STRICT PROPERTY (M) IN BANACH SPACES

YUNAN CUI*

Abstract. A new property, namely strict property (M), that implies the Opial property is introduced. We discuss relations between this property and some other well known properties. We also prove that Cesaro sequence spaces have strict property (M).

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Key words. fixed point property, Cesaro sequence spaces; property (M) ; strict property (M).

1. Introduction. Let $(X, \|\cdot\|)$ be a real Banach space, and X^* be the dual space of X. Let $B(X)$, $S(X)$ be the closed unit ball and the unit sphere of X, respectively.

DEFINITION 1.1. A Banach space X has *property* (M) if whenever $x_n \stackrel{w}{\to} 0$ then $\psi_{(x_n)}(x) := \limsup_{n\to\infty} \|x_n - x\|$ is a function of $\|x\|$ only.

Property (M) was introduced by Kalton (see [9]). It is an essential ingredient in his characterization of those separable Banach spaces X for which the compact operators $K(X)$ form an M-ideal in the algebra of all bounded linear operators, $L(X)$. That is

$$L(X)^* = (K(X)^{\perp} \oplus V)_1, \text{ for some closed subspace } V.$$

It was immediately recognized as a prime candidate for a sufficiency condition for the weak fixed point property but it was not until 1997 that García-Falset and Sims (see [6]) proved that a Banach space with property (M) has the weak fixed point property, i.e., every nonexpansive mapping T from a weakly compact and closed set $A \subset X$ into itself has a fixed point in A.

Since in a Banach space X, property (M) implies that for every weakly null sequence $\{x_n\} \subset X$ and $x \in X$ we have $\psi_{(x_n)}(tx)$ is an increasing function of t on $[0, \infty)$, (see [6]), it is clear that property (M) is equivalent to $x_n \stackrel{w}{\to} 0$ and $\|u\| \le \|v\|$ implying that $\limsup_{n\to\infty} \|x_n + u\| \le \limsup_{n\to\infty} \|x_n + v\|$.

Suppose that X has property (M), $x_n \stackrel{w}{\to} 0$ and $\|u\| \le \|v\|$. Choose $\lambda \ge 1$ such that $\|\lambda u\| = \|v\|$ then

$$\limsup_{n\to\infty} \|x_n + v\| = \limsup_{n\to\infty} \|x_n - v\|$$

$$= \limsup_{n\to\infty} \|x_n + \lambda u\| \ge \limsup_{n\to\infty} \|x_n - u\|.$$

We now introduce strict property (M).

DEFINITION 1.2. A Banach space X has *strict property* (M) if X has property (M) and $\limsup_{n\to\infty} \|x_n + u\| < \limsup_{n\to\infty} \|x_n + v\|$ whenever $x_n \stackrel{w}{\to} 0$ and $\|u\| < \|v\|$.

Similar conditions on weak* null sequences in X^* are called *property* (M^*).

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*Department of Mathematics, Harbin University of Science and technology, Harbin 150080, P. R. China, email `cuiya@hkd.hrbust.edu.cn`

DEFINITION 1.3. A Banach space X is said to have the *Opial property* if every weakly null sequence $\{x_n\} \subset X$ satisfies

$$\liminf_{n\to\infty} \|x_n\| \le \liminf_{n\to\infty} \|x_n + x\|$$

for every $x \in X$ $(x \neq 0)$, (see [15]).

It is clear that a Banach space with strict property (M) has the *Opial* property. Hence a Banach space with strict property (M) has weak normal structure. It is well known that c_0 fails to have weak normal structure but it has property (M). This means that strict property (M) is essentially stronger than property (M).

DEFINITION 1.4. A Banach space X has *property* (M_p), $(1 \le p < \infty)$, if

$$\limsup_{n\to\infty} \|x_n + x\|^p = \limsup_{n\to\infty} \|x_n\|^p + \|x\|^p$$

for all $x \in X$, whenever $x_n \overset{w}{\to} 0$. Property (M_∞) is the requirement that

$$\limsup_{n\to\infty} \|x_n + x\| = \max\{\limsup_{n\to\infty} \|x_n\|, \|x\|\}.$$

for all $x \in X$, whenever $x_n \overset{w}{\to} 0$.

It is clear that a Banach space with property (M_p) $(1 \le p < \infty)$ has strict property (M).

