OSCILLATIONS IN PARABOLIC NEUTRAL SYSTEMS*

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Abstract

Sufficient conditions are obtained for the oscillation of all solutions of the homogeneous Neumann and Dirichlet boundary value problems associated with the neutral parabolic system

$$\frac{\partial}{\partial t}\left[u_i(x,t) - c_i u_i(x,t-\tau)\right] - D_i \nabla^2 u_i(x,t) + \sum_{j=1}^{m} a_{ij} u_j(x,t-\sigma_j) = 0$$

for $i = 1, 2, \cdots, m$; $x \in \Omega \subset R^n$, $t > 0$ where ∇^2 denotes the Laplacian in R^n.

1. Introduction

There has been increased activity in the investigation of oscillations of solutions of lumped parameter scalar neutral delay differential equations. It is known that a necessary and sufficient condition for oscillation of all solutions of the scalar neutral delay differential equations of the type

$$\frac{d}{dt}\left[u(t) - cu(t-\tau)\right] + au(t-\sigma) = 0 \tag{1.1}$$

is that the associated characteristic equation

$$\lambda(1 - ce^{-\lambda\tau}) + ae^{-\lambda\sigma} = 0 \tag{1.2}$$

has no real roots; for more details of this and several related extensions we refer to Grammatikopoulos et al.[6-9], Kulenovic et al. [14], Stavroulakis [17], Jiong [12], Gopalsamy and Zhang [5]. Quite recently Arino and Gyori[1] have established that a necessary and sufficient condition for the oscillation of the vector matrix system

$$\frac{d}{dt}\left[y(t) - \sum_{j=1}^{m} B_j y(t-\sigma_j)\right] = \sum_{j=1}^{m} A_j y(t-\tau_j) \tag{1.3}$$

(A_j and B_j are $n \times n$ matrices, σ_j, τ_j are nonnegative numbers and y is an n-vector) is that the associated characteristic equation

$$\det\left[\lambda\left(I - \sum_{j=1}^{m} B_j e^{-\lambda\sigma_j}\right) - \sum_{j=1}^{m} A_j e^{-\lambda\tau_j}\right] = 0 \tag{1.4}$$

has no real roots. In applications it is often desirable to obtain sufficient conditions in terms of the parameters (coefficients, delays etc.) of the equations themselves and this involves further analysis of the characteristic equations such as (1.4). In fact it is a nontrivial task to obtain sufficient conditions for (1.4) to have or not to have real roots when $m > 1$. A special case of (1.3) has been considered by Gyori and Ladas [11] in the form

$$\frac{d}{dt}\left[y_i(t) - p_i y_i(t-\tau)\right] + \sum_{k=1}^{m}\left[\sum_{j=1}^{n} q_{ij}^{(k)} y_j(t-\sigma_k)\right] = 0 \tag{1.5}$$

for $i = 1, 2, \cdots, n$ and easily verifiable sufficient conditions for the oscillation of the system (1.5) have been obtained.

The purpose of this article is to derive sufficient conditions for all solutions of the linear systems of the type

$$\frac{\partial}{\partial t}[u_i(x,t) - c_i u_i(x,t-\tau)] + \sum_{j=1}^{m} a_{ij}(x,t)u_j(x,t-\sigma_j) = D_i \nabla^2 u_i(x,t) \qquad 1.6$$
$$i = 1,2,\cdots,m; \; x \in \Omega \subset R^n$$

where D_i, c_i, τ, σ_i are nonnegative numbers and ∇^2 is the Laplacian in R^n; Ω is a bounded domain with a smooth boundary $\partial\Omega$. For an analysis of the stability characteristics of neutral parabolic and hyperbolic equations we refer to Datko [3].

