Chapter 3

$U(1) \times U(1)$ stability of the $(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ Kasner metrics.

In this Chapter we present the proof of Theorem 1.5.2, namely that the singularity of $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ (or permutation thereof) Kasner metrics is stable under $U(1) \times U(1)$ symmetric perturbations. The problem reduces to establishing *a priori* estimates for a Lorentzian harmonic–type map from two–dimensional Minkowski space–time to the unit two–dimensional hyperboloid of constant negative curvature. We shall start by analyzing the harmonic map equations, the geometric interpretation of the estimates proved below will be given in Section 3.5.

3.1 Introduction — notation

Let M be a Riemannian manifold with scalar product $\langle\ ,\ \rangle$. Let $t_0 < 0$, let $x(t,\theta)$: $[t_0, 0) \times S^1 \to M$ satisfy

$$\frac{DX_t}{Dt} - \frac{DX_\theta}{D\theta} = -\frac{X_t}{t} \tag{3.1.1}$$

where $X_t \equiv \frac{\partial x(t,\theta)}{\partial t}$, $X_\theta \equiv \frac{\partial x(t,\theta)}{\partial \theta}$, D denotes the Levi-Civita connection of $\langle\ ,\ \rangle$,

$$\frac{D}{D\theta} \equiv D_{X_\theta} \equiv X_\theta^A D_A, \quad \frac{D}{Dt} \equiv D_{X_t} \equiv X_t^A D_A.$$

$\langle Y \rangle$ or $|Y|$ will be used to denote $\sqrt{\langle Y, Y\rangle}$; we shall identify S^1 with $\{\theta \in [0, 2\pi]_{|\mathrm{mod}\, 2\pi}\}$. We shall throughout use the notation

$$X^{(k)} = \left(\frac{D}{D\theta}\right)^{(k-1)} X_\theta \,,$$

we shall always assume $X_t(t_0, \cdot)$, $X_\theta(t_0, \cdot) \in H_1(S^1)$. $K(\; , \;)$ denotes the curvature tensor of $\langle \, , \, \rangle$ defined by

$$K(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \,,$$

We use $|K|$ to denote an upper bound on the sectional curvatures, and $|DK|$, etc. to denote the Riemannian norm of the tensor DK, etc. The matrix $\eta_{\mu\nu}$ will denote the two dimensional Minkowski metric, $\eta_{\mu\nu}\, dx^\mu\, dx^\nu = -dt^2 + d\theta^2$. For $|t| < \pi$ it will often be convenient to use the coordinates

$$u = t + \theta \,, \qquad v = t - \theta \,,$$

so that

$$\frac{\partial}{\partial u} = u^\mu\, \partial_\mu = \tfrac{1}{2}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right), \quad \frac{\partial}{\partial v} = v^\mu\, \partial_\mu = \tfrac{1}{2}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}\right)$$

$\eta_{uv} = -\frac{1}{2}$, $\eta^{uv} = -2$.

3.2 Pointwise Estimates

In this Section we shall prove some rough pointwise estimates as $t \to 0$ for solutions of (3.1.1). The ideas of the proofs are inspired by some unpublished work by V. Moncrief; the author is grateful to V. Moncrief for making his results available prior to publication.

Lemma 3.2.1 *$X^{(k)}$ satisfies the equation*

$$\left[\left(\frac{D}{Dt}\right)^2 - \left(\frac{D}{D\theta}\right)^2\right] X^{(k)} = -\frac{1}{t}\frac{DX^{(k)}}{Dt} + N^{(k)} \,. \tag{3.2.1}$$

If for all multi–indices $0 \le |\alpha| \le k-1$ we have

$$|D^\alpha X_t| \le C_1\, |t|^{-|\alpha|-1}, \quad |D^\alpha X_\theta| \le C_1\, |t|^{-|\alpha|-1} \,,$$

and if

$$|K| + |DK| + \ldots |D^{|\alpha|-1} K| \le C_2 \ ,$$

then

1. $$|N^{(k)}| \le F(C_1, C_2, k)\, |t|^{-k-2} \ .$$

2. *We also have the estimate*

$$|N^{(k)}| \le \sum_{i=1}^{k} F_1(C_1, C_2, i, k)\, |X^{(i)}|\, |t|^{i-k-2} \tag{3.2.2}$$

where F, F_1 are some constants depending upon the arguments listed.

Proof: Applying $\left(\frac{D}{D\theta}\right)^k$ to both sides of (3.1.1) one obtains

$$\left[\left(\frac{D}{Dt}\right)^2 - \left(\frac{D}{D\theta}\right)^2\right] X^{(k)} = -\frac{1}{t}\frac{DX^{(k)}}{Dt} + L^{(k)} - \frac{M^{(k)}}{t} \ ,$$

with

$$\begin{aligned} L^{(k)} &= \left(\frac{D}{Dt}\right)^2 X^{(k)} - \left(\frac{D}{D\theta}\right)^k \frac{DX_t}{Dt} \ , \\ M^{(k)} &= \left(\frac{D}{D\theta}\right)^k X_t - \frac{D}{Dt} X^{(k)} \ . \end{aligned}$$

We have the recurrence relations for $k \ge 1$

$$\begin{aligned} L^{(k+1)} &= \frac{DL^{(k)}}{D\theta} + \frac{D}{Dt}(K(X_t, X_\theta)\, X^{(k)}) + K(X_t, X_\theta)\, \frac{DX^{(k)}}{Dt} \\ M^{(k+1)} &= \frac{DM^{(k)}}{D\theta} + K(X_\theta, X_t)\, X^{(k)} \end{aligned}$$

with

$$L^{(0)} = M^{(0)} = M^{(1)} = 0 \ , \qquad L^{(1)} = K(X_\theta, X_t)\, X_t \ ,$$

and part 1 follows by induction. To prove part 2 one shows by induction that there exists a set of linear operators $A^{(k,i)}$ such that

$$N^{(k)} = \sum_{i=1}^{k} A^{(k,i)}(X^{(i)})$$

(*e.g.* $A^{(1,1)}(Y) = K(Y, X_t)\, X_t$) and the bounds on $|A^{(k,i)}|$ are established by an induction argument. □

Lemma 3.2.2 *Let $T_{\mu\nu}$ be a symmetric traceless tensor, let $j_\mu = T_\mu{}^\nu{}_{,\nu}$. We have*

$$|T_{tt}(t_1,\theta)| \le \sup_{\psi\in[\theta-t_1+t_0,\theta+t_1-t_0]} (|T_{tt}|(t_0,\theta)+|T_{t\theta}|(t_0,\theta)) \tag{3.2.3}$$

$$+\int_{t_0}^{t_1} \sup_{\psi\in[\theta-t_1+t,\theta+t_1-t]} (|j_t|(t,\theta)+|j_\theta|(t,\theta))\,dt\,. \tag{3.2.4}$$

