By Lemma 4.5, $\kappa\,\|g\| \leq \|q\|$ for each $(J,g) \in q$; thus $\|q\| = 0$ implies $q = 0$. Also,

$$\|\lambda q\| = \inf\{\|g\| \mid (J,g) \in \lambda q\} \leq \inf\{\|\lambda f\| \mid (I,f) \in q\} = |\lambda|\,\|q\|.$$

But, if $(J,g) \sim (I,\lambda f)$, then $\|\lambda f_{|I\cap J}\| = \|g_{|I\cap J}\| \leq \|g\|$. Thus, $\|\lambda q\| \geq |\lambda|\,\|q\|$.
If $(I_1, f_1) \in q_1$ and $(I_2, f_2) \in q_2$ then

$$\begin{aligned}\|f_1 + f_2\| &= \sup\{\|f_1(x) + f_2(x)\| \mid x \in I_1 \cap I_2,\ \|x\| = 1\}\\ &\leq \sup\{\|f_1(x)\| \mid x \in I_1,\ \|x\| = 1\} + \sup\{\|f_2(x)\| \mid x \in I_2,\ \|x\| = 1\}\\ &= \|f_1\| + \|f_2\|,\end{aligned}$$

and

$$\begin{aligned}\|f_1 f_2\| &= \sup\{\|f_1(f_2(x))\| \mid x \in I_2 I_1,\ \|x\| = 1\}\\ &\leq \|f_1\|\ \sup\{\|f_2(x)\| \mid x \in I_2,\ \|x\| = 1\}\\ &= \|f_1\|\,\|f_2\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\inf\{\|f\| \mid (I,f) \in q_1 + q_2\} &\leq \inf\{\|f_1 + f_2\| \mid (I_1, f_1) \in q_1,\ (I_2, f_2) \in q_2\}\\ &\leq \inf\{\|f_1\| + \|f_2\| \mid (I_1, f_1) \in q_1,\ (I_2, f_2) \in q_2\}\\ &= \inf\{\|f_1\| \mid (I_1, f_1) \in q_1\} + \inf\{\|f_2\| \mid (I_2, f_2) \in q_2\},\end{aligned}$$

that is, $\|q_1 + q_2\| \leq \|q_1\| + \|q_2\|$.
Similarly,

$$\begin{aligned}\inf\{\|f\| \mid (I,f) \in q_1 q_2\} &\leq \inf\{\|f_1 f_2\| \mid (I_1, f_1) \in q_1,\ (I_2, f_2) \in q_2\}\\ &\leq \inf\{\|f_1\|\,\|f_2\| \mid (I_1, f_1) \in q_1,\ (I_2, f_2) \in q_2\}\\ &= \inf\{\|f_1\| \mid (I_1, f_1) \in q_1\} \cdot \inf\{\|f_2\| \mid (I_2, f_2) \in q_2\},\end{aligned}$$

that is, $\|q_1 q_2\| \leq \|q_1\|\,\|q_2\|$.

Proof of 4.1 (b). Let $a \in A$. If $(I, f) \sim (A, L_a)$, then, for each $x \in I$ with $\|x\| = 1$,

$$\|f\| = \|L_{a|I}\| \geq \sup\{\|azx\| \mid z \in A, \|z\| = 1\} = \|M_{a,x}\| \geq \kappa\,\|a\|.$$

Therefore, $\|a\| \geq \|[(A, L_a)]\| \geq \kappa\,\|a\|$ and the mapping $a \mapsto [(A, L_a)]$ is a topological homomorphism from A into $Q(A)$.

Take $p, q \in Q(A)$ with $\|p\| = \|q\| = 1$. By definition of the norm, there is $(I, f) \in p$ with $\|f\| \geq 1$. Thus, for every $0 < \varepsilon < 1$, there is $x \in I$, $\|x\| = 1$ such that $\|px\| = \|f(x)\| \geq 1 - \varepsilon$. Similarly, there is $y \in A$, $\|y\| = 1$ such that $\|qy\| \geq 1 - \varepsilon$. By Lemma 3.1, $\|M_{px,qy}\| \geq \kappa\,\|px\|\,\|qy\| \geq \kappa(1-\varepsilon)^2$. Therefore, on $Q(A)$,

$$\|M_{p,q}\| \geq \|R_y\|\,\|M_{p,q}\|\,\|L_x\| \geq \|R_y M_{p,q} L_x\| \geq \kappa\,\|M_{px,qy}\| \geq \kappa^2(1-\varepsilon)^2.$$

Hence, $\|M_{p,q}\| \geq \kappa^2$ and thus $Q(A)$ is ultraprime.