Oscillations of scalar parabolic equations have been considered before by Kreith and Ladas [13] and Yoshida [18]. However nonscalar parabolic delay systems have not been hitherto considered in the literature on oscillation of delay systems. We shall use the following definition:

Definition. *A scalar valued function* $w : (\Omega \cup \partial\Omega) \times [0,\infty) \mapsto R$ *is called oscillatory if for each positive number* T *there exists a point* $(x_o, T_o) \in \Omega \times [T,\infty)$ *such that* $w(x_o, T_o) = 0$; *a vector valued function,* $\vec{u} : \Omega \times [0,\infty) \mapsto R^n$ *is said to be oscillatory if at least one component of* $\vec{u}$ *is oscillatory in the sense of oscillation of a scalar function. A vector* $\vec{u} : \Omega \times [0,\infty) \mapsto R^n$ *is said to be nonoscillatory if each of its components is nonoscillatory.*

We remark that the above definition is one of the several possible ways of generalising the oscillations of a scalar valued function of one variable.

2. Neumann neutral parabolic system

We consider the following initial boundary value problem

$$\left.\begin{array}{l} \dfrac{\partial}{\partial t}[u_i(x,t) - c_i u_i(x,t-\tau)] + \displaystyle\sum_{j=1}^{m} a_{ij}(x,t)u_j(x,t-\sigma_j) = D_i \nabla^2 u_i(x,t); \\ \qquad\qquad\qquad\qquad x \in \Omega;\; t > 0 \\ \qquad \dfrac{\partial}{\partial n} u_i(x,t) = 0 \quad \text{on} \quad \partial\Omega \\ u_i(x,s) = \phi_i(x,s), \quad x \in \Omega \cup \partial\Omega;\; s \in [-(\sigma^* + \tau), 0],\; i = 1,2,\cdots,m \end{array}\right\} \qquad 2.1$$

where $\sigma^* = \max \sigma_1, \sigma_2, \cdots, \sigma_m$, $\frac{\partial}{\partial n}$ denotes the outward normal derivative. We recall that if

$$u_i : \Omega \times [0, \infty) \mapsto R$$

is nonoscillatory then there exists a $T_o > 0$ such that

$$\left.\begin{array}{lll} v_i(t) = \int_\Omega u_i(x,t)\,dx \neq 0 & \text{for} & t > T_o\,,\, i = 1, 2, \cdots, m \\ |u_i(x,t)| > 0 & \text{for} & x \in \Omega\,,\, t > T_o. \end{array}\right\} \qquad 2.2$$

Also we note from Green's formula and the homogeneous Neumann boundary condition

$$\int_\Omega \nabla u_i(x,t)\,dx = \int_{\partial\Omega} \frac{\partial u_i}{\partial n}\,dS = 0; \qquad i = 1, 2, \cdots, m. \qquad 2.3$$

Theorem 2.1. *Suppose the following are satisfied;*

1. $c_i \in [0, 1)$; $\tau, \sigma_i \in [0, \infty)$; $i = 1, 2, \cdots, m$.

2. a_{ij} $(i, j = 1, 2, \cdots, m)$ *are bounded continuous functions on* $\Omega \times [0, \infty)$ *such that*

$$\min_{1 \le i \le m} \left[\alpha_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{m} \beta_{ji} \right] > 0 \qquad 2.4$$

where

$$\alpha_{ii} = \inf_{\substack{x \in \Omega \\ t \ge 0}} \left\{ a_{ii}(x,t) \right\}; \qquad \beta_{ji} = \sup_{\substack{x \in \Omega \\ t \ge 0}} \left\{ |a_{ji}(x,t)| \right\}. \qquad 2.5$$

3. *at least one of*

$$\textit{either} \quad c_o e^{\delta\tau} + \frac{\mu e^{\delta\sigma_o}}{\delta} > 1 \quad \textit{for} \quad \delta \in (0, \infty) \quad \textit{or} \quad \mu\sigma_o > \frac{1 - c_o}{e} \qquad 2.6$$

holds where

$$\left.\begin{array}{l} c_o = \min\{c_1, c_2, \cdots, c_m\}, \qquad \sigma_o = \min\{\sigma_1, \sigma_2, \cdots, \sigma_m\} \\ \mu = \min\limits_{1 \le i \le m} \left\{ \alpha_{ii} - \sum\limits_{\substack{j=1 \\ j \neq i}}^{m} \beta_{ji} \right\}. \end{array}\right\} \qquad 2.7$$

Then all nontrivial solutions of (2.1) are oscillatory.