Proof. Let $\bar T_{\mu\nu}(\mu,\nu) = T_{\mu\nu}(t=\frac{u+v}{2},\theta=\frac{u-v}{2})$, let $\bar T_{uu} = \bar T_{\mu\nu}\,u^\mu\,u^\nu$, etc. We have

$$\bar T_{uu,v} = -\tfrac{1}{2}\,\bar\jmath_u\,,\quad \bar T_{vv,u} = -\tfrac{1}{2}\,\bar\jmath_v\,,\quad \bar\jmath_\mu(u,v) = j_\mu\left(\frac{u+v}{2},\frac{u-v}{2}\right)$$

therefore

$$\begin{aligned}\bar T_{uu}(u_1,v_1) &= -\tfrac{1}{2}\int_{v_1-\lambda}^{v_1}\bar\jmath_u(u_1,v)\,dv + \bar T_{uu}(u_1,v_1-\lambda)\,,\\ \bar T_{vv}(u_1,v_1) &= -\tfrac{1}{2}\int_{u_1-\lambda}^{u_1}\bar\jmath_v(u,v_1)\,du + \bar T_{vv}(u_1-\lambda,v_1)\,,\end{aligned}$$

adding these equations, setting $\lambda = 2(t_1-t_0)$, one is by elementary manipulations led to

$$\begin{aligned}T_{tt}(t_1,\theta_1) = &-\tfrac{1}{2}\textstyle\int_{t_0}^{t_1}\{(j_t+j_\theta)(t,\theta_1+t_1-t)+(j_t-j_\theta)(t,\theta_1-t_1+t)\}\,dt\\ &+\tfrac{1}{2}\,(T_{tt}+T_{t\theta})(t_0,\theta_1+t_1-t_0)+\tfrac{1}{2}\,(T_{tt}-T_{t\theta})(t_0,\theta_1+t_0-t_1)\,,\end{aligned} \tag{3.2.5}$$

and the result follows. □

Let us recall Gromwall's lemma:

Lemma 3.2.3 *Let $f,x\in C^1([t_0,0))$, $y\in C^0([t_0,0))$, $y\ge 0$, satisfy for $t\in[t_0,0)$*

$$f(t)\le x(t)+\int_{t_0}^t y(s)\,f(s)\,ds\,.$$

Then

$$f(t)\le x(t_0)\exp\left(\int_{t_0}^t y(s)\,ds\right)+\int_{t_0}^t\frac{dx}{dt}(s)\,\exp\left\{\int_s^t y(u)\,du\right\}ds\,.$$

Proposition 3.2.1 *Let $x(t_0,\theta)\in C^k(S^1)$, $k\ge 1$, $X_t(t_0,\theta)\in C^{k-1}(S^1)$. For all $t\ge t_0$ we have*[1]

[1]The proof of point a) of Proposition 3.2.1 is a slight variation of an unpublished argument of V. Moncrief.

a)

$$\left(|X_t|^2+|X_\theta|^2\right)(t,\theta)\le 2\Big\{\sup_{\psi\in[\theta-t+t_0,\theta+t-t_0]}\left(|X_t|^2+|X_\theta|^2\right)(t_0,\psi)\Big\}\left(\frac{t_0}{t}\right)^2. \quad (3.2.6)$$

b) If $k\ge 2$, and if

$$|K|+|DK|+\ldots+|D^{(k-2)}K|\le C_2$$

then there exist constants C depending only upon the arguments listed such that, for all $1\le|\alpha|\le k$,

$$|D^\alpha x|(t,\theta)\le C(|\alpha|,t_0,C_2,\|X_\theta(t_0)\|_{C^{|\alpha|-1}},\|X_t(t_0)\|_{C^{|\alpha|-1}})\,|t|^{-|\alpha|}. \quad (3.2.7)$$

Proof. Let

$$T^{(k)}_{\mu\nu}=|t|\{\langle X^{(k)}_\mu,X^{(k)}_\nu\rangle-\tfrac12\,\eta_{\mu\nu}\langle X^{(k)\alpha},X^{(k)}_\alpha\rangle\}$$

(adding a subscript means differentiation). We have

$$j^{(k)}_\mu=T^{(k)\nu}_{\mu\ \ ,\nu}=\tfrac12\,\delta^0_\mu(\langle X^{(k)}_\theta\rangle^2-\langle X^{(k)}_t\rangle^2)-|t|\langle X^{(k)}_\mu,N^{(k)}\rangle+\epsilon^{(k)}|t|\langle X^\nu,K(X_\nu,X_\mu)X^{(k)}\rangle\,, \quad (3.2.8)$$

$\epsilon^{(k)}=0$ if $k=0$, $\epsilon^{(k)}=1$ otherwise. For $k=0$ it follows

$$j^{(0)}_\theta=0\,,\quad |j^{(0)}_t|=\tfrac12\,|\langle X_t\rangle^2-\langle X_\theta\rangle^2|\le\frac{T^{(0)}_{tt}}{|t|}\,.$$

For $t_0\le t\le t_1$ let

$$\begin{aligned}f^{(k)}(t)&=\sup_{\psi\in[\theta-t_1+t,\theta+t_1-t]}T^{(k)}_{tt}(t,\psi)\,,\\ h^{(k)}(t)&=\sup_{\psi\in[\theta-t_1+t,\theta+t_1-t]}|T^{(k)}_{t\theta}|(t,\psi)\,.\end{aligned}$$

From Lemma 3.2.2 we have

$$f^{(0)}(t)\le f^{(0)}(t_0)+h^{(0)}(t_0)+\int_{t_0}^t\frac{f^{(0)}(s)}{|s|}\,ds\,,$$

so that Gromwall's lemma with $x(t) = f^{(0)}(t_0) + h^{(0)}(t_0)$, $y(t) = \frac{1}{|t|}$, gives

$$\begin{aligned} \forall t_0 \leq t_1 < 0 \quad f^{(0)}(t_1)|t_1| &\leq (f^{(0)}(t_0) + h^{(0)}(t_0))|t_0| \\ &\leq 2f^{(0)}(t_0)|t_0| \end{aligned} \tag{3.2.9}$$

which is equation (3.2.6). To obtain part (b) we shall proceed by induction, suppose therefore that (3.2.7) holds for $|\alpha| \leq k-1$. (3.2.8) and Lemma 3.2.1 yield

$$\begin{aligned} |j_t^{(k)}| &\leq \frac{T_{tt}^{(k)}}{|t|} + |t||X_t^{(k)}||N_k| + C_2|t||X_t||X_\theta|^2|X^{(k)}| \\ &\leq \frac{T_{tt}^{(k)}}{|t|} + \{(2)^{-\frac{1}{2}}|X_t^{(k)}|\}\{2^{\frac{1}{2}}F|t|^{-k-1}\} + C|t|^{-k-2} \\ &\leq \frac{3}{2}\frac{T_{tt}^{(k)}}{|t|} + \tilde{F}^2|t|^{-2k-2}, \\ |j_\theta^{(k)}| &\leq \frac{T_{tt}^{(k)}}{2|t|} + \tilde{F}^2|t|^{-2k-2}, \end{aligned}$$

with some constant $C = C(C_1, C_2)$, $\tilde{F}^2 = F^2 + C|t_o|^k$, and Lemma 3.2.2 gives

$$f^{(k)}(t) \leq f^{(k)}(t_0) + h^{(k)}(t_0) + \int_{t_0}^{t}\left(\frac{2f^{(k)}(s)}{|s|} + 2\tilde{F}^2 s^{-2k-2}\right)ds\,\alpha\,.$$