The *central closure* of a prime algebra A is defined as $AZ(Q_a(A)) \subseteq Q_a(A)$, and A is called *centrally closed* if A coincides with its central closure. As an immediate consequence of Theorem 4.1 we have the following.

COROLLARY 4.7. *Every ultraprime normed algebra is centrally closed.*

EXAMPLES. Suppose that $A = K(H)$, the C*-algebra of all compact operators on some Hilbert space H. Since A is topologically simple, $Q(A)$ is isometrically isomorphic with the space of all continuous right A-module homomorphisms, i.e. all left centralizers of A. It follows that $Q(A) = L(H)$. In this case, $Q_a(A) = \mathrm{End}_A(F(H))$, the algebra of all right A-module homomorphisms on the finite rank operators (cf. [25], ex. XII.2.3).

Now let $A = L(H)$. If $f \in \mathrm{Hom}_A(I, A)$ and I is closed, it follows easily from the existence of an approximate identity in I that $f = L_a$ for some $a \in A$. From this and the fact that every non-zero ideal is ultraweakly dense in A, we deduce that $Q(A) = A$.

Clearly, similar arguments apply to factors of type II or III, i.e. $Q(A) = A$ and $Q(I) = A$ for each non-zero closed ideal I of the factor A (cf. Proposition 3.6).

5. APPLICATIONS TO ELEMENTARY OPERATORS

This section is devoted to applications of the concept of an ultraprime algebra to operator theory. If A is an algebra, its *algebra of elementary operators*, $\mathcal{E}\ell(A)$, consists of all linear mappings $x \mapsto \sum_{j=1}^{n} a_j x b_j$, where a_j, b_j are finitely many elements in A. Denoting the opposite algebra of A by A^{op}, there is a unique canonical epimorphism θ from $A \otimes A^{\mathrm{op}}$ onto $\mathcal{E}\ell(A)$ such that $\theta(a \otimes b) = M_{a,b}$. If A is a Banach algebra, then θ extends to a contraction from the projective tensor product $A \hat{\otimes} A^{\mathrm{op}}$ into $L(A)$. The kernel of θ has been determined in several special cases. In [8] Fong and Sourour proved that if A is either $L(E)$ for a Banach space E or $C(H)$, the Calkin algebra on a separable Hilbert space H, then θ is injective. Synnatzschke [28] obtained the same result under the assumption that A is a primitive algebra containing sufficiently many one-dimensional elements. In [21], we proved that, in the case of a C*-algebra A, θ is an isomorphism if and only if A is prime. In fact, the proof we used there can be modified easily to cover the case of an arbitrary ultraprime normed algebra.

THEOREM 5.1. *Let A be an ultraprime normed algebra. Then θ is an isomorphism from $A \otimes A^{\mathrm{op}}$ onto $\mathcal{E}\ell(A)$.*

If $u \in A \otimes A^{\mathrm{op}}$ is represented as $u = \sum_{j=1}^{n} a_j \otimes b_j$ with $\{b_1, \ldots, b_n\}$ linearly independent, then $\theta(u) = \sum_{j=1}^{n} M_{a_j,b_j} = 0$ implies $a_j = 0$ for all $1 \leq j \leq n$, whence $u = 0$. This is proved exactly as in [21], Part I, Thm 4.1; we only have to observe that every A-bimodule homomorphism from an ideal of A into A is a multiplication by a complex number (Theorem 4.1 and Proposition 3.4).

In the same vein, Corollaries 4.2 and 4.3 of [21], Part I as well as Prop. 4.6 and Cor. 4.7 take over to the case of an ultraprime normed algebra. In particular, $\mathcal{E}\ell(A) = L(A)$ for every finite dimensional prime algebra.