Proof. Our plan of proof is to suppose that there exists a nonoscillatory solution of the system (2.1) and then show that this leads to a contradiction. Accordingly let us suppose that (2.1) has a nonoscillatory solution. This implies that there exists a $T_o > 0$ such that

$$|u_i(x,t)| > 0 \qquad \text{for} \qquad x \in \Omega; \quad t > T_o, \quad i = 1, 2, \cdots, m \tag{2.8}$$

and if we let

$$v_i(t) = \int_\Omega |u_i(x,t)|\,dx, \qquad i = 1, 2, \cdots, m \tag{2.9}$$

then for $t > T = T_o + \sigma^* + \tau$

$$\left.\begin{aligned}\frac{d}{dt}\Big[v_i(t) - c_i v_i(t-\tau)\Big] &+ \int_\Omega a_{ii}(x,t)|u_i(x,t-\sigma_i)|\,dx \\ &\le \int_\Omega \sum_{\substack{j=1\\ j\neq i}}^{m} |a_{ij}(x,t)\,|u_j(x,t-\sigma_j)|\,dx \\ &\quad + |D_i \int_\Omega \nabla^2 u_i(x,t)\,dx\,|; \quad i = 1, 2, \cdots, m.\end{aligned}\right\} \tag{2.10}$$

Adding all the inequalities in (2.10) and simplifying (using (2.3)),

$$\begin{aligned}\frac{d}{dt}\Big[\sum_{i=1}^{m} v_i(t) - \sum_{i=1}^{m} c_i v_i(t-\tau)\Big] \\ + \int_\Omega \sum_{i=1}^{m}\Big[a_{ii}(x,t) - \sum_{\substack{j=1\\ j\neq i}}^{m} |a_{ji}(x,t)|\Big]|u_i(x,t-\sigma_i)|\,dx \\ \le 0; \qquad i = 1, 2, \cdots, m\end{aligned} \tag{2.11}$$

and therefore

$$\frac{d}{dt}\Big[\sum_{i=1}^{m} v_i(t) - \sum_{i=1}^{m} c_i v_i(t-\tau)\Big] + \mu \sum_{i=1}^{m} v_i(t-\sigma_i) \le 0. \tag{2.12}$$

We shall now show that $v_i\,(i = 1, 2, \cdots, m)$ remain uniformly bounded for $t \ge 0$. An integration of (2.12) on $[T, t)$ leads to

$$\begin{aligned}\sum_{i=1}^{m} v_i(t) - \sum_{i=1}^{m} c_i v_i(t-\tau) + \mu \int_T^t \sum_{i=1}^{m} v_i(s-\sigma_i)\,ds \\ \le \sum_{i=1}^{m} v_i(T) + \sum_{i=1}^{m} c_i |v_i(T-\tau)|.\end{aligned} \tag{2.13}$$

For convenience we let

$$\alpha = \sum_{i=1}^{m} v_i(T) + \sum_{i=1}^{m} c_i |v_i(T-\tau)|. \tag{2.14}$$

From (2.13) and (2.14),

$$\begin{aligned}\sum_{i=1}^{m} v_i(t) &\leq \alpha + \sum_{i=1}^{m} c_i v_i(t-\tau) \\ &\leq \alpha + c\sum_{i=1}^{m} v_i(t-\tau); \qquad c = \max\{c_1, c_2, \cdots, c_m\}.\end{aligned} \tag{2.15}$$