From Gromwall's Lemma one obtains

$$f^{(k)}(t) \leq (f^{(k)}(t_0) + h^{(k)}(t_0))\frac{t_0^2}{t^2} + \frac{2\tilde{F}^2}{2k-1}(|t|^{-2k-1} - |t_0|^{-2k+1}t^{-2}), \tag{3.2.10}$$

so that the result for all the derivatives of the form

$$\left(\frac{D}{D\theta}\right)^{i-1}X_\theta, \qquad \frac{D}{Dt}\left(\frac{D}{D\theta}\right)^{i-2}X_\theta$$

follows. The estimates for the remaining derivatives can be obtained by e.g. commuting all the t derivatives to the left, and then using (3.2.1) to replace pairs of t-derivatives by pairs of θ derivatives. □

Remark. If $|X_\theta| \leq C|t|^{\lambda-1}$ for some $\lambda > 0$, then a simple modification of the above proof gives

$$|D^\alpha X_\theta| \leq C|t|^{\lambda-|\alpha|-1}. \tag{3.2.11}$$

It is tempting to conjecture that this is a sharp estimate: (3.2.11) is indeed the best one can expect, since this behaviour is displayed by the maps considered in Appendix B, with any $0 \leq \lambda < 1$.

3.3 Integral "Decay" Estimates

Proposition 3.3.1 *Let $x \in C^i([t_0, 0) \times S^1)$ and let $X_\theta(t_0, \cdot)$, $X_t(t_0, \cdot) \in H_i(S^1)$, $i \geq 1$. There exist constants depending only upon the arguments listed such that*

1. $\forall \quad 1 \leq |\alpha| \leq i+1$,

$$g^{(\alpha)}(t) \equiv \oint d\theta \, |t|^{2|\alpha|} |D^\alpha x|^2 \leq C\left(|\alpha|, \|X_\theta(t_0)\|_{H_{|\alpha|-1}(S^1)}, \|X_t(t_0)\|_{H_{|\alpha|-1}(S^1)}, t_0\right). \tag{3.3.1}$$

2. *If at least one differentiation is a θ differentiation we have*

$$\lim_{t \to 0} g^{(\alpha)}(t) = 0 . \tag{3.3.2}$$

3. *If at least one differentiation is a θ differentiation then $\frac{g^{(\alpha)}(t)}{|t|} \in L^1([t_0, 0])$ and*

$$\int_{t_0}^{0} \frac{g^{(\alpha)}(s)}{|s|} \, ds \leq C'\left(|\alpha|, t_0, \|X_\theta(t_0)\|_{H_{|\alpha|-1}(S^1)}, \|X_t(t_0)\|_{H_{|\alpha|-1}(S^1)}\right). \tag{3.3.3}$$

4. *$g^{(t)}(s)$ tends to a limit as s goes to zero.*

Remark: The results above are close to being sharp, because, as shown in Appendix B (*cf.* Proposition B.1.1), for any $\epsilon \in [0, 1)$ there exist solutions of (3.1.1) such that we have $|X_\theta| \approx Ct^{\epsilon - 1}$, $|X_{\theta\theta}| \approx Ct^{\epsilon-2}$, etc.

Proof. Let

$$T^{(k)}_{\mu\nu} = |t|^{2k+2} \left\{ \langle X^{(k)}_\mu, X^{(k)}_\nu \rangle - \tfrac{1}{2} \eta_{\mu\nu} \langle X^{(k)\alpha}, X^{(k)}_\alpha \rangle \right\},$$

$$e^{(k)}(t) = \oint d\theta \, T^{(k)}_{tt} .$$

We have

$$\mathcal{J}_t^{(k)} = \mathcal{T}_t^{(k)\nu}{}_{,\nu} = |t|^{2k+1}\{k\langle X_t^{(k)}\rangle^2 + (k+1)\langle X_\theta^{(k)}\rangle^2\} - |t|^{2k+2}\langle X_t^{(k)}, N^{(k)}\rangle$$
$$+ \epsilon^{(k)}|t|^{2k+2}\langle X_\theta, K(X_\theta, X_t)X^{(k)}\rangle\ , \qquad (3.3.4)$$

$\epsilon^{(k)} = 0$ for $k = 0$, $\epsilon^{(k)} = 1$ otherwise, so that for $t_1 > t_0$

$$e^{(k)}(t_1) = e^{(k)}(t_0) - \int_{t_0}^{t_1} dt \oint d\theta\, \mathcal{J}_t^{(k)} \qquad (3.3.5)$$

which for $k = 0$ reads

$$e^{(0)}(t_1) = e^{(0)}(t_0) - \int_{t_0}^{t_1} dt \oint d\theta\, |t|\langle X_\theta\rangle^2 \qquad (3.3.6)$$

so that $e^{(0)}(t)$ is strictly decreasing and therefore tends to a limit, $e^{(0)}(0)$, which gives (3.3.1) for $\alpha = \theta$ or $\alpha = t$. (3.3.6) and Lebesgue monotone convergence theorem imply $\oint d\theta\, |t|\langle X_\theta\rangle^2 \in L^1([t_0, 0])$. To show (3.3.3) for higher derivatives we shall proceed by induction, suppose therefore that for $1 \le k \le i-1$

$$\int_{t_0}^{0} \frac{e^{(k)}(t)}{|t|}\, dt < \infty\,. \qquad (3.3.7)$$

Part 2 of Lemma 3.2.1 gives

$$\begin{aligned} \big||t|^{2k+2}\langle X_t^{(k)}, N^{(k)}\rangle\big| &\le (|t|^{k+\frac12}|X_t^{(k)}|)(|t|^{k+\frac32}|N^{(k)}|) \\ &\le \tfrac12 |t|^{2k+1}|X_t^{(k)}|^2 + C\sum_{i=1}^{k} |t|^{2i-1}|X^{(i)}|^2 \end{aligned} \qquad (3.3.8)$$

for some constant C. From Proposition 3.2.1, point 2, we have $\epsilon^{(k)}|X_\theta||K(X_\theta, X_t)X^{(k)}| \le \tilde{C}|t|^{-k-3}$ for some constant $\tilde{C}$, so that from (3.3.5) we have

$$\begin{aligned} e^{(k)}(t_1) \le\ & e^{(k)}(t_0) - \int_{t_0}^{t_1} dt \oint d\theta\, |t|^{2k+1}\{(k-\tfrac12)\langle X_t^{(k)}\rangle^2 + (k+1)\langle X_\theta^{(k)}\rangle^2\} \\ & + C\sum_{i=1}^{k}\int_{t_0}^{t_1} dt \oint d\theta\, |t|^{2i-1}|X^{(i)}|^2 + 2\pi\tilde{C}(t_1 - t_0)\,. \end{aligned} \qquad (3.3.9)$$