The so-called 'range inclusion problem' for elementary operators asks for criteria on $S \in \mathcal{E}\ell(A)$ to satisfy $SA \subseteq J$, where J is some prescribed ideal of A. The case

$A = L(H)$ was studied to some extent in [1], [5] and [6]. As an immediate consequence of Theorem 5.1 we have:

COROLLARY 5.2. *Let J be an ultraprime ideal of the normed algebra A and let $S \in \mathcal{E}\ell(A)$. Then $SA \not\subseteq J$ if and only if there is a representation $S = \sum_{j=1}^{n} M_{a_j,b_j}$ with $\{a_1,\ldots,a_n\}$ and $\{b_1,\ldots,b_n\}$ both linearly independent modulo J.*

Combining this with the last example of Section 3 we obtain the following.

COROLLARY 5.3. *Let J be any closed ideal of a factor A and let $S \in \mathcal{E}\ell(A)$. Then $SA \not\subseteq J$ if and only if there is a representation $S = \sum_{j=1}^{n} M_{a_j,b_j}$ with $\{a_1,\ldots,a_n\}$ and $\{b_1,\ldots,b_n\}$ both linearly independent modulo J.*

In [21], the C*-version of Theorem 5.1 is also applied to characterize completely positive and (weakly) compact elementary operators.

In the remainder of this section, we will add some observations on the spectra of elementary operators. To this end, A will always denote a unital Banach algebra. We use the notion of 'joint spectrum' of an n-tuple $a \in A^n$ as introduced by Harte ([9], cf. also [10], Chap. 11); however, we prefer the notation used in [21], Part I. In particular, $P\sigma(S)$ and $P\sigma_r(S)$ stand for the *point* and the *compression spectrum* of $S \in \mathcal{E}\ell(A)$, respectively, while $AP\sigma(S)$ resp. $AP\sigma_r(S)$ denotes the *approximate point* resp. *defect spectrum* of S. When necessary, we may indicate by $\sigma^{\mathcal{E}\ell(A)}(S)$ with respect to which algebra the spectrum is computed. An n-tuple $a = (a_1,\ldots,a_n)$ is called *commuting* if the set $\{a_1,\ldots,a_n\}$ is commutative. If X, Y are subsets of $\mathbb{C}^n$, then $X \circ Y := \{\sum_{j=1}^{n} \xi_j\eta_j \mid \xi_j \in X,\ \eta_j \in Y\}$.

PROPOSITION 5.4. *Let A be a unital Banach algebra, $a, b \in A^n$ and let $S = \sum_{j=1}^{n} M_{a_j,b_j} \in \mathcal{E}\ell(A)$.*

If A is prime, then

(1) $$P\sigma(a) \circ P\sigma_r(b) \subseteq P\sigma(S)$$

and

$$(2) \qquad P\sigma_r(a) \circ P\sigma(b) \subseteq P\sigma_r(S).$$

If A is ultraprime, then

$$(3) \qquad AP\sigma(a) \circ AP\sigma_r(b) \subseteq AP\sigma(S)$$

and

$$(4) \qquad AP\sigma_r(a) \circ AP\sigma(b) \subseteq AP\sigma_r(S).$$

If, in addition, a and b are commuting n-tuples, then

$$(5) \qquad AP\sigma(a) \circ AP\sigma(b) \cup AP\sigma_r(a) \circ AP\sigma_r(b) \subseteq \sigma(S)$$

and

$$(6) \qquad AP\sigma(a) \circ AP\sigma_r(b) \supseteq AP\sigma(S).$$

Finally, if A is a prime C-algebra and a, b are commuting, then*

$$(7) \qquad AP\sigma_r(a) \circ AP\sigma(b) \supseteq AP\sigma_r(S).$$

The inclusions (1), (2) and (3), (4) were proved under the assumption that A is a prime C*-algebra in [21], Part I, Thm 3.8, and the argument used takes over to the general case almost verbatim. However, we offer an alternative argument for the inclusion (3) (an analogous one which proves (4) is omitted). Though it may be less direct, it emphasizes the connection to Theorem 5.1.