Now if we let

$$V(t) = \sum_{i=1}^{m} v_i(t) \tag{2.16}$$

then we have from (2.15),

$$V(t) \leq \alpha + cV(t-\tau); \qquad t > T.$$

We now let

$$\rho(t) = \sup_{s\in[-\tau,t]} V(s)$$

and derive that

$$\rho(t) \leq \alpha + c\rho(t) + \max_{s\in[-\tau,T]} V(s)$$

from which it will follow that

$$\rho(t) \leq \frac{\left(\alpha + \max_{s\in[-\tau,T]} V(s)\right)}{1-c}. \tag{2.17}$$

The uniform boundedness of V and hence that of v_i, $i = 1, 2, \cdots, m$ follows from (2.17); that is

$$\int_\Omega |u_i(x,t)|\, dx \qquad \text{is bounded for} \qquad t \geq 0\,,\, i = 1, 2, \cdots, m. \tag{2.18}$$

We next show that $\dot{v}_i$, $i = 1, 2, \cdots, m$ remain bounded for all $t \geq 0$. We note from (2.10) that

$$\dot{v}_i(t) - c_i\dot{v}_i(t-\tau) \leq \sum_{\substack{j=1\\ j\neq i}}^{m} \beta_{ij} v_j(t-\sigma_j). \tag{2.19}$$

We let

$$p_i(t) = \sup_{s\in(0,t)} \dot{v}_i(s)$$

and derive from (2.19) that

$$p_i(t) \le cp_i(t) + \sup_{s\in[-\tau,t]} \sum_{\substack{j=1 \\ j\neq i}}^{m} \beta_{ij} v_j(s) \qquad t > 0$$

which implies that

$$p_i(t) \le \left(\frac{1}{1-c}\right)\left[\sup_{s\in[-\tau,t]} \sum_{\substack{j=1 \\ j\neq i}}^{m} \beta_{ij} v_j(s)\right], \qquad t > 0. \tag{2.20}$$

From the uniform boundedness of $v_j\,(j = 1, 2, \cdots, m)$ follows that of p_i, $i = 1, 2, \cdots, m$; the uniform boundedness of $\dot{v}_i$, $i = 1, 2, \cdots, m$ is now immediate. We note from (2.13) that

$$\mu \int_T^t v_i(s-\sigma_i)\, ds \le \mu \int_T^t \sum_{i=1}^{m} v_i(s-\sigma_i)\, ds \le \alpha + \sum_{i=1}^{m} c_i v_i(t-\tau) < \infty \tag{2.21}$$

and therefore $v_i \in L_1[T, \infty)$. From the uniform boundedness of $\dot{v}_i$, follows the uniform continuity of v_i, $i = 1, 2, \cdots, m$. By Barbalat's lemma (see Corduneanu [2]) we can conclude that

$$\lim_{t\to\infty} v_i(t) = 0; \qquad i = 1, 2, \cdots, m. \tag{2.22}$$

Using (2.22) and integrating both sides of (2.12) on (t, ∞),

$$-\left[\sum_{i=1}^{m} v_i(t) - \sum_{i=1}^{m} c_i v_i(t-\tau)\right] + \mu \int_t^\infty \sum_{i=1}^{m} v_i(s-\sigma_i)\, ds \le 0$$

and so

$$\sum_{i=1}^{m} v_i(t) \ge \sum_{i=1}^{m} c_i v_i(t-\tau) + \mu \int_t^\infty \sum_{i=1}^{m} v_i(s-\sigma_i)\, ds \tag{2.23}$$

$$\ge c_o \sum_{i=1}^{m} v_i(t-\tau) + \mu \int_{t-\sigma_o}^\infty \sum_{i=1}^{m} v_i(s)\, ds. \tag{2.24}$$

We now let

$$z(t) = \sum_{i=1}^{m} v_i(t)$$

and derive from (2.24),

$$z(t) \geq c_o z(t-\tau) + \mu \int_{t-\sigma_o}^{\infty} z(s)\, ds. \tag{2.25}$$

Define a sequence $\{z_j\}$ as follows:

$$\left.\begin{aligned} z_o(t) &= z(t) \\ z_{j+1} &= \begin{cases} c_o z_j(t-\tau) + \mu \int_{t-\sigma_o}^{\infty} z_j(s)\, ds; & t > T \\ z(t) - z(T) + c_o z_j(T-\tau) + \mu \int_{T-\sigma_o}^{\infty} z_j(s)\, ds; & t \leq T. \end{cases} \end{aligned}\right\} \tag{2.26}$$

