INTEGRABLE SYSTEMS AND THE GEOMETRY OF DIFFERENTIAL OPERATORS

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Abstract. *The problem of characterising completely integrable nonlinear partial differential equations (p.d.e) among all p.d.e. is a major current problem in nonlinear dynamics. It is widely believed that the Painlevé property is a key to the ultimate resolution of this problem. This paper suggests an alternative approach to the characterisation problem which relies on a study of the geometry of differential operator spaces.*

1. A Brief Historical Introduction to Integrability

It is now some 40 years since M.D. Kruskal 's groundbreaking work on the lack of thermalisation (ergodicity) in certain nonlinear dynamical systems following the numerical experiments of Fermi, Pasta and Ulam (see [1]). Kruskal found that upon use of a continuous approximation in a mechanical system consisting of a series of masses coupled by springs obeying a nonlinear force law, it was possible to describe the dynamics by the nonlinear p.d.e.

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0. \tag{1.1}$$

This equation is known as the Korteweg-de Vries (KdV) equation after the two Dutch scientists who first discovered it in 1895 in connection with a problem in fluid dynamics. In the early sixties, Kruskal and Zabusky conducted a number of numerical experiments which eventually showed [2] that if the initial condition for the for p.d.e. (1.1) is such that two solitary waves of different amplitudes were placed so that only their exponential tails overlapped and the wave with the larger amplitude is to the left of the one with the smaller amplitude then the larger wave eventually overtakes the smaller and passes through it to re-emerge with its identity intact. Because of this "particle-like" interaction of the two solitary waves, completely unexpected for a nonlinear equation, Kruskal and Zabusky called these solitary wave solutions *solitons*.

In the years that have followed since this early work of Kruskal and Zabusky a remarkable picture has emerged of the a class of p.d.e. which are particularly privileged within the set of all p.d.e.. The first major result came in 1967 when Gardner, Greene, Kruskal, and Miura [3] obtained the global-in-time solution of the characteristic initial value problem for the KdV equation for initial data which decays sufficiently rapidly at $|x|=\infty$. They did this by introducing a novel analytic method of solution which eventually came to be called the *inverse scattering transform* and which was later recognised to be a generalisation of the Fourier transform to the nonlinear domain. In later work [4], the above mentioned authors were able to obtain a complete characterisation of that subset of the allowed initial data which generates the soliton solutions first observed numerically by Kruskal and Zabusky. Moreover, using the inverse scattering transform, the authors were able to compute soliton solutions explicitly in finite terms of known functions.

All these remarkable results galvanised the mathematical community and an intense search began in order to discover whether p.d.e. other than the KdV were amenable to the inverse scattering transform. By the mid-seventies many such classes of equations had been discovered and furthermore these classes were seen to possess a rich geometric and analytic structure. Included among the properties of theses equations are: (1) A generalisation of the Fourier transform to nonlinear initial value problems; (2) A Hamiltonian structure on an infinite dimensional phases space; (3) An infinite number of constants of the motion in involution with respect to the Hamiltonian structure in (2); (4) Existence of Backlund maps mapping solutions to solutions; (5) Existence of stable "particle-like" solutions now known as solitons; (6) Existence of Lax pairs; (7) Existence of nontrivial prolongation structures; plus much more. It was observed that if a given p.d.e. has any one of these properties then it has them all and such p.d.e. came to be called *completely integrable.* For the purposes of this article we will say that a p.d.e. which can be solved by the inverse scattering transform is completely integrable.

This name is rather appropriate since in the classical theory of dynamical systems developed by Liouville, Jacobi and others in the 19th century a (Hamiltonian) dynamical system was said to be completely integrable if a sufficient number of constants of the motion in involution could be found which could then be used to put the system into canonical form (action-angle variables) and the system integrated. It was shown rather early in the development of the subject that this structure *persists* in the p.d.e. setting with the difference that the natural phase space is *infinite dimensional.* There has thus emerged, in the past twenty years or so, a theory of Hamiltonian systems with the classical dynamical systems - for example, geodesic flow on an ellipsoid - occupying the comparatively more straightforward theory of *finite* dimensional systems and p.d.e. that of *infinite* dimensional systems. For excellent introductions to these intriguing topics the reader is refered to [5],[6].