By hypothesis the integrand of the last integral at the right-hand side of (3.3.9) is in $L^1([t_o, 0])$; therefore by Lebesgue monotone convergence theorem

$$\int_{t_0}^{0} dt \oint d\theta\, |t|^{2k+1}(\langle X_t^{(k)}\rangle^2 + \langle X_\theta^{(k)}\rangle^2) < \infty\,. \qquad (3.3.10)$$

(3.3.10) implies that there exists a sequence $t_i \to 0$ such that $e^{(k)}(t_i) \to 0$, and from (3.3.5) we have

$$e^{(k)}(t) = e^{(k)}(t_i) - \int_t^{t_i} dt \oint d\theta \, \mathcal{J}_t^{(k)} . \tag{3.3.11}$$

(3.3.4), (3.3.7), (3.3.8) and (3.3.10) imply that $\mathcal{J}_t^{(k)}$ is in $L^1([t_0, 0] \times S^1)$ so that we may pass to the limit $t_i \to 0$ to obtain

$$e^{(k)}(t) = - \int_t^{0} dt \oint d\theta \, \mathcal{J}_t^{(k)} . \tag{3.3.12}$$

(3.3.12) shows in particular that $\lim_{t\to 0} e^{(k)}(t) = 0$, which is (3.3.2). Finally let $h(t) = \oint |t|^2 |X_\theta|^2(t,\theta)$. For $t_2 > t_1$ we have

$$h(t_2) - h(t_1) = \int_{t_1}^{t_2} k(t)\, dt \,,$$

$$k(t) = -\frac{2h(t)}{|t|} + 2 \oint d\theta \, t^2 \langle X_\theta, X_{\theta t}\rangle \,,$$

by what has been said $k(t) \in L^1([t_0, 0])$ and an argument along the lines of the proof of (3.3.12) shows that (3.3.2) holds for $\alpha = \theta$. The estimate (3.3.3) follows from (3.3.10) by commuting pairs of t derivatives with pairs of θ derivatives using equation (3.1.1). □

3.4 Pointwise "Decay" Estimates

For $t_0 \leq t \leq 0$ let $C_{t_0}^t$ denote the solid truncated light cone

$$C_{t_0}^t = \{(s,\theta), t_0 \leq s \leq t, s \leq \theta \leq -s\} .$$

Let $B(t)$ denote the "space ball", $B(t) = \{(s,\theta) : s = t, t \leq \theta \leq -t\}$. Let $R_{t_0}^t$, $L_{t_0}^t$ be the right and left truncated light-rays from $(0,0)$, *cf.* Figure 3.4.1:

$$\begin{aligned} L_{t_0}^t &= \{(s,\theta) : t_0 \leq s \leq t, \theta = s\} \\ R_{t_0}^t &= \{(s,\theta) : t_0 \leq s \leq t, \theta = -s\} \end{aligned}$$

By proposition 3.2.1 we can define

$$v_0 = \sup_{C_{t_0}^0} |t||X_t| < \infty . \tag{3.4.1}$$

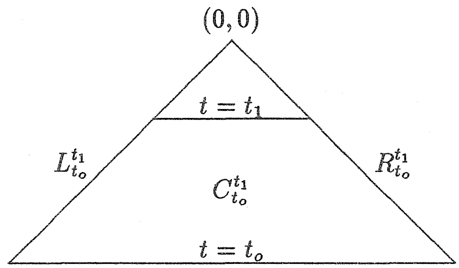

Figure 3.4.1: Truncated light cone.

Lemma 3.4.1 *Let*

$$\| X_t^{(k)}(t_0) \|_{H_k(S^1)} + \| X_\theta^{(k)}(t_0) \|_{H_k(S^1)} \leq M \ .$$

There exist t_1 independent constants $C_1(M, t_0)$, $C_1(k, M, t_0)$, $C_2(k, M, t_0)$ such that for all $t_0 \leq t_1 < 0$, $k \geq 1$, we have

a)

$$\int_{C_{t_0}^{t_1}} |t|^2 \{4\langle X_{\theta\theta}\rangle^2 + \langle X_{\theta t}\rangle^2\} \leq 6 \int_{C_{t_0}^{t_1}} (|K|v_0^2)^2 |X_\theta|^2 + C\,,$$

b)

$$\int_{C_{t_0}^{t_1}} |t|^{2k} \{\langle X_\theta^{(k)}\rangle^2 + \langle X_t^{(k)}\rangle^2\} \leq (|K|v_0^2)^2 C_1(k) \int_{C_{t_0}^{t_1}} |X_\theta|^2 + C_2(k)\,,$$

and $|K|$ is defined by

$$|K| = \sup_{p\in M} |K|(p) = \sup_{\substack{p\in M \\ A,B,C,D\in T_pM}} \frac{|\langle K(A,B)C, D\rangle|}{|A||B||C||D|}\,,$$

where K is the curvature tensor of $(M, \langle\ ,\ \rangle)$.

Proof. Let

$$T_{\mu\nu}^{(k)} = |t|^{2k}(\langle X_\mu^{(k)}, X_\nu^{(k)}\rangle - \tfrac{1}{2}\,\eta_{\mu\nu}\,\langle X^{(k)\alpha}, X_\alpha^{(k)}\rangle)\,.$$

We have

$$
\begin{aligned}
T^{(k)}_{uu,v} &= -\tfrac{1}{2} T^{(k)\nu}_{\mu}{}_{,\nu}\, u^{\mu} \\
&= -\frac{|t|^{2k-1}}{4}\{(2k-1)\langle X^{(k)}_{\theta}, X^{(k)}_{t}\rangle + (k-1)\langle X^{(k)}_{t}\rangle^{2} + k\langle X^{(k)}_{\theta}\rangle^{2}\} \\
&\quad + \frac{|t|^{2k}}{2}\langle X^{(k)}_{u}, N^{(k)}\rangle + \frac{1}{2}\epsilon^{(k)}|t|^{2k}\langle X_{\alpha}, K(X_u, X^{\alpha})X^{(k)}\rangle , \\
T^{(k)}_{vv,u} &= -\tfrac{1}{2} T^{(k)\nu}_{\mu}{}_{,\nu}\, v^{\mu} \\
&= -\frac{|t|^{2k-1}}{4}\{(1-2k)\langle X^{(k)}_{\theta}, X^{(k)}_{t}\rangle + (k-1)\langle X^{(k)}_{t}\rangle^{2} + k\langle X^{(k)}_{\theta}\rangle^{2}\} \\
&\quad + \frac{|t|^{2k}}{2}\langle X^{(k)}_{v}, N^{(k)}\rangle + \frac{1}{2}\epsilon^{(k)}|t|^{2k}\langle X_{\alpha}, K(X_v, X^{\alpha})X^{(k)}\rangle ,
\end{aligned}
$$