One can see from (2.25) and (2.26) that

$$0 \leq \cdots \leq z_{j+1}(t) \leq z_j(t) \leq z_{j-1}(t) \leq \cdots \leq z_2(t) \leq z_1(t) \leq z_o(t)$$

for $t > T$ and therefore the sequence $\{z_j\}$ converges as $j \to \infty$ to a limit say z^* on $[T, \infty)$; such a limit also satisfies

$$z^*(t) = \begin{cases} c_o z^*(t-\tau) + \mu \int_{t-\sigma_o}^{\infty} z^*(s)\, ds; & t > T \\ z(t) - z(T) + c_o z^*(t-\tau) + \mu \int_{T-\sigma_o}^{\infty} z^*(s)\, ds; & t \leq T. \end{cases} \tag{2.27}$$

We can verify that z^* is an eventually positive solution of (2.27) as follows;

$$\begin{aligned} z^*(t) &> c_o z^*(t-\tau) \\ &> c_o^2 z^*(t-2\tau) \\ &\cdots\cdot \\ &> c_o^k z^*(t-k\tau) \\ &= z^*(t_o) \exp\left[\left(\frac{t-t_o}{\tau}\right) \ell n[c_o]\right] \\ &= \alpha e^{-\beta t} \qquad \text{(say)} \end{aligned}$$

for some $\alpha > 0$ and $\beta > 0$ since z^* has the tail left end of z. We also have from (2.27),

$$\begin{aligned} z^*(t) &= c_o z^*(t-\tau) + \mu \int_{t-\sigma_o}^{\infty} z^*(s)\, ds \\ &> c_o \alpha e^{-\beta(t-\tau)} + \mu \int_{t-\sigma_o}^{\infty} \alpha e^{-\beta s}\, ds \\ &= \left(c_o e^{\beta\tau} + \frac{\mu e^{\beta\sigma_o}}{\beta}\right) \alpha e^{-\beta t} > \alpha e^{-\beta t} \end{aligned}$$

if either

$$c_o e^{\beta\tau} + \frac{\mu e^{\beta\sigma_o}}{\beta} > 1 \qquad \text{or} \qquad c_o + \mu\sigma_o e > 1.$$

Thus (2.27) has a bounded nonoscillatory solution and hence its characteristic equation has a real nonpositive root since this is a necessary condtion. The characteristic equation associated with (2.27) is

$$\lambda(1 - c_o e^{-\lambda\tau}) + \mu e^{-\lambda\sigma_o} = 0. \tag{2.28}$$

It is easy to see that $\lambda = 0$ is not a root and so we can let $\lambda = -q$ for some $q > 0$. Then

$$q = q c_o e^{q\tau} + \mu e^{q\sigma_o}$$

and therefore

$$1 = c_o e^{q\tau} + \mu \frac{e^{q\sigma_o}}{q} \geq c_o + \mu e \sigma_o. \tag{2.29}$$

But (2.29) contradicts our hypothesis (3) and hence the result follows. This completes the proof.

3. Dirichlet neutral parabolic system

We consider the oscillations of the following neutral parabolic system with homogeneous Dirichlet boundary conditions;

$$\left.\begin{aligned} \frac{\partial}{\partial t}\Big[u_i(x,t) - c_i u_i(x,t-\tau)\Big] &+ \sum_{j=1}^{m} a_{ij}(x,t) u_j(s, t-\sigma_j) \\ &= \sum_{j=1}^{m} D_{ij} \nabla^2 u_j(x,t); \quad x \in \Omega;\ t > 0 \\ u_i(x,s) = 0, &\qquad x \in \partial\Omega, \quad t \in [-(\sigma^* + \tau), 0] \\ u_i(x,s) = \phi_i(x,s), &\qquad s \in [-(\sigma^*+\tau), 0], \quad x \in \Omega \cup \partial\Omega \\ i = 1, 2, \cdots, m; &\qquad \sigma^* = \max(\sigma_1, \sigma_2, .., \sigma_m) \end{aligned}\right\} \tag{3.1}$$