A natural question that was posed rather early in the history of the subject is: what is it that *really* distiguishes a completely integrable p.d.e. from any other and how general are completely integrable p.d.e.? For example, if we consider a generalisation of the KdV equation (1.1) in the form

$$\frac{\partial u}{\partial t} + u^p \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad p \in \mathbb{R}, \tag{1.2}$$

then it is known that (1.2) is completely integrable if $p = 1$ or 2 but for no other values of p ! Examples like this one indicate that whatever meaning we might ultimately ascribe to it, complete integrability is likely to be a relatively rare phenomenon among nonlinear p.d.e.. So the question is: does there exist some invariant or other intrinsic property of a given p.d.e. which will *apriori* indicate that the equation in question is completely integrable ? This is what is meant here by the characterisation problem for completely integrable systems.

One path that has been followed since the late 1970s in an attempt to address the characterisation problem is that of a study of the so called *Painlevé Property*. An ordinary differential equation (o.d.e) is said to have the Painlevé property if the only moveable singular points of its solutions are poles. The term 'moveable' is used to mean that the position of the singular points is determined by the initial conditions. Some motivation for the study of this rather strange property will be given in a moment. It was observed by Ablowitz, Ramani and Segur in 1978 that every symmetry reduction of thus far known completely integrable p.d.e. had the Painlevé property and this fact led these authors to conjecture that this is a generic property of completely integrable p.d.e. which may be used to settle the characterisation problem [7]. Later, the procedure for ascribing the Painlevé property to a p.d.e. was improved obviating the need for going through symmetry reductions [8], however, these details don't concern us here.

Though the evidence to date indicates that the Painlevé property is a strong indicator of complete integrability there are currently no theorems of sufficient generality that would give a clear connection between the Painlevé property and complete integrability. Moreover, the conjecture as stated by Ablowitz, Ramani and Segur is false since counter-examples are known. In subsequently weakened versions of the conjecture a statement is made to the effect that the Painlevé property is to hold "possibly after a change of coordinates". This means that the Painlevé property is a property of the p.d.e. *and* the coordinate system that the p.d.e. happens to be in making the Painlevé property a non-intrinsic property whereas integrability is surely and intrinsic property of the p.d.e. itself. Nonetheless, there is general agreement and substantial results to indicate that despite the above mentioned difficulties the Painlevé property is destined to play a key role in the ultimate resolution of the characterisation problem. In fact, it was precisely the Painlevé property that Sonya Kovalevsky used in her famous study of the Euler equations of a rotating rigid body which allowed her to unify the three known cases of integrable spinning tops (Euler, Lagrange, equal moments of inertia) and add her own fourth case now known as the Kovalevsky top. In this new case she was able to find a sufficient number of constants of the motion which she then used to completely integrate the equations and describe the motion in terms of elliptic functions. The Kovalevsky top was particularly interesting physically because it represented the only case of a solvable nonsymmetric spinning body. For this work Kovalevsky won the 1888 Bordin Prize of the French Academy and her work helped to stimulate Painlevé's research.

2. Geometry of Hyperbolic Differential Operators and the Generalised Toda Lattice

One of the most striking features of the theory of integrable systems is its strong and often unexpected connections to a variety of geometric structures. Connections of integrable systems to generalised symmetries, prolongation structures, symplectic structures and the twistor correspondence of Penrose are well known. In this section I will describe a less well known geometric structure associated with one important integrable system: the generalised Toda lattice. At the end I will return to the characterisation problem outlined in the previous section.

