

Subelliptic operators on Lie groups

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1 Introduction

Hörmander's analysis [Hör] of hypoelliptic operators lent impetus to the development of the theory of subelliptic operators on Riemannian manifolds or on Lie groups. Recent results have been surveyed in the article by Jerison and Sánchez-Calle [JSC], the lecture notes of Varopoulos, Saloff-Coste and Coulhon [VSC] or the books by Davies [Dav] and Robinson [Rob]. The Lie group theory, which has been extensively studied for subelliptic operators with real coefficients, concentrates on analyzing the heat semigroup generated by the operator and the corresponding heat kernel. In this review we describe the main results of this theory under slightly more general assumptions than hitherto.

Throughout the sequel we adopt the general notation of [Rob]. In particular G denotes a d -dimensional Lie group which we may assume to be connected because all analysis takes place on the connected component of the identity. The Lie algebra of G is denoted by \mathfrak{g} . Furthermore (\mathcal{X}, G, U) is used for a continuous representation of G on the Banach space \mathcal{X} by bounded operators $g \mapsto U(g)$. Both strong and weak* continuity are considered. Moreover if $a_i \in \mathfrak{g}$ then A_i denotes the generator of the one-parameter subgroup $t \mapsto U(e^{-ta_i})$ of the representation. The C^n -subspaces \mathcal{X}'_n of the representation (\mathcal{X}, G, U) with respect to a subbasis $a_1, \dots, a_{d'}$ of \mathfrak{g} is the common domain of all monomials M_m of order $m \leq n$ in the generators $A_1, \dots, A_{d'}$. The C^n -norm is defined by

$$\|x\|'_n = \sup_{0 \leq m \leq n} \|M_m x\|$$

where the supremum is over all the monomials and $M_0 = I$. The C^∞ -elements \mathcal{X}'_∞ of the subbasis are then defined by

$$\mathcal{X}'_\infty = \bigcap_{n \geq 1} \mathcal{X}'_n .$$

Similarly if a_1, \dots, a_d is a full vector space basis of \mathfrak{g} we use the notation \mathcal{X}_n and $\|\cdot\|_n$ for the corresponding C^n -subspace and norm and denote the C^∞ -elements by \mathcal{X}_∞ .

If the representation (\mathcal{X}, G, U) is strongly continuous, \mathcal{X}^* is the dual of \mathcal{X} and $U(g)^*$ the adjoint of $U(g)$ then one has a dual representation (\mathcal{X}^*, G, U_*) , where

$$U_*(g) = U(g^{-1})^* ,$$

which is weakly* continuous. Alternatively if (\mathcal{X}, G, U) is weakly* continuous, \mathcal{X}_* is the predual of \mathcal{X} and $U(g)^*$ is the adjoint of $U(g)$ on \mathcal{X}_* one has a dual representation (\mathcal{X}_*, G, U_*) which is strongly continuous. We will denote both cases with the common notation (\mathcal{F}, G, U_*) with $\mathcal{F} = \mathcal{X}^*$, or \mathcal{X}_* .

The theory of subelliptic operators is constructed from a Lie algebraic basis $a_1, \dots, a_{d'}$ of \mathfrak{g} , i.e., a finite sequence of linearly independent elements of \mathfrak{g} whose Lie algebra generates \mathfrak{g} . Thus there is an integer r such that $a_1, \dots, a_{d'}$ together with all commutators

$(\text{ada}_{i_1}) \dots (\text{ada}_{i_{n-1}})(a_{i_n})$, $i_j = 1, \dots, d'$, where $n \leq r$, span the vector space \mathfrak{g} . The smallest integer r with this property is referred to as the rank of the algebraic basis and a vector space basis is defined to have rank one. Note that if the algebraic basis $a_1, \dots, a_{d'}$ is completed to a full vector space basis $a_1, \dots, a_{d'}, \dots, a_d$ and $\mathcal{X}'_n, \mathcal{X}_n$ denote the corresponding C^n -subspaces then

$$\mathcal{X}'_{nr} \subseteq \mathcal{X}_n \subseteq \mathcal{X}'_n$$

for all $n \geq 1$ where the inclusions are continuous embeddings of Banach spaces. Hence $\mathcal{X}_\infty = \mathcal{X}'_\infty$. Although the subspaces \mathcal{X}'_n for $n < \infty$ depend on the choice of algebraic basis the subspaces \mathcal{X}'_∞ and \mathcal{X}_∞ are basis independent.