where ϕ_i, $i = 1, 2, .., m$ are continuous functions defined on the domain specified above. We use the following well known property of the Laplacian operator with homogeneous Dirichlet boundary condition; *there exists a positive number say δ and a strictly positive* *function* $\psi : \Omega \mapsto (0,\infty)$ *such that*

$$\left.\begin{aligned} \nabla\psi(x) + \delta\psi(x) = 0\,; \qquad & x \in \Omega \\ \psi(x) = 0\,; \qquad & x \in \partial\Omega \\ \psi(x) > 0\,; \qquad & x \in \Omega. \end{aligned}\right\} \qquad 3.2$$

If $\{u_1, u_2, \cdots, u_m\}$ is a nonoscillatory solution of (3.1) then the vector $\{u_1\psi, u_2\psi, \cdots, u_m\psi\}$ is also nonoscillatory such that

$$sign\{u_j(x,t)\} = sign\{u_j(x,t)\psi(x)\}; \qquad j = 1,2,\cdots,m;\ t > T$$

for some positive T.

Theorem 3.1. *Assume that the following hold:*

1. $c_i \in [0,1)$; $\tau \in [0,\infty)$; $\sigma_i \in (0,\infty)$; $D_{ii} \in [0,\infty)$; $D_{ij} \in (-\infty,\infty)$; $i,j = 1,2,\cdots,m$.
2. a_{ij}, $i,j = 1,2,\cdots,m$ *are bounded continuous functions such that*

$$\alpha_{ii} = \inf_{\substack{x\in\Omega \\ t\geq 0}} \left\{a_{ii}(x,t)\right\}; \qquad \beta_{ij} = \sup_{\substack{x\in\Omega \\ t\geq 0}} \left\{|a_{ij}(x,t)|\right\}; \quad i,j = 1,2,\cdots,m.$$

3. *either*

$$c_o e^{q\tau} e^{\lambda\tau} + c_o q e^{q\tau}\left(\frac{e^{\lambda\tau}}{\lambda}\right) + \mu e^{q\sigma_o}\left(\frac{e^{\lambda\sigma_o}}{\lambda}\right) > 1 \quad \textit{for} \quad \lambda \in (0,\infty) \qquad 3.3$$

or

$$\left(\mu\sigma_o e^{q\sigma_o} + c_o q e^{q\tau}\tau\right) > \frac{1 - c_o e^{q\tau}}{e} \qquad 3.4$$

holds where

$$\mu = \min_{1\leq i\leq m}\left[\alpha_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{m} \beta_{ji}\right] > 0; \qquad 3.5$$

$$q = \min_{1\leq i\leq m}\left[D_{ii} - \sum_{\substack{j=1 \\ j\neq i}} |D_{ji}|\right]\delta > 0 \qquad 3.6$$

$$c_o = \min(c_1, c_2, \cdots, c_m), \qquad \sigma_o = \min(\sigma_1, \sigma_2, \cdots, \sigma_m).$$

Then all nontrivial solutions of (3.1) are oscillatory.

Proof. Details of proof need only some minor modifications of those of Theorem 3.1 and we shall be brief. Suppose $\{u_1, u_2, \cdots, u_m\}$ is a nonoscillatory solution of (3.1). We multiply both sides of the partial differential equation (3.1) by $\psi(x)$ and integrate both sides of the resulting equations over the domain Ω; we then define v_i such that

$$v_i(t) = \int_\Omega u_i(x,t)\psi(x)\,dx, \qquad i = 1, 2, \cdots, m \tag{3.7}$$