In the second volume of his treatise on the differential geometry of surfaces in $\mathbb{R}^3$, Darboux [9] studied, in great detail, the linear hyperbolic partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} + \alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} + \gamma u = 0 \tag{2.1}$$

where α, β and γ are functions of x and y. Darboux's main interest in equation (2.1) arose from his interest in the differential geometry of certain surfaces embedded in $\mathbb{R}^3$ known as *focal* surfaces. If $(x, y) \to (u_1, u_2, u_3)$ represents a parametrisation of the surface then it is known that the u_i satisfy equations of the form (2.1) with α, β, γ being certain functions of the metric on the surface. Darboux was interested in a classification of focal surfaces induced from a classification of the equations of the form (2.1). Because of the difficulty of the latter classification, especially with the tools then available to Darboux, his program was never completed. This problem has recently been addressed and the basic equivlalence problem for equation (2.1) has been solved thanks to the Cartan equivalence method [10]. We will now briefly describe the classifying group that Darboux had in mind and which was used to solve the previously mentioned equivalence problem. We will only outline some of the results and draw the readers attention to reference [10] for more details.

Denote by $\mathcal{D}$ the set of differential operators

$$\partial^2_{xy} + \alpha \partial_x + \beta \partial_y + \gamma,$$

where α, β and γ are smooth functions of x and y. Darboux essentially defined two pseudogroups of transformations that act on $\mathcal{D}$. We will denote these by G and H. The pseudogroup G is defined by the conjugation

$$L \to \lambda \cdot L = \lambda^{-1} L \lambda, \quad \lambda \in C^\infty(\mathbb{R}^2). \tag{2.2}$$

Let w be the transformation $(x, y) \to (\xi(x), \eta(y))$ where ξ and η are any smooth functions such that w is a local diffeomorphism. The pseudogroup H action is the coordinate transformation

$$L \to |\mathrm{J}(w)|^{-1} w_\star L, \tag{2.3}$$

where $\mathrm{J}(w)$ is the Jacobian of w and $\star$ is the usual push forward map. Unfortunately we cannot here go into the origins of these pseudogroup actions and how they relate to the original geometric question raised by Darboux. Our main interest here will be to show how they relate to certain completely integrable nonlinear p.d.e.. We note that the two actions G and H commute so that we are able to form the direct product of pseudogroups $\mathcal{G} = G \times H$ and in general consider the $\mathcal{G}$-action on $\mathcal{D}$. Firstly, however, let us consider the G-action on $\mathcal{D}$. One of the key ingredients in the relationship between integrable systems and the set of operators $\mathcal{D}$ are certain invariants classically known as the *Laplace invariants.* These are functions $f : \mathcal{D} \to C^\infty(\mathbb{R}^2)$ satisfying

$$f(\lambda \cdot L) = f(L), \quad \forall \lambda \in G.$$

Two such functions are known

$$h(L) = \alpha_x + \alpha\beta - \gamma, \quad k(L) = \beta_y + \alpha\beta - \gamma.$$

In fact it is not difficult to show that h and k form a complete set of invariants in the sense that two operators L and $\bar{L} \in \mathcal{D}$ are G-equivalent if and only if their

Laplace invariants agree. That is, there exists $\lambda \in G$ such that $\bar{L} = \lambda \cdot L$ if and only if $(h,k)(L) = (h,k)(\bar{L})$. Thus, h and k label the G-orbits in $\mathcal{D}$; (see [10]).

Darboux also inquired about certain other automorphisms of $\mathcal{D}$ which do not in general preserve G-orbits. Two such which were studied by Darboux are known as the *Laplace transformations.* These are essentially Backlund maps mapping solutions of $Lu = 0$ to solutions of $L_1 u_1 = 0$ where $L, L_1 \in \mathcal{D}$ and where L_1 is defined from the linear system

$$\begin{pmatrix} \partial_x + \beta & -h(L) \\ -1 & \partial_y + \alpha \end{pmatrix} \begin{pmatrix} u_1 \\ u \end{pmatrix} = 0, \tag{2.5}$$