Proof of (3). Let $p(\zeta_1, \ldots, \zeta_{2n}) = \sum_{j=1}^{n} \zeta_j \zeta_{j+n} \in \mathbf{C}[\zeta_1, \ldots, \zeta_{2n}]$. Using the well-known fact that $AP\sigma(a) = P\sigma((a)_{\mathcal{U}})$ and $AP\sigma_r(a) = P\sigma_r((a)_{\mathcal{U}})$, we obtain

$$\begin{aligned}
AP\sigma(a) \circ AP\sigma_r(b) &= P\sigma((a)_{\mathcal{U}}) \circ P\sigma_r((b)_{\mathcal{U}}) \\
&= p\left(P\sigma^{\hat{A}_{\mathcal{U}}}((a)_{\mathcal{U}}) \times P\sigma_r^{\hat{A}_{\mathcal{U}}}((b)_{\mathcal{U}})\right) \\
&= p\left(P\sigma^{\hat{A}_{\mathcal{U}}}((a)_{\mathcal{U}}) \times P\sigma^{\hat{A}_{\mathcal{U}}^{\mathrm{op}}}((b)_{\mathcal{U}})\right) \\
&\subseteq p\left(P\sigma^{\hat{A}_{\mathcal{U}} \otimes \hat{A}_{\mathcal{U}}^{\mathrm{op}}}((a)_{\mathcal{U}} \otimes 1, 1 \otimes (b)_{\mathcal{U}})\right)
\end{aligned}$$

by [10], 11.7.5

$$\subseteq P\sigma^{\hat{A}_{\mathcal{U}} \otimes \hat{A}_{\mathcal{U}}^{\mathrm{op}}}\left(p((a)_{\mathcal{U}} \otimes 1, 1 \otimes (b)_{\mathcal{U}})\right)$$

by the spectral mapping theorem ([10], 11.2.2)

$$= P\sigma^{\hat{A}_{\mathcal{U}} \otimes \hat{A}_{\mathcal{U}}^{\mathrm{op}}}\Big(\big(\sum_{j=1}^{n} a_j \otimes b_j\big)_{\mathcal{U}}\Big)$$

$$= P\sigma^{\mathcal{E}\ell(\hat{A}_{\mathcal{U}})}(\hat{S}_{\mathcal{U}})$$

by Theorem 5.1 and Proposition 3.9

$$\subseteq P\sigma^{L(\hat{A}_{\mathcal{U}})}(\hat{S}_{\mathcal{U}})$$

$$= AP\sigma(S).$$

The inclusion (7) was proved in [21], Part I, Thm 3.9, and the argument for (5) can also be extracted easily from the proof of *ibid.*, Thm 3.9.

Proof of (6). Writing $L_a = (L_{a_1}, \ldots, L_{a_n})$ and $R_b = (R_{b_1}, \ldots, R_{b_n})$ we have

$$AP\sigma(L_a, R_b) \subseteq AP\sigma(L_a) \times AP\sigma(R_b) = AP\sigma(a) \times AP\sigma_r(b)$$

by [10], 11.6.1. Applying the spectral mapping theorem for commuting n-tuples (*ibid.*, 11.3.4) we obtain

$$AP\sigma(S) = AP\sigma(p(L_a, R_b)) = p(AP\sigma(L_a, R_b)) \subseteq AP\sigma(a) \circ AP\sigma_r(b).$$

REMARKS. 1. Observe that inclusion (6) is valid without any additional hypothesis on A.

2. The additional assumption in (7) is due to the fact that a bounded surjective operator on a Banach space need not be right invertible, that is,

$$0 \in \sigma_r(T) = AP\sigma_r(L_T) \not\Rightarrow 0 \in AP\sigma_r(T).$$

3. The inclusions (1) and (2) cannot be reversed even for the algebra $L(H)$, when H is an infinite dimensional Hilbert space.

4. In [18], Lumer and Rosenblum started the spectral analysis of elementary operators on the basis of (joint) topological divisors of zero. In fact, they used the ultraprimeness of $L(E)$ in the form of condition (a) of Lemma 3.1 in order to describe $\sigma(S)$ in the case where $a_j = f_j(a)$, $b_j = g_j(b)$, $a, b \in L(E)$ and f_j, g_j are holomorphic functions on $\sigma(a)$, $\sigma(b)$ respectively ([18], Thm 10).

5. In a recent paper [4], Eschmeier proved that, if $a, b \in L(E)^n$ are commuting n-tuples and σ_T denotes the Taylor joint spectrum, then $\sigma\left(\sum_{j=1}^{n} M_{a_j,b_j}\right) = \sigma_T(a) \circ \sigma_T(b)$.

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