$\epsilon^{(k)} = 0$ if $k = 0$, $\epsilon^{(k)} = 1$ otherwise, therefore

$$
\begin{aligned}
\frac{\partial}{\partial u}(uT^{(k)}_{vv}) + \frac{\partial}{\partial v}(vT^{(k)}_{uu}) &= \frac{|t|^{2k}}{2}\left\{(k+1)|X^{(k)}_{\theta}|^{2} + k|X^{(k)}_{t}|^{2} + \frac{(2k-1)\theta}{|t|}\langle X^{(k)}_{\theta}, X^{(k)}_{t}\rangle\right\} \\
-\frac{|t|^{2k+1}}{2}\langle X^{(k)}_{t} + \frac{\theta}{|t|}X^{(k)}_{\theta}, N^{(k)}\rangle &- \frac{1}{2}\epsilon^{(k)}|t|^{2k}\langle X_{\alpha}, K(X_t + \frac{\theta}{|t|}X_{\theta}, X^{\alpha})X^{(k)}\rangle . \quad (3.4.2)
\end{aligned}
$$

Integrating (3.4.2) over $C^{t_1}_{t_0}$ yields, for $k \geq 1$,

$$
\begin{aligned}
\int_{C^{t_1}_{t_0}} |t|^{2k}(3|X^{(k)}_{\theta}|^{2} + |X^{(k)}_{t}|^{2}) &- 2\int_{C^{t_1}_{t_0}} |t|^{2k+1}\left\langle X^{(k)}_{t} + \frac{\theta}{|t|}X^{(k)}_{\theta}, N^{(k)}\right\rangle \\
\leq \quad & |t|^{2k+1}\int_{B(t)}\left(|X^{(k)}_{t}|^{2} + |X_{\theta}|^{2} + \frac{2\theta}{|t|}\langle X^{(k)}_{t}, X^{(k)}_{\theta}\rangle\right) d\theta\Big|^{t_1}_{t_0} \\
& + 2\epsilon^{(k)}\int_{C^{t_1}_{t_0}}\left||t|^{2k}\langle X_{\alpha}, K(X_t + \frac{\theta}{|t|}X_{\theta}, X^{\alpha})X^{(k)}\rangle\right| , \qquad (3.4.3)
\end{aligned}
$$

where $f(t)|^{t_1}_{t_0} = f(t_1) - f(t_0)$. For $k = 1$ we have

$$
|N^{(1)}| = |K(X_{\theta}, X_t)X_t| \leq |K|\,|X_t|^{2}\,|X_{\theta}|
$$

and straightforward manipulations lead to

$$
\int_{C^{t_1}_{t_0}} |t|^{2}\{2|X_{\theta\theta}|^{2} + \tfrac{1}{2}|X_{\theta t}|^{2}\} \leq 3\int_{C^{t_1}_{t_0}} (|K|\,v_0^{2})^{2}\,|X_{\theta}|^{2} + C ,
$$

and we have used proposition 3.2.1 to estimate the integrals at the right hand side of (3.4.3) by a constant C. Lemma 3.2.1 part 2 and an induction argument yield similarly

$$
\int_{C^{t_1}_{t_0}} |t|^{2k}\{|X^{(k)}_{\theta}|^{2} + |X^{(k)}_{t}|^{2}\} \leq C_1(k)\,(|K|\,v_0^{2})^{2}\int_{C^{t_1}_{t_0}} |X_{\theta}|^{2} + C_2(k) ,
$$

which had to be established. □

Remark. If $k=0$ we have $N^{(0)}=0$ and (3.4.2) integrated over $C^{t_1}_{t_0}$ yields

$$\int_{C^{t_1}_{t_0}} \left(|X_\theta|^2 - \frac{\theta}{|t|}\langle X_\theta, X_t\rangle\right) \le C \tag{3.4.4}$$

for some t_1–independent constant C.

Proposition 3.4.1 *Let* $t_0 < 0$, *let* $x(t_0,\cdot) \in C^k(S^1)$, $X_t(t_0,\cdot) \in C^{k-1}(S^1)$, $k \ge 2$, *suppose that either*

$$\int_{C^0_{t_0}} |X_\theta|^2 < \infty \tag{3.4.5}$$

or

$$\int_{C^0_{t_0}} |\langle X_\theta, X_t\rangle| < \infty\,.$$

Then for all $0 \le |\alpha| \le k-1$

$$\lim_{\substack{(t,\theta)\to(0,0)\\ (t,\theta)\in C^0_{t_0}}} t^{|\alpha|+1}|D^\alpha X_\theta| = 0\,.$$

Proof. (3.4.4) shows that without loss of generality we can suppose that (3.4.5) holds. Lemma 3.4.1, part a) shows that

$$\int_{C^0_{t_0}} t^2(|X_{\theta\theta}|^2 + |X_{\theta t}|^2) < \infty\ ,$$

therefore for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $|t| \le \delta$

$$\int_{C^{t/2}_{2t}} t^2(|X_{\theta\theta}|^2 + |X_{\theta t}|^2) + \int_{C^{t/2}_{2t}} |X_\theta|^2 \le \epsilon\,. \tag{3.4.6}$$

For $0 \le t \le \min(|t_0|, \pi/2)$ let $f_t : C^{-\frac{1}{2}}_{-2} \to I\!R^+ \cup \{0\}$ be defined by

$$C^{-\frac{1}{2}}_{-2} \ni (s,\theta) \to f_t(s,\theta) = t|X_\theta|(ts, t\theta)\,.$$

By (3.4.6) and by the pointwise estimates of proposition 3.2.1 for any $p \ge 2$ and for $t \le \delta$ we have

$$\begin{aligned} \int_{C^{-\frac{1}{2}}_{-2}} |\partial_\mu f_t|^p \ &\le\ C(p)\,\epsilon\,, \\ \int_{C^{-\frac{1}{2}}_{-2}} |f_t|^2 \ &\le\ \epsilon\,, \end{aligned} \tag{3.4.7}$$

so that setting p equal to, say, 3, one obtains by Sobolev inequality

$$\forall\,(s,\theta)\in C_{-2}^{-\frac{1}{2}} \qquad |f_t(s,\theta)|\leq C'\epsilon\,.$$

so that

$$\forall\,|\theta|\leq|t| \qquad |t|\,|X_\theta(t,\theta)|\leq C'\epsilon\,.$$

The higher derivatives estimates follow in a similar way from Lemma 3.4.1, part b). □

Proposition 3.4.2 *Suppose that $|K|v_0^2<\frac{1}{3\sqrt{6}}$. Then the conclusion of proposition 3.4.1 holds.*