upon elimination of u. This then provides us with a map $L \to L_1$ which we label $\mathcal{L}_1$ and which defines one of the Laplace transformations. The other classically known transformation maps solutions of $Lu = 0$ to solutions of $L_{-1}u_{-1} = 0$ where L_{-1} is defined by the linear system

$$\begin{pmatrix} \partial_x + \beta & -1 \\ -k(L) & \partial_y + \alpha \end{pmatrix} \begin{pmatrix} u \\ u_{-1} \end{pmatrix} = 0, \tag{2.6}$$

upon elimination of u. Thus we get a map $L \to L_{-1}$ which we label $\mathcal{L}_{-1}$ and which defines the second of the Laplace transformations. As the name suggests, these transformations apparently go back to the work of Laplace but the history is not very clear.

Most of the connections between the set $\mathcal{D}$ and integrable systems of nonlinear partial differential equations arises from the interaction between the pseudogroup $\mathcal{G}$ and the Laplace transformations $\mathcal{L}_1$ and $\mathcal{L}_{-1}$. An obvious question is: how are the Laplace invariants of the images of L under the Laplace transformations related to the Laplace invariants of L. Darboux wrote down the formulas

$$h(L_1) = 2h(L) - k(L) - \partial_{xy}^2 \ln |h(L)|, \tag{2.7}$$

$$k(L_1) = h(L), \qquad h(L_{-1}) = k(L), \tag{2.8}$$

$$k(L_{-1}) = 2k(L) - h(L) - \partial_{xy}^2 \ln |k(L)|, \tag{2.9}$$

where $L_1 := \mathcal{L}_1(L), L_{-1} := \mathcal{L}_{-1}(L)$. From these formulas it is easy to see that the Laplace transformations, while not in general inverses of each other, are inverses on the quotient space, $Q = \mathcal{D}/G$ of G-orbits in $\mathcal{D}$; for example, $h(L_{1,-1}) = k(L_1) = h(L)$; and so on. All this naturally leads to another question, also dealt with by Darboux, namely, is it possible to give a complete description of all those operators $L \in \mathcal{D}$ which eventually return to the G-orbit of L under iteration of $\mathcal{L}_1$ and $\mathcal{L}_{-1}$? Thus Darboux posed the problem of studying subsets $\mathcal{P}_n \in \mathcal{D}$ satisfying

$$h(L_n) = h(L), \qquad k(L_n) = k(L), \qquad \forall L \in \mathcal{P}_n. \tag{2.10}$$

Such operators will be called n-periodic. The actual manner in which Darboux introduced this question was by defining the *Lapalce-Darboux sequence of operators*

$$\ldots L_{-3}, L_{-2}, L_{-1}, L, L_1, L_2, L_3, L_4 \ldots$$

obtained by repeated application of the Laplace transformations. In order to study the n-periodic subsets of $\mathcal{D}$, we think of the Laplace-Darboux sequences as being defined on the quotient space $Q = \mathcal{D}/G$ and thus each L_i is a representative from a G-orbit.

Definition. Let n be a nonnegative integer. The n^{th} *order Laplace-Darboux map*

$$\mathbb{L}^{(n)} : \mathcal{D} \to C^{\infty}(\mathbb{R}^2)^2$$

is defined by

$$\mathbb{L}^{(n)}(L) = ((h \circ \mathcal{L}_1^n)(L), (k \circ \mathcal{L}_{-1}^n)(L)).$$

We have,

Proposition 1.
L is n-periodic if and only if

$$\mathbb{L}^{(n)}(L) = \mathbb{L}^{(0)}(L). \tag{2.11}$$

Thus, we can use the Laplace-Darboux map to study the n-periodic operators $\mathcal{P}_n$.
There is a very pretty description of $\mathcal{P}_1$:

Proposition 2 (Darboux; see [10]).
$\mathcal{P}_1$ is the $\mathcal{G}$-orbit of the Klein-Gordon operator. That is,

$$L \in \mathcal{P}_1 \quad \textit{if and only if} \quad L = \sigma \cdot (\partial_{xy}^2 - 1) \quad \textit{for some} \quad \sigma \in \mathcal{G}.$$