There is a canonical modulus associated with each algebraic basis $a_1, \dots, a_{d'}$ which is defined by considering absolutely continuous paths $\gamma: [0, 1] \rightarrow G$ from the identity $e \in G$ to $g \in G$ such that the tangents are almost everywhere in the span of $a_1, \dots, a_{d'}$. Then if ψ is a smooth function over G and $A_1, \dots, A_{d'}$ are the generators of left translations there are tangential coordinates $\gamma_1, \dots, \gamma_{d'}$ such that

$$\frac{d\psi(\gamma(t))}{dt} = \sum_{i=1}^{d'} \gamma_i(t) (A_i \psi)(\gamma(t)) .$$

The modulus $|g|'$ of g is defined in terms of these coordinates by

$$|g|' = \inf_{\gamma(0)=e, \gamma(1)=g} \int_0^1 dt \left(\sum_{i=1}^{d'} \gamma_i(t) \right)^{1/2} .$$

It follows that if

$$B'_\rho = \{g \in G : |g|' < \rho\}$$

there is an integer $D' \geq d$ and $c', C' > 0$ such that

$$c' \rho^{D'} \leq |B'_\rho| \leq C' \rho^{D'}$$

for all $\rho \in \langle 0, 1 \rangle$. The integer D' is defined to be the local dimension of the algebraic basis $a_1, \dots, a_{d'}$. For a proof see [NSW] Theorem 4 and [Rob] Lemma IV.2.3.

If the algebraic basis $a_1, \dots, a_{d'}$ is again completed to a full vector space basis $a_1, \dots, a_{d'}, \dots, a_d$ one can repeat the above definitions and obtain a modulus $|\cdot|$ which automatically satisfies

$$|g| \leq |g|'$$

for all $g \in G$. Moreover if

$$B_\rho = \{g \in G : |g| < \rho\}$$

then

$$c\rho^d \leq |B_\rho| \leq C\rho^d$$

for all $\rho \in \langle 0, 1 \rangle$ and some $c, C > 0$ with d the dimension of G . The two moduli are equivalent for large values. Explicitly there is a $b \geq 1$ such that

$$|g| \leq |g|' \leq b|g|$$

for all g with $|g| \geq 1$. Near the identity the two measures are, however, distinct. Nevertheless there is a $b' > 0$ such that

$$|g| \leq |g|' \leq b'|g|^{1/r}$$

whenever $|g| \leq 1$ where r is the rank of the algebraic basis by [NSW], Proposition 1.1.

Much of the subsequent analysis is on function spaces over the group. We let L_p denote the usual spaces $L_p(G; dg)$ formed with respect to left Haar measure dg . Moreover, $L_{\bar{p}}$ denotes the corresponding spaces $L_p(G; d\hat{g})$ with respect to right Haar measure $d\hat{g}$. Then the modular function Δ satisfies $d\hat{g} = \Delta^{-1}dg$. The C^n -subspaces of L_p and $L_{\bar{p}}$ corresponding to left translations are denoted by $L'_{p;n}$ and $L'_{\bar{p};n}$.

2 Heat semigroups

Let $a_1, \dots, a_{d'}$ be an algebraic basis of the Lie algebra \mathfrak{g} and $A_1, \dots, A_{d'}$ the corresponding generators associated with the continuous representation (\mathcal{X}, G, U) . Next let $C = (c_{ij})$ be a $d' \times d'$ -matrix with real entries and strictly positive real part $\Re C = (C + C^*)/2$. Thus we do not assume that C is symmetric, but the real part is of course symmetric. Then for $c_0, c_1, \dots, c_{d'} \in \mathbb{C}$ introduce the densely-defined subelliptic operator H by $D(H) = \mathcal{X}'_2$ and

$$H = - \sum_{i,j=1}^{d'} c_{ij} A_i A_j + \sum_{i=1}^{d'} c_i A_i + c_0 I \quad (1)$$

Further define the formal adjoint H^\dagger of H by

$$H^\dagger = - \sum_{i,j=1}^{d'} c_{ji} A_i A_j - \sum_{i=1}^{d'} \bar{c}_i A_i + \bar{c}_0 I$$

with $D(H^\dagger) = D(H)$. Note that if H^\dagger is defined relative to the dual representation (\mathcal{F}, G, U_*) then H^\dagger is the restriction of the adjoint of H to \mathcal{F}'_2 . In all the subsequent statements of this section the subelliptic operators are assumed to be of the foregoing type.

The first result is an extension of results given in [Rob] Section IV.4.

Theorem 2.1 *Let (\mathcal{X}, G, U) be a continuous representation and $a_1, \dots, a_{d'}$ an algebraic basis.*

- I. *The subelliptic operator H defined by (1) is closable and its closure \bar{H} generates a continuous holomorphic semigroup S on \mathcal{X} with the property $S_t \mathcal{X} \subseteq \mathcal{X}_\infty$ for all $t > 0$.*
- II. *The action of S is determined by a U -integrable kernel K , i.e. there is a family $(K_t)_{t>0}$ of functions over G forming a convolution semigroup with respect to left Haar measure dg such that*

$$S_t = \int_G dg K_t(g) U(g)$$

for all $t > 0$.