To obtain the weak fixed point property in certain Banach spaces, García-Falset introduced in [4] the following coefficient.

$$R(X) = \sup\left\{\liminf_{n\to\infty} \|x_n + x\| : \{x_n\} \subset B(X), x_n \overset{w}{\to} 0, x \in B(X)\right\}.$$

He proved that a Banach space X with $R(X) < 2$ has the weak fixed point property (see [4] and [5]). A Banach space X is nearly uniformly smooth (NUS) if for every $\varepsilon > 0$ there is $\eta > 0$ such that if $t \in (0, \eta)$ and (z_n) is a basic sequence in $B(X)$, then there exists $k > 1$ such that $\|x_1 + tx_k\| \le 1 + t\varepsilon$ (see [17]). A natural generalization of this notion is WNUS whenever it satisfies the above condition with for some $\varepsilon \in (0, 1)$ in place of for every $\varepsilon > 0$ (see [5]).

Recall that a sequence $\{x_n\}$ is said to be an *ε-separate sequence* for some $\varepsilon > 0$ if

$$sep(\{x_n\}) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon.$$

An important property that implies weak normal structure is the following property. A Banach space X is said to have the *uniform Kadec-Klee* property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if x is a weak limit of a norm one ε-separate sequence then $\|x\| < 1 - \delta$.

It is well known that a Banach space X is WNUS if and only if it is reflexive and $R(X) < 2$ (see [4]) and a Banach space X is NUS if and only if it is reflexive and its dual space has the uniform Kadec-Kleeproperty, (see [8]).

Let l^0 denote the set of all real sequences and N the set of natural numbers.

DEFINITION 1.5. A *Köthe sequence space* is a subspace X of l^0 containing an element $x = \{x(i)\} \in X$ with $x(i) > 0$ and such that for every $x \in l^0$ and $y \in X$ with $|x(i)| \leq |y(i)|$ for all $i \in \mathbb{N}$ we have $x \in X$ and $||x|| \leq ||y||$.

DEFINITION 1.6. Let X be a Köthe sequence space. We say that the norm is *absolutely continuous* at $x = \{x(i)\}$ if $\lim_{n\to\infty} ||(0, \cdots, 0, x(n), x(n+1), \cdots)|| = 0$. Let X_a denote the subspace consisting of those elements x at which the norm is continuous. We say that X has *absolutely continuous norm* if $X_a = X$.

DEFINITION 1.7. A Köthe sequence space X is said to have the *Fatou property* if for every sequence $\{x_n\} \subset X$ and $y \in X$ satisfying $|x_n(i)| \uparrow |y(i)|$ for all $i \in N$ we have $||x_n|| \to ||y||$.

A Cesaro sequence space was introduced by J. S. Shue in 1970, (see [8]). It is, for example, useful for the theory of matrix operators. For $1 < p < \infty$, the *Cesaro* sequence space (ces_p, for short) is defined by

$$ces_p = \left\{ x \in l^0 : ||x|| = \left(\sum_{n=1}^{\infty} \left[\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right]^p \right)^{\frac{1}{p}} \right\}.$$

We will subsequently consider some geometric properties of ces_p.

2. Results.

Theorem 2.1. *If a Banach space X has property (M_p) $(1 \leq p < \infty)$, then $R(X) = 2^{\frac{1}{p}}$.*

Proof. For every weakly null sequence $\{x_n\} \subset S(X)$ and $x \in S(X)$, we get

$$\limsup_{n\to\infty} ||x_n - x||^p = \limsup_{n\to\infty} ||x_n||^p + ||x||^p = 2.$$

So,

$$\liminf_{n\to\infty} ||x_n - x|| \leq \limsup_{n\to\infty} ||x_n - x|| = 2^{\frac{1}{p}}.$$

It is clear that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\limsup_{i\to\infty} ||x_{n_i} - x|| = \limsup_{n\to\infty} ||x_n - x||^p = 2^{\frac{1}{p}},$$

so $R(X) = 2^{\frac{1}{p}}$. □

We recall that a separable Banach space X is (weakly) stable (see [12]) if for any pair of (weakly null sequences) bounded sequence $\{u_n\}$, $\{v_n\}$ in X,

$$\lim_{n\to\infty} \lim_{m\to\infty} ||u_m + v_n|| = \lim_{m\to\infty} \lim_{n\to\infty} ||u_m + v_n||$$

whenever either side exists.