The remaining sets $\mathcal{P}_n$ are much less trivial to characterise. For instance, for $\mathcal{P}_2$ we have

Proposition 3 (Darboux; see [10]).
If $L \in \mathcal{P}_2$, then there exists $\sigma \in \mathcal{G}$ such that $\psi = \ln|h(\sigma \cdot L)|$ satisfies the sinh-Gordon equation

$$\partial_{xy}^2 \psi = \sinh \psi.$$

Clearly, Proposition 3 is much less satisfying as a description of the 2-periodic operators than is Proposition 2 as a description of the 1-periodic operators. Similar results are easily proved for all the higher order periodic operators showing that the periodic operators are in correspondence with the periodic generalised Toda latttice equations (see [20]), all of which, like the sinh-Gordon equation are integrable. In the general nonperiodic case with $L \notin \ker(\mathbb{L}^{(n)})$ for all $n \geq 0$ then $u_n = \ln|h(L_n)| = \ln|k(L_{n+1})|$ satisfy the A_∞-generalised Toda lattice equations[12]

$$\partial_{xy}^2 u_n = -e^{u_{n-1}} + 2e^{u_n} - e^{u_{n+1}} \tag{2.12}$$

where A_∞ is one of the affine Lie algebras [21].

Example. An easy calculation shows that the operator $L = \partial_{xy}^2 + A(x)\partial_x - 1$, where A'' is not identically zero, is not periodic nor in the kernel of any Laplace-Darboux map. We also have,

$$h(L_n) = (n+1)A' + 1, \quad k(L_{-n}) = 1 - nA',$$

and hence

$$u_n = \ln |(n+1)A' + 1|,$$

is a (degenerate) solution of the A_∞-generalised Toda lattice (2.12). We can go a little further here since the $\mathcal{G}$-action on $\mathcal{D}$ induces a symmetry of the A_∞-Toda equations in the sense that if $u_n = \ln |h(L_n)|$ satisfies equation (2.12) then so does $\bar{u}_n = \ln |h(\sigma \cdot L_n)| = \ln |h((f'g')^{-1} L_n))|$ for $\sigma \in \mathcal{G}$, where the functions involved in the pseudogroup H are denoted by f and g. (Note that it is easy to show that the $\mathcal{G}$-action commutes with the Laplace-Darboux maps-see [10]). For

$$\partial^2_{\bar{x}\bar{y}} + e^{\bar{u}_{n-1}} - 2e^{\bar{u}_n} + e^{\bar{u}_{n+1}}$$

$$= (f'g')^{-1}(\partial^2_{xy} + e^{u_{n-1}} - 2e^{u_n} + e^{u_{n+1}}) = 0.$$

We thereby obtain in this case a solution of the Toda equations (2.12) depending upon three arbitrary functions

$$u_n = \ln |\phi(x)\psi(y)((n+1)F(x) + 1)|.$$

It is an interesting problem to attempt to characterise those operators in $\mathcal{D}$ which are not periodic and not in the kernel of any Laplace-Darboux map. This will lead, among other things, to (possibly new) solutions of the lattice (2.12). A start on this problem has been made in [10].

A very different class of integrable systems arise from the consideration of other $\mathcal{G}$-invariant subsets of $\mathcal{D}$. So far we have given some elementary results associated with the periodic fixed points of the Laplace-Darboux maps, $\mathcal{P}_n$ and we have alluded to the kernels of the latter. Again it is not difficult to show that the kernels are also $\mathcal{G}$-invariant and it transpires that these operators are associated with the *free end, finite, generalised Toda lattice for an even number of particles.* We see this as follows. Consider a line of unit masses with nearest neighbor interactions. Denote by $\Phi(x)$ the potential energy of the interactions where x is the distance between nearest neighbors. Then Newton's equations of motion for the coupled masses are

$$\frac{d^2 x_n}{dt^2} = \Phi'(x_{n+1} - x_n) - \Phi'(x_n - x_{n-1}),$$

where x_n is the deviation of the n^{th} mass from its equilibrium position. Postulating that the potential energy of the interaction is given by

$$\Phi(x) = e^{-x},$$

leads to the system

$$\frac{d^2 x_n}{dt^2} = -e^{-(x_{n+1} - x_n)} + e^{(x_n - x_{n-1})}. \tag{2.13}$$