Proof. Without loss of generality we may suppose $-\pi\leq t_0$. Let

$$C_{t_0}^{t\pm}=\{(s,\theta)\in C_{t_0}^t,\pm\theta\geq 0\}\,.$$

Integrating $(\frac{d}{d\theta}-\frac{d}{dt})(\theta|X_\theta|^2)$ over $C_{t_0}^{t_1+}$ and $(\frac{d}{d\theta}+\frac{d}{dt})(\theta|X_\theta|^2)$ on $C_{t_0}^{t_1-}$, adding the resulting identities one obtains

$$\int_{C_{t_0}^{t_1}}\langle X_\theta\rangle^2=\int_{C_{t_0}^{t_1}}2\langle X_\theta,|\theta|X_{\theta t}-\theta X_{\theta\theta}\rangle-\int_{B(t)}|\theta|\langle X_\theta\rangle^2\Big|_{t_0}^{t_1}, \tag{3.4.8}$$

the estimates of Lemma 3.4.1, part 1, and (3.4.8) give a t_1-independent bound C_1 for $\int_{B(t)}(\ )\big|_{t_0}^{t_1}$, therefore

$$\begin{aligned}
\int_{C_{t_0}^{t_1}}\langle X_\theta\rangle^2 &\leq C_1+2\int_{C_{t_0}^{t_1}}|t||X_\theta|(|X_{\theta t}|+|X_{\theta\theta}|)\\
&\leq C_1+2\Big(\int_{C_{t_0}^{t_1}}\langle X_\theta\rangle^2\Big)^{\frac{1}{2}}\Big\{\Big(\int_{C_{t_0}^{t_1}}|t|^2|X_{\theta t}|^2\Big)^{\frac{1}{2}}+\Big(\int_{C_{t_0}^{t_1}}|t|^2|X_{\theta\theta}|^2\Big)^{\frac{1}{2}}\Big\}\\
&\leq C_1+(\epsilon_1+\epsilon_2)\int_{C_{t_0}^{t_1}}\langle X_\theta\rangle^2+\frac{1}{\epsilon_1}\int_{C_{t_0}^{t_1}}|t|^2|X_{\theta t}|^2+\frac{1}{\epsilon_2}\int_{C_{t_0}^{t_1}}|t|^2|X_{\theta\theta}|^2\\
&\leq C_2+\left[\epsilon_1+\frac{6(v_0^2|K|)^2}{\epsilon_1}+\epsilon_2+\frac{3(v_0^2|K|)^2}{2\epsilon_2}\right]\int_{C_{t_0}^{t_1}}\langle X_\theta\rangle^2,
\end{aligned}$$

where we have used the Schwartz inequality and Lemma 3.4.1, point a). Setting $\epsilon_1=\sqrt{6}\,v_0^2|K|$, $\epsilon_2=\sqrt{\frac{3}{2}}\,v_0^2|K|$ one obtains

$$(1-3\sqrt{6}\,v_0^2|K|)\int_{C_{t_0}^{t_1}}\langle X_\theta\rangle^2\leq C_2\,,$$

so that for $|K|v_0^2 < (3\sqrt{6})^{-1}$

$$\forall\, t_0 \le t < 0 \qquad \int_{C_{t_0}^t} \langle X_\theta\rangle^2 \le C_3 = \frac{C_2}{(1 - 3\sqrt{6}v_0^2|K|)} ,$$

and the Lebesgue monotone convergence theorem implies that the hypotheses of proposition 3.4.1 hold. □

Propositions 3.2.1 and 3.4.2 imply:

Corollary 3.4.1 *Let x be a C^i, $i \ge 2$ solution of (3.1.1) and suppose that*

$$(|X_t|^2 + |X_\theta|^2)(t_0, \theta) < \frac{1}{6^{\frac{3}{2}} t_0^2 |K|} .$$

Then the conclusion of proposition 3.4.1 holds.

3.5 The Stability Theorem.

Let (Σ, g, K) be $U(1) \times U(1)$ symmetric Cauchy data, $\Sigma \approx T^3$, let $X_a = X^i_a \frac{\partial}{\partial x^i}$, $a = 1, 2$ be the Killing vectors generated be the $U(1) \times U(1)$ action. It has been shown in [32] that if one assumes

$$(K_{ij} - g^{k\ell}K_{k\ell}\, g_{ij})\, X^j_a = 0 \quad \left(\Longleftrightarrow \qquad c_a = \epsilon_{\alpha\beta\gamma\delta} X_1^\alpha X_2^\beta \nabla^\gamma X_a^\delta = 0\right) , \qquad (3.5.1)$$

then there exists a coordinate system $\{t \in (-\infty, 0),\ \theta,\ x^a \in [0, 2\pi]_{|\mathrm{mod}\, 2\pi}, a = 1, 2\}$ in which the metric takes the form

$$ds^2 = \gamma_{\mu\nu}dx^\mu dx^\nu = e^{2B}(-dt^2 + d\theta^2) + \lambda|t|\, n_{ab}\, (dx^a + g^a d\theta)(dx^b + g^b d\theta) , \qquad (3.5.2)$$

$$n_{ab}\, dx^a dx^b = (\cosh\rho + \cos\phi\, \sinh\rho)\, (dx^1)^2 + 2 \sinh\rho \sin\phi\, dx^1 dx^2$$

$$+ (\cosh\rho - \cos\phi\, \sinh\rho)\, (dx^2)^2 ,$$

$$B = B(t, \theta), \quad \rho = \rho(t, \theta), \quad \phi = \phi(t, \theta) ,$$

where λ and g^a are real constants, $\lambda > 0$. For a metric of the form (3.5.2) the dynamical part of Einstein equations reduces (*cf. e.g.* [32], or [62]) to harmonic–map–type equations

(3.1.1) for a map $x(t,\theta) = (\rho(t,\theta), \phi(t,\theta)) : (-\infty, 0) \times S^1 \to \mathcal{H}^2$, where $\mathcal{H}^2 \approx I\!R^2$ is the unit hyperboloid with the metric

$$ds^2 = d\rho^2 + \sinh^2 \rho \, d\phi^2 \,.$$

We also have the constraint equations

$$\frac{\partial B}{\partial t} = -\frac{1}{4t} + \frac{t}{4}[|X_t|^2 + |X_\theta|^2] \,, \tag{3.5.3}$$

$$\frac{\partial B}{\partial \theta} = \frac{t}{2}\langle X_t, X_\theta \rangle \,. \tag{3.5.4}$$

The main result of this chapter is the following:

Theorem 3.5.1 *Let $\Sigma \approx T^3$ and let (g_o, K_o) be Cauchy data for a Kasner metric with exponents $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$, or permutation thereof. There exists $\epsilon > 0$ such that for all $U(1) \times U(1)$ symmetric Cauchy data $(g, K) \in C^\infty(\Sigma)$ satisfying (3.5.1), for which*

$$||(g - g_o, K - K_o)||_{H_1(\Sigma)\oplus L^2(\Sigma)} < \epsilon \,,$$

the maximal globally hyperbolic Hausdorff development (M, γ) of (Σ, g, K) is future inextendible. Moreover on every future inextendible timelike curve in M the curvature scalar $|R_{\alpha\beta\gamma\delta}\, R^{\alpha\beta\gamma\delta}|$ tends to infinity in finite proper time.