and note that there exists a $T > 0$ such that $|v_i(t)| > 0$ for $t > T$ and $i = 1, 2, \cdots, m$ due to the nonoscillatory nature of $\{u_1, u_2, \cdots, u_m\}$. We derive from (3.1) and (3.7),

$$\begin{aligned} \frac{d}{dt}\Big[v_i(t) - c_i v_i(t-\tau)\Big] + \Big[\alpha_{ii} v_i(t-\sigma_i) - \sum_{\substack{j=1 \\ j\neq i}}^{m} \beta_{ij} v_j(t-\sigma_j)\Big] \\ \leq \sum_{j=1}^{m} D_{ij} \int_\Omega \nabla^2 u_j(x,t)\psi(x)\,dx \\ \leq -D_{ii}\delta v_i(t) + \delta \sum_{\substack{j=1 \\ j\neq i}}^{m} |D_{ij}| v_j(t). \end{aligned} \tag{3.8}$$

In the derivation of (3.8) we have used Green's formula and (3.2) so as to obtain

$$\begin{aligned} \int_\Omega \nabla^2 u_j(x,t)\psi(x)\,dx &= \int_\Omega u_j(x,t)\,\nabla^2\,\psi(x)\,dx \\ &= -\delta \int_\Omega u_j(x,t)\psi(x)\,dx \\ &= -\delta v_j(t); \qquad j = 1, 2, \cdots, m. \end{aligned} \tag{3.9}$$

Adding all the inequalities in (3.8) and simplifying further,

$$\frac{d}{dt}\Big[\sum_{j=1}^{m} v_i(t) - \sum_{j=1}^{m} c_i v_i(t-\tau)\Big] + \mu \sum_{i=1}^{m} v_i(t-\sigma_i) + q \sum_{i=1}^{m} v_i(t) \leq 0. \tag{3.10}$$

As in the case of the proof of Theorem 2.1, one can show that (3.10) leads to an inequality of the form

$$w(t) \geq c_o w(t-\tau) + \mu \int_{t-\sigma_o}^{\infty} w(s)\,ds + q \int_t^{\infty} w(s)\,ds \tag{3.11}$$

where

$$w(t) = \sum_{i=1}^{m} v_i(t) \qquad \text{for} \qquad t > T + \sigma^* + \tau;$$

one can show (as in the proof of Theorem 2.1) that there exists an eventually positive bounded solution of the scalar neutral equation

$$\frac{d}{dt}\left[w(t) - c_o w(t-\tau)\right] + \mu w(t-\sigma_o) + q w(t) = 0. \qquad 3.12$$

We let

$$U(t) = w(t)e^{-qt} \qquad 3.13$$

and note that U is an eventually positive bounded nonoscillatory solution of

$$\frac{d}{dt}\left[U(t) - c_o e^{q\tau} U(t-\tau)\right] + c_o e^{q\tau} q U(t-\tau) + \mu e^{q\sigma_o} U(t-\sigma_o) = 0. \qquad 3.14$$

A necessary condition for (3.14) to have a bounded nonoscillatory solution is that the associated characteristic equation

$$z\left[1 - c_o e^{q\tau} e^{-z\tau}\right] + c_o e^{q\tau} q e^{-z\tau} + \mu e^{q\sigma_o} e^{-z\sigma_o} = 0 \qquad 3.15$$

has a nonpositive root; since $z = 0$ is not a root, we let $z = -\lambda$ for some $\lambda > 0$ and derive from (3.15),

$$1 = c_o e^{q\tau} e^{\lambda\tau} + c_o e^{q\tau} q\left(\frac{e^{\lambda\tau}}{\lambda}\right) + \mu e^{q\sigma_o}\left(\frac{e^{\lambda\sigma_o}}{\lambda}\right) \qquad 3.16$$

$$\geq c_o e^{q\tau} + \left(c_o e^{q\tau} q\tau + \mu e^{q\sigma_o}\sigma_o\right)e. \qquad 3.17$$

But (3.16) and (3.17) contradict respectively (3.3) and (3.4); thus (3.1) cannot have a nonoscillatory solution and this completes the proof.

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