If we now define $r_n = x_n - x_{n+1}$ then equations (2.13) become

$$\frac{d^2 r_n}{dt^2} = -e^{-r_{n+1}} + 2e^{-r_n} - e^{r_{n-1}}. \tag{2.14}$$

If we now consider solutions of the A_∞ generalised Toda equations (2.12) depending only upon the variable $t = x - y$ by setting $u_n = -r_n(x-y)$ then we find that $r_n(t)$ satisfies equation (2.14). Thus the A_∞ Toda equations as discovered by the Laplace-Darboux map admits the Toda lattice equations (2.14) as a symmetry reduction. Historically it was the dynamical system (2.13) that was first studied and shown to admit soliton and other wave solutions [11]. In 1973, Flashcka [22] succeeded in finding a Lax pair for the periodic version of the infinite lattice, that is, the lattice equations (2.13) with the periodicity restriction $x_n = x_{n+N}$ for all $N \geq 2$; thereby demonstrating the integrability of the periodic Toda lattice equations. The integrability of the generalised Toda lattice (2.12) was first establised by Mikhailov in [12].

As mentioned earlier, a very different class of nonlinear integrable p.d.e. arises if, instead of the periodic operators $\mathcal{P}_n$, we consider the operators belonging to the kernels of the Laplace-Darboux maps of various orders. These operators are related to the free end generalised Toda lattice equations for an even number of masses. For example, consider the subset $\ker(\mathbb{L}^{(2)})$. This entails a 'double termination' of the Laplace Darboux-sequence and leads to the free end generalised Toda lattice for *four* masses as follows. ¿From equations (2.7)-(2.9), we have that $L \in \ker(\mathbb{L}^{(2)})$ if and only if

$$2h(L_1) - k(L_1) - \partial^2_{xy} \ln |h(L_1)| = 0, \tag{2.15}$$

$$2k(L_{-1}) - h(L_{-1}) - \partial^2_{xy} \ln |k(L_{-1})| = 0. \tag{2.16}$$

Combining equations (2.15) and (2.16) with equations (2.7) and (2.9) and upon using equations (2.8) to eliminate $h(L_{-1})$ and $k(L_1)$ we are led to a system of four equations in the four unknowns $h(L_1), h(L), k(L)$ and $k(L_{-1})$ which determine the G-equivalence classes of operators in $\ker(\mathbb{L}^{(2)})$. Letting $y_1 = \ln |h(L)|, y_2 = \ln |h(L)|, y_3 = \ln |k(L)|, y_4 = \ln |k(L_{-1})|$ we have shown that $L \in \ker(\mathbb{L}^{(2)})$ if and only if

$$\partial^2_{xy} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} e^{y_1} \\ e^{y_2} \\ e^{y_3} \\ e^{y_4} \end{pmatrix} \tag{2.17}$$

Thus, an operator $L \in \mathcal{D}$ belongs to the kernel of $\mathbb{L}^{(2)}$ if and only if equations (2.17) are satisfied. Now equations (2.17) may be obtained as a generalisation of the Toda equations (2.13) imposing the boundary conditions $x_0 = -\infty$ and $x_5 = \infty$ meaning that the Toda lattice is given the boundary conditions that the interaction potential to the left of x_1 and to the right of x_4 is zero. This represents the so called free end Toda lattice and equations (2.17) are called the generalised free end Toda lattice equations. Furthermore, because the matrix in equations (2.17) is the Cartan matrix of the Lie algebra of the Lie group A_4, they are also known as the A_4 Toda lattice equations.