Remark: If one assumes that the Cauchy data are given directly in the form appropriate for a metric of the form (3.5.2), then it is sufficient to assume that $(g, K) \in H_1(\Sigma) \oplus L^2(\Sigma)$. It should be pointed out that the construction which leads from general coordinates to the coordinates of (3.5.2) decreases the degree of differentiability of the components of the metric tensor.

To prove Theorem 3.5.1 we shall need the following:

Proposition 3.5.1 *Let Γ be a future inextendible timelike curve in a vacuum space–time with a metric of the form (3.5.2), then Γ reaches $t = 0$ in finite proper time.*

Proof: Let $\Gamma = \{x^\mu(s)\}$, where s is a proper time along Γ, with $t(s)$ being an increasing function of s. From $\gamma_{\mu\nu}x^\mu x^\nu = -1$ we have

$$e^B \frac{dt}{ds} \geq 1 \ . \tag{3.5.5}$$

The constraint equation (3.5.3) and Proposition 3.2.1 give

$$\frac{-1+E_1}{4t} \leq \frac{\partial B}{\partial t} \leq -\frac{1}{4t} \ ,$$

where

$$E_1 = \sup_{t \geq t_o} t^2[|X_t|^2 + |X_\theta|^2] < \infty$$

which implies, for some constant C,

$$C^{-1}|t|^{(E_1-1)/4} \leq e^B \leq C|t|^{-1/4} \ ,$$

so that (3.5.5) implies, for $s_2 \geq s_1$,

$$C|t|^{-1/4}\frac{dt}{ds} \geq 1 \quad \Longrightarrow \quad s_2 \leq s_1 + 4C(|t(s_1)|^{3/4} - |t(s_2)|^{3/4})/3 \ ,$$

thus Γ reaches $t = 0$ in finite proper time. $\square$

Proof of Theorem 3.5.1: Consider the map $x_o(t,\theta) = (\rho_o(t,\theta), \phi_o(t,\theta)) = (0,0)$; it is easily seen that x_o solves the dynamic equations (3.1.1), integrating (3.5.2) one finds that the corresponding metric is the Kasner metric with exponents $(p_1,p_2,p_3) = (\frac{2}{3},\frac{2}{3},-\frac{1}{3})$. It follows from Corollary 3.4.1 that for all $x(t_o,\theta)$, $X_t(t_o,\theta)$ satisfying

$$t_o^2[|X_t|^2 + |X_\theta|^2] < 6^{-3/2}$$

we have

$$\lim_{t\to 0} t\,|X_\theta| = \lim_{t\to 0} t^2|X_{\theta\theta}| = \lim_{t\to 0} t^2|X_{t\theta}| = 0 \ , \tag{3.5.6}$$

moreover from Proposition 3.2.1, point a) it follows that

$$v(t,\theta) \equiv t|X_t|(t,\theta) \leq 2^{1/2}6^{-3/4} < 1 \ . \tag{3.5.7}$$

A SHEEP calculation gives

$$R^{\hat\mu}{}_{\hat\nu\hat\alpha\hat\beta} = \frac{e^{-2B}}{8t^2}\{(1 - t^2|X_t|^2)\,A^{\hat\mu}{}_{\hat\nu\hat\alpha\hat\beta} + B^{\hat\mu}{}_{\hat\nu\hat\alpha\hat\beta}\} \ , \tag{3.5.8}$$

where hats refer to the orthonormal frame

$$e^{\hat{t}} = e^{-B}dt \ ,$$

$$e^{\hat{\theta}} = e^{-B}d\theta \ ,$$

$$e^{\hat{1}} = (\lambda|t|)^{1/2}e^{\rho/2}\left[\cos(\phi/2)(dx + g^1 d\theta) + \sin(\phi/2)(dy + g^2 d\theta)\right] \ ,$$

$$e^{\hat{2}} = (\lambda|t|)^{1/2}e^{-\rho/2}\left[-\sin(\phi/2)(dx + g^1 d\theta) + \cos(\phi/2)(dy + g^2 d\theta)\right] \ ,$$

with $x \equiv x^1$, $y \equiv x^2$, and where the non-vanishing components of $A^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}\hat{\beta}}$ are

$$A^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = 2\,, \quad A^{\hat{t}}{}_{\hat{1}\hat{t}\hat{1}} = v_\rho - 1\,, \quad A^{\hat{t}}{}_{\hat{2}\hat{t}\hat{2}} = -v_\rho - 1\,, \quad A^{\hat{t}}{}_{\hat{1}\hat{t}\hat{2}} = v_\phi\,,$$

$$A^{\hat{\theta}}{}_{\hat{1}\hat{\theta}\hat{1}} = -v_\rho - 1\,, \quad A^{\hat{\theta}}{}_{\hat{2}\hat{\theta}\hat{2}} = v_\rho - 1\,, \quad A^{\hat{\theta}}{}_{\hat{1}\hat{\theta}\hat{2}} = -v_\phi\,, \quad A^{\hat{1}}{}_{\hat{2}\hat{1}\hat{2}} = 2\,, \tag{3.5.9}$$

$$v_\rho \equiv t\rho_{,t}\,, \quad v_\phi = t\sinh\rho\,\phi_{,t}\,,$$

while for $B^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}\hat{\beta}}$ the following estimations hold

$$|B^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}\hat{\beta}}| \le C\left[t^2|X_\theta|^2(1 + |t||X_t| + |t||X_\theta|) + t^2|X_t||X_\theta|(1 + |t||X_t|) + t^2|X_{\theta\theta}| + t^2|X_{t\theta}|\right] \ . \tag{3.5.10}$$

(3.5.6) – (3.5.7) allow us to neglect all the terms involving $B^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}\hat{\beta}}$ when calculating $\alpha \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ to obtain

$$\alpha \approx \frac{e^{-4B}}{4t^4}(1 - t^2|X_t|^2)^2(3 + t^2|X_t|^2) \ .$$

Let

$$\tilde{a} = 2B + \tfrac{1}{2}\ln|t|, \quad \tilde{a}_o = \sup_\theta a(t_0, \theta) \ . \tag{3.5.11}$$

By equation (3.5.3) the function $\tilde{a}$ is monotonically decreasing, therefore

$$e^{-4B} \ge |t|e^{-2\tilde{a}_0} \tag{3.5.12}$$

which together with (3.5.7) implies that there exists $\epsilon > 0$ such that for $t \ge t_1$, t_1 large enough, we have

$$|\alpha(t,\theta)| > \frac{\epsilon}{|t|^3} \ .$$

By Proposition 3.5.1 every future inextendible timelike curve reaches $t = 0$ in finite time, and Proposition C.2.4 establishes our claims. $\square$

The following result proves existence of curvature singularities in polarized Gowdy metrics on T^3 without smallness hypotheses:

Proposition 3.5.2 *Let x be a C^1 solution of (3.1.1) such that $X_\theta(t_0, \cdot)$, $X_t(t_0, \cdot) \in H_1(S^1)$, let*

$$\alpha = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$$

be the quadratic curvature scalar of the associated Gowdy space-time. Suppose that $|t||X_t|(t, \cdot)$ does not converge to 1 in $L^2(S^1)$ as t goes to zero. Then there exists a sequence of points (t_i, θ_i), $t_i \to 0$, such that

$$|\alpha(t_i, \theta_i)| > \frac{\epsilon}{|t_i|^3}$$

for some $\epsilon > 0$.