- III. *The kernel K_t is a C^∞ -function for all $t > 0$ and the function $t \mapsto K_t(g)$ is analytic for all $g \in G$.*

- IV. *There exist $a, b > 0$ and $\omega \geq 0$ such that*

$$|K_t(g)| \leq at^{-D'/2} e^{\omega t} e^{-b(|g|')^2/t}$$

for all $g \in G$ and $t > 0$.

V. The kernel is pointwise positive if and only if the coefficients of H are real and then there are $a', b' > 0$ and $\omega' \geq 0$ such that

$$K_t(g) \geq a't^{-D'/2} e^{-\omega't} e^{-b'(|g|')^2/t}$$

for all $g \in G$ and $t > 0$.

The statements of the theorem can be improved in several respects. We will comment further on this after discussing two different but related methods of proving the result.

The conceptually simplest proof is based on Ouhabaz' recent extension [Ouh] of the theory of Dirichlet forms to sectorial forms. The starting point is the observation that one can associate with the operator H on L_2 a sectorial form h with $D(h) = L'_{2,1}$ and

$$h(\varphi, \psi) = \sum_{i,j=1}^{d'} c_{ij}(A_i\varphi, A_j\psi) + \sum_{i=1}^{d'} c_i(\varphi, A_i\psi) + c_0(\varphi, \psi)$$

for $\varphi, \psi \in D(h)$. This form determines in a canonical manner a closed extension \widetilde{H} of H which generates a continuous holomorphic semigroup on L_2 . But then Ouhabaz' results, in particular Theorem 2.7 and the discussion in Section 4 of [Ouh], imply that the semigroup S generated by \widetilde{H} interpolates between the L_p -spaces. In particular the semigroup extends to a continuous semigroup on each of the L_p -spaces and then, by an easy argument, to a continuous semigroup on each of the $L_{\widehat{p}}$ -spaces. Moreover, the holomorphy of S on L_2 implies that $S_t L_2 \subseteq D(\widetilde{H}) \subseteq D(h) = L'_{2,1}$ for all $t > 0$. Similarly, $S_t L_2 \subseteq L'_{2,1}$ for all $t > 0$. Therefore one can repeat the differential inequality arguments of [Rob] Chapter 4 based on the Nash inequalities to deduce that S_t , $t > 0$, is a bounded operator from L_1 to L_2 and then by a duality argument involving the formal adjoint of H it is a bounded operator from L_2 to L_∞ . Combination of these results implies that S_t is bounded from L_1 to L_∞ and then by the Dunford-Pettis theorem it must have an integral kernel. Subsequently, the perturbed form of the differential inequality arguments as given in [Rob] Section IV.2 yield Gaussian bounds on the kernel. In these latter calculations it is important that the matrix of leading coefficients of H are real but symmetry of the matrix is not of consequence. Once the existence of a kernel with Gaussian bounds is established one can associate a continuous semigroup S with the general representation (\mathcal{X}, G, U) by the definition

$$S_t = \int_G dg K_t(g) U(g)$$

for all $t > 0$. Moreover the arguments used in Step 4 of the proof of Theorem IV.4.5 in [Rob] establish that the generator of S is exactly the closure of H . The only elements of this proof are general functional analysis and interpolation theory.

The second method of proof is a rearrangement of the arguments used in Section IV.4 of [Rob]. One begins by considering the operator H_0 associated with left-translations on the real Banach space $C_0(G)$ with $c_0 = c_1 = \dots = c_{d'} = 0$. If $\varphi \in D(H_0)$ is real valued and $\varphi(g) = \|\varphi\|_\infty$ then $((A_i A_j - A_j A_i)\varphi)(g) = \sum_{k=1}^d c_{i,j}^k (A_k \varphi)(g) = 0$. So the $d' \times d'$ -matrix D_g with entries $-(A_i A_j \varphi)(g)$ is real-valued, symmetric and positive definite. Hence

$$\begin{aligned} \operatorname{Re}((H_0 \varphi)(g)) &= \operatorname{Re}(\operatorname{Tr}(C D_g)) \\ &= \operatorname{Tr}(D_g^{1/2} (\Re C) D_g^{1/2}) \geq 0 \end{aligned}$$

Thus H_0 is dissipative and closable. Then the arguments of [Rob] apply directly to H_0 and the theorem can be established for this operator. The first-order terms are subsequently handled by perturbation theory. Note that this proof goes beyond conventional functional analysis by its use of Bony's results [Bon] on subelliptic differential operators.

Note that in both cases one needs supplementary arguments to establish the statement that $S_t\mathcal{X} \subseteq \mathcal{X}_\infty$. This requires an extension of Hörmander's proof of hypoellipticity to the operator H .

One feature of the theorem which can be substantially improved is the upper