The generalised Toda lattice equations have been the subject of numerous investigations in recent years. In particular we mention the work of Leznov and Saveliev [13] who discovered the connection between the generalised Toda lattice equations (both free end and periodic) and Lie algebra representations. We also note the monumental work of Kostant [23] along this direction which came out at about the same time as reference [22]. In the case of the free end lattice they were able to obtain the "general solution" in terms of $2r$ arbitrary functions were r is the number of particles in the lattice thereby generalising the well known formula for the Liouville equation

$$\partial_{xy}^2 u = e^u,$$

which corresponds to the A_1 Toda lattice.

Apart from the work of the classical authors, connections between the A_n Toda lattice and linear wave equations has been recently discussed by Torrence [14]. This work of Torrence aims to find all operators which possess the so called characteristic propogation property and which are shown to be in correspondence with the A_n Toda equations.

Remarkably, thanks to a deep result of E. Vessiot [16] on those second order hyperbolic p.d.e. in the plane that can be integrated by the method of Darboux, the set $\ker(\mathbb{L}^{(1)})$ can be explicitly parametrised in terms of four arbitrary functions of one variable [15]. (Reference [15] also fills in a gap in Vessiot's program of classifying the scalar Darboux integrable p.d.e. in the plane.) Thus, the results of [15,16] provide a geometric method for the construction of the general solution of the A_2 Toda equations and may be compared to the result of Leznov and Saveliev for these equations. The extension of this result to the explicit parametrisation of $\ker(\mathbb{L}^{(n)})$ for all $n \geq 2$ is currently in the making. This will provide a geometric construction of the general solution of the A_{2m} Toda lattice, where $2m$ is the number of masses in the lattice. The relationship between operators in $\mathcal{D}$ so constructed and the characteristic propogation property studied by Torrence remains to be explored.

3. Concluding Remarks

In the previous section we briefly explored a conection between the set of differential operators $\mathcal{D}$, the generalised Toda lattice and a number of the automorphisms of $\mathcal{D}$. This highlights a by now ubiquitous situation for integrable nonlinear systems, namely, their connections with geometric structures, often in very unexpected ways. Examples of these include the twistor correspondence of Penrose (see for example [17]) and the generalisation to p.d.e. of Liouville-Arnol'd integration (see for example the collection [18]). In this paper I have tried to point out one further connection which to date has not been very much explored. And it is intriguing that a construction so far removed from dynamics as the geometry of the set of operators $\mathcal{D}$ and the associated Laplace-Darboux map leads to integrable dynamical systems: the generalised Toda lattice and its various reductions. Our work in this area is directed toward understanding the basic geometry of the Laplace-Darboux map and to ask the question: where do the nonlinear integrable systems "come from" ? For example, at this stage the definition of the Laplace-Darboux map does not appear to be well motivated. The Laplace transformations $\mathcal{L}_1$ and $\mathcal{L}_{-1}$ are also difficult to motivate in any natural way. Other problems include a more complete description of the periodic subsets $\mathcal{P}_n$, $n \geq 2$, in terms of $\mathcal{G}$-orbits just as for $\mathcal{P}_1$ (Proposition 2). Answers to these and similar questions will guide further work and attempts to extend the Laplace-Darboux map to new operator spaces and new integrable systems. In fact, encouraging results in this direction have already been obtained by Weiss [24] however the underlying mechanism by which the integrable nonlinear equations are being encoded is still unclear. As mentioned earlier, a first step in this direction is the solution of the $\mathcal{G}$-equivalence problem for $\mathcal{D}$ treated in [10] using the Cartan equivalence method. Here the all the local differential invariants have been constructed and a start has been made on obtaining an explicit description of all the $\mathcal{G}$- equivalence classes. The extent to which such an analysis will shed light on the characterisation problem for integrable systems remains an intriguing open question.

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