Theorem 2.2. *Let X be a separable stable Banach space with property (M) and suppose that X contains no copy of l_1. Then X has WNUS.*

Proof. Since a stable Banach space is weakly sequentially complete (see [12]) and X contains no copy of l_1, we conclude that X is reflexive. By Theorem 3.10 in [9], there exists $p \in (1, \infty)$ such that X has property (M_p). Using Theorem 2.1, we obtain that $R(X) < 2$. Hence X has WNUS. $\square$

Corollary 2.1. *Let X be a separable stable Banach space with property (M) and suppose that X contains no copy of l_1. Then X has the fixed point property.*

Theorem 2.3. *If X is a weakly stable Banach space with strict property (M) and contains no copy of l_1. then X has the uniform Kadec–Klee property.*

Proof. Suppose that X fails to have the uniform Kadec–Klee property. Then there exists an $\varepsilon_0 > 0$ such that for any $0 < \delta < 1 - (1 - \varepsilon_0^p)^{\frac{1}{p}}$ there exists a sequence $\{x_n\} \subset S(X)$ with $sep(\{x_n\}) \geq \varepsilon_0$ and $x \in X$ such that $x_n \xrightarrow{w} x$ and $\|x\| \geq 1 - \delta$.

It is clear that the sequence $\{x_n - x\}$ is weakly null and $sep(\{x_n - x\}) \geq \varepsilon_0$. By the Bessaga–Pelczynski selection principle we may, without loss of generality, assume that the sequence is a basic sequence. Put $X_0 = \overline{span}(\{x_n - x\})$. Then X_0 is separable, contains no copy of l_1 and has strict property (M). Using Proposition 3.9 in [9], there exists $p \in (1, \infty]$ and a normalized weakly null sequence $\{z_n\}$ such that for every $u \in X_0$ and every $\alpha \in \mathbf{R}$,

$$\lim_{n\to\infty} \|u + \alpha z_n\|^p = \|u\|^p + |\alpha|^p$$

when $p < \infty$ or

$$\lim_{n\to\infty} \|u + \alpha z_n\| = \max\{\|u\|, |\alpha|\}$$

when $p = \infty$.

Since X_0 has strict property (M), we have $1 < p < \infty$.

Now let $(\omega_n) \subset X_0$ be any normalized weakly null sequence generating a type. That is, $\lim_{n\to\infty} \|u + \omega_n\|$ exits for all $n \in N$. Then we can define $\gamma(\alpha, \beta) = \lim_{n\to\infty} \|\alpha u + \beta \omega_n\|$, where $u \in S(X_0)$. Then by weak stability, if $\alpha, \beta \in \mathbf{R}$,

$$\gamma(\alpha, \beta) = \lim_{n\to\infty} \lim_{m\to\infty} \|\alpha z_n + \beta_n \omega_n\|$$

$$= \lim_{m\to\infty} \lim_{n\to\infty} \|\alpha z_n + \beta_n \omega_n\| = (|\alpha|^p + |\beta|^p)^{\frac{1}{p}}.$$

Hence

$$1 = \|x_n\| = \|x + x_n - x\| = \gamma(\|x\|, \|x_n - x\|)$$

$$= (\|x\|^p + \|x_n - x\|^p)^{\frac{1}{p}} \geq ((1 - \delta)^p + \varepsilon^p)^{\frac{1}{p}} > 1,$$

a contradiction. $\square$

Theorem 2.4. *Let X be a separable Banach space and X^* have property (M^*). Then X has strict property (M).*

Proof. By Proposition 2.3 in [9], we get that X. Next, we will prove that X has strict property (M). For any $u, v \in X$ with $\|u\| > \|v\|$ and every weakly null sequence $(x_n) \subset X$, we pick up $x_n^* \in S(X^*)$ such that $\langle v + x_n, x^* \rangle = \|v + x_n\|$ for all $n \in N$. By passing to a subsequence we may suppose that $x_n^* \xrightarrow{w^*} x^*$. Take $u^* \in S(X^*)$ such that $< u, u^* >= \|u\| > \|v\| \geq \left\langle v, \frac{x^*}{\|x^*\|} \right\rangle$ and put $\omega^* = \|x^*\| u^*$. Then

$$\limsup_{n\to\infty} \|v + x_n\| = \limsup_{n\to\infty} \langle v + x_n, x^* \rangle$$

$$= \limsup_{n\to\infty} \left(\langle v, x^* \rangle + \langle x_n, x_n^* - x^* \rangle \right)$$

$$< \limsup_{n\to\infty} \left(\langle u, \omega^* \rangle + \langle x_n, x_n^* - x^* \rangle \right)$$