Remarks:

1. Note that a sufficient condition for convergence in $L^2(S^1)$ is pointwise convergence, so that Proposition 3.5.2 implies in particular that if $|t||X_t|(t, \cdot)$ converges pointwise to something different from 1 as t tends to zero, then there must be a curvature singularity somewhere on the boundary $t = 0$.

2. The proof of Proposition 3.5.2 does not imply existence of a singularity on the whole boundary $t = 0$ since there may be subsets of the set $t = 0$ on which $|t||X_t|(t, \cdot)$ converges to 1. It might happen that the metric is extendible through such subsets — this occurs indeed for some polarized Gowdy metrics [37] [36].

Proof. From the proof of Theorem 3.5.1 one obtains

$$\alpha = \frac{e^{-4B}}{4t^4} \{(1 - t^2|X_t|^2)^2(3 + t^2|X_t|^2) + \beta_1 + \beta_2\} , \tag{3.5.13}$$

$$\beta_1 = \frac{1}{8}(1 - t^2|X_t|^2) B_{\hat\mu\hat\nu\hat\alpha\hat\beta} A^{\hat\mu\hat\nu\hat\alpha\hat\beta} \,, \qquad \beta_2 = \frac{1}{16} B_{\hat\mu\hat\nu\hat\alpha\hat\beta} B^{\hat\mu\hat\nu\hat\alpha\hat\beta} \,,$$

therefore, by (3.5.12),

$$4 \oint \alpha\, |t|^3 \, d\theta \geq e^{-2\tilde a_0} \left[3 \oint (1 - |t|^2 |X_t|^2)^2 d\theta - \left| \oint \beta_1 \, d\theta \right| - \left| \oint \beta_2 \, d\theta \right| \right] \,, \qquad (3.5.14)$$

where $\tilde a_o$ has been defined in (3.5.11). Suppose that $1 - |t|^2 |X_t|^2$ does not converge to 0 in $L^2(S^1)$, therefore there exists $\epsilon > 0$ and a sequence $t_i \nearrow 0$ such that

$$\oint \left(1 - |t_i|^2 |X_\theta|^2 (t_i, \theta)\right)^2 \, d\theta > 8\pi e^{2\tilde a_0} \, \epsilon \,,$$

Proposition 3.2.1 and (3.5.10) imply,

$$\forall \quad \hat\mu, \hat\nu, \hat\alpha, \hat\beta \qquad |B_{\hat\mu\hat\nu\hat\alpha\hat\beta}| \leq C \,, \qquad (3.5.15)$$

which together with proposition 3.3.1, point 2, gives

$$\oint |B_{\hat\mu\hat\nu\hat\alpha\hat\beta}|^2 (t, \theta) \, d\theta \ \xrightarrow{t \to 0} \,.$$

It follows that there exists $t(\epsilon) < 0$ such that for all $0 > t > t(\epsilon)$ we have

$$\left| \oint \beta_2 (t, \theta) \, d\theta \right| d\theta < 8\pi e^{2\tilde a_0} \, \epsilon \,.$$

From $2xy \leq \delta x^2 + \delta^{-1} y^2$ it follows that

$$\oint 2 B_{\hat\mu\hat\nu\hat\alpha\hat\beta} A^{\hat\mu\hat\nu\hat\alpha\hat\beta} \, d\theta \leq \sum \oint \delta \, |A_{\hat\mu\hat\nu\hat\alpha\hat\beta}|^2 \, d\theta + \sum \oint \delta^{-1} |B_{\hat\mu\hat\nu\hat\alpha\hat\beta}|^2 \, d\theta \,,$$

which can be made arbitrarily small by an appropriate choice of δ for t large enough, so that for $t > t(\epsilon)$ we can also require

$$\left| \oint \beta_1 (t, \theta) \, d\theta \right| < 8\pi e^{2\tilde a_0} \, \epsilon \,.$$

It follows that for $t_i > t(\epsilon)$ we have

$$\oint \alpha(t_i, \theta) \, d\theta > \frac{2\pi\epsilon}{|t_i|^3} \,,$$

therefore there exist points θ_i such that

$$\alpha(\theta_i, t_i) > \frac{\epsilon}{|t_i|^3} \,.$$

□

Appendix A

On the "hyperboloidal initial data", and Penrose conditions.

Let us briefly recall the conformal framework introduced by Penrose [104] to describe the behaviour of physical fields at null infinity. Given a, say vacuum, smooth "physical" space-time $(\tilde{M}, \tilde{\gamma})$ one associates to it a smooth "unphysical space-time" (M, γ) and a smooth function Ω on M, such that $\tilde{M}$ is a subset of M and

$$\Omega|_{\tilde{M}} > 0 , \quad \gamma_{\mu\nu}|_{\tilde{M}} = \Omega^2 \tilde{\gamma}_{\mu\nu} , \tag{A.0.1}$$

$$\Omega|_{\partial\tilde{M}} = 0 , \tag{A.0.2}$$

$$d\Omega(p) \neq 0 \quad \text{for} \quad p \in \partial\tilde{M} , \tag{A.0.3}$$

where $\partial\tilde{M}$ is the boundary of $\tilde{M}$ in M (it should be stressed that in this section a notation *inverse* to that used in 1.6 is used: tilded quantities denote the physical ones, while non-tilded quantities denote the unphysical (conformally rescaled) ones). It is common usage in general relativity to use the symbol $\mathcal{I}$ for $\partial\tilde{M}$, and we shall sometimes do so. If Σ is a hypersurface in M, by $\mathcal{I}^+$ we shall denote the connected component of $\mathcal{I}$ which intersects the causal future of Σ. The hypothesis of smoothness of (M, γ, Ω) and the fact that $(\tilde{M}, \tilde{\gamma})$ is vacuum imposes several restrictions on various fields; if one defines (*cf.* [104])

$$P_{\mu\nu} = \tfrac{1}{2} \left(R_{\mu\nu} - \tfrac{1}{6} R \gamma_{\mu\nu} \right) , \tag{A.0.4}$$