$$= \limsup_{n\to\infty} \langle u + x_n, \omega^* + x_n^* - x^* \rangle$$

$$\leq \limsup_{n\to\infty} \|u + x_n\| \, \|\omega^* + x_n^* - x^*\|$$

$$= \limsup_{n\to\infty} \|u + x_n\| \, \|x^* + x_n^* - x^*\|$$

since $\limsup_{n\to\infty} \|\omega^* + x_n^* - x^*\| = \limsup_{n\to\infty} \|x^* + x_n^* - x^*\| = 1$ by property (M^*). Hence

$$\limsup_{n\to\infty} \|v + x_n\| < \limsup_{n\to\infty} \|u + x_n\| .$$

$\square$

Corollary 2.2. *A reflexive Banach space X has strict property (M) if and only if it has property (M).*

Theorem 2.5. *For the following conditions on the Banach space X we have*

$$(1) \Rightarrow (2) \Rightarrow (3).$$

(1) X has strict property (M).

(2) If $x_n \xrightarrow{w} 0$ then for each $x \in X$ we have $\psi_{(x_n)}(tx)$ is a strictly increasing function of t on $[0, \infty)$.

(3) X satisfies the Opial condition.

Proof. The proof is similar to the proof of proposition 2.1 in [6]. $\square$

Let X be a Köthe sequence space. We define a new property, namely *weak property* (M) in X as follows: if $x_n \xrightarrow{w} 0$ and $u, v \in X$ with $|u(i)| \leq |v(i)|$ then $\limsup_{n\to\infty} \|v + x_n\| \leq \limsup_{n\to\infty} \|u + x_n\|$.

Theorem 2.6. *Let X be a Köthe sequence space with the Fatou property. Then X has weak property (M) if and only if X has absolutely continuous norm.*

Proof. Necessity. Suppose that X does not have an absolutely continuous norm. Then there exists $\varepsilon_0 > 0$ and $x_0 \in S(X)$ such that

$$\left\| \sum_{i=n+1}^{\infty} x_0(i) e_i \right\| \geq \varepsilon_0$$

for any $n \in N$, where $e_i = (0, 0, \cdots, \overset{ith}{1}, 0, \cdots)$.

Take $n = 0$. Since X has the Fatou property, there is $n_1 \in \mathrm{N}$ such that

$$\left\| \sum_{i=1}^{n_1} x_0(i) e_i \right\| \geq \frac{3\varepsilon_0}{4}.$$

Notice that

$$\lim_{m\to\infty} \left\| \sum_{i=n_1+1}^{m} x_0(i) e_i \right\| \geq \varepsilon_0,$$

so there exists $n_2 > n_1$ such that

$$\left\| \sum_{i=n_1+1}^{n_2} x_0(i) e_i \right\| \geq \frac{3\varepsilon_0}{4}.$$

In this way, we get a sequence $\{n_i\}$ of natural numbers such that

$$\left\| \sum_{j=n_i+1}^{n_{i+1}} x_0(i) e_i \right\| \geq \frac{3\varepsilon_0}{4}, \ i = 1, 2, \cdots.$$

Put $x_i = \sum_{j=n_i+1}^{n_{i+1}} x_0(i) e_i$. Then

(1) $\|x_i\| \geq \frac{3\varepsilon_0}{4}$ for all $i \in \mathrm{N}$;

(2) $x_i \xrightarrow{w} 0$ as $i \to \infty$. It is well known that for any Köthe space X the dual space X^* is isometric to $X' \oplus S$, where S is the space of all singular functionals over X , i.e., functionals which vanish on the subspace $X_a = \{x \in X :$ the norm is absolutely continuous at $x\}$ and $X' = \{y \in l^0 : \sum_{i=1}^{\infty} x(i)y(i) < \infty$ for all $x \in X\}$. This means that every $f \in X^*$ is uniquely represented in the form

$$f = T_y + \varphi,$$

where $\varphi \in S$ and for $y \in X'$ the function T_y is defined by

$$T_y(x) = \sum_{i=1}^{\infty} x(i)y(i)$$

for all $x \in X$.

Consider $\sum\limits_{i=1}^{\infty} x(i)y(i) < \infty$. We have

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} x_n(j)y(j) = \lim_{i\to\infty} \sum_{j=n_i+1}^{n_{i+1}} x_i(j)y(j) = 0.$$

Take $i_0 \in N$ large enough so that $\left\|\sum\limits_{i=i_0+1}^{\infty} x_0(i)e_i\right\| \le \frac{4\varepsilon_0}{3}$. Put $z_0 = \sum\limits_{i=i_0+1}^{\infty} x_0(i)e_i$ and $z_i = -2x_i$ for all $i \in N$. Then

(3) $\|z_i + z_0\| = \|z_0\| \le \frac{4\varepsilon_0}{3}$ for any i large enough and $\|z_i\| = 2\|x_i\| \ge \frac{3}{2}\varepsilon_0$. This contradicts X having the weak property (M).

Sufficiency. Let $\varepsilon > 0$ be given. For any $u, v \in X$ with $|u(i)| \le |v(i)|$, there exists $i_0 \in \mathrm{N}$ such that $\left\|\sum\limits_{i=i_0+1}^{\infty} v(i)e_i\right\| < \epsilon$. For every weakly null sequence $\{x_n\} \subset X$ there exists $n_0 \in \mathrm{N}$ such that $\left\|\sum\limits_{i=i_0+1}^{\infty} v(i)e_i\right\| < \epsilon$. Hence

$$\|x_n + u\| = \left\|\sum_{i=1}^{\infty} (x_n(i) + u(i))\, e_i\right\|$$

$$\le \left\|\sum_{i=1}^{i_0} u(i)e_i + \sum_{i=i_0+1}^{\infty} x_n(i)e_i\right\| + 2\epsilon$$

$$\le \left\|\sum_{i=1}^{i_0} v(i)e_i + \sum_{i=i_0+1}^{\infty} x_n(i)e_i\right\| + 2\epsilon$$

$$\le \left\|\sum_{i=1}^{\infty} (x_n(i) + v(i))\, e_i\right\| + 4\epsilon = \|x_n + u\| + 4\epsilon.$$

By the arbitrariness of ϵ, we get that $\limsup\limits_{n\to\infty} \|x_n + u\| \le \limsup\limits_{n\to\infty} \|x_n + v\|$. □

Corollary 2.3. *Let X be a Köthe sequence space with the Fatou property. If X has strict property (M), then X has absolutely continuous norm.*

Lemma 2.1. *Let $y, z \in ces_p$. Then for any $\varepsilon > 0$ and $L > 0$, there exists $\delta > 0$ such that*

$$\left| \|y+z\|^p - \|y\|^p \right| < \varepsilon$$

whenever $\|y\|^p \leq L$ *and* $\|z\|^p \leq \delta$ *(see [1]).*

Theorem 2.7. *The Cesaro sequence spaces have property (M_p) $(1 < p < \infty)$.*

Proof. Let $\{x_n\}$ be a weak null sequence and let $x \in X$. Given $\varepsilon > 0$. Take $r = \max\left\{ \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p, \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p : n \in \mathcal{N} \right\}$. By lemma 2.1, there exists $\delta \in (0, \epsilon)$ such that

$$\left| \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i) + y(i)| \right)^p - \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p \right| < \epsilon,$$

whenever $\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p < L$ and $\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |y(i)| \right)^p \leq \delta$.

Take $i_0 \in \mathrm{N}$ such that $\sum_{n=i_0+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p < \delta$. Since $x_n \xrightarrow{w} 0$, there exists $m_0 \in \mathrm{N}$ such that $\sum_{n=1}^{i_0} \left(\frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p < \delta$ whenever $m > m_0$. Hence

$$\|x_m + x\|^p = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x_m(i) + x(i)| \right)^p$$

$$= \sum_{n=1}^{i_0} \left(\frac{1}{n} \sum_{i=1}^{n} |x_m(i) + x(i)| \right)^p + \sum_{n=i_0+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x_m(i) + x(i)| \right)^p$$

$$\leq \sum_{n=1}^{i_0} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p + \sum_{n=i_0+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p + 2\epsilon$$

$$\leq \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p + \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p + 4\epsilon.$$

In same a way, we can get

$$\|x_m - x\|^p \geq \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p + \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p - 4\epsilon.$$

By the arbitrariness of ϵ, we have

$$\limsup_{m\to\infty} \|x_m - x\|^p = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p + \limsup_{m\to\infty} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p,$$

i.e.,

$$\limsup_{m\to\infty} \|x_m - x\|^p = \|x\|^p + \limsup_{m\to\infty} \|x_m\|^p .$$

The proof is complete. $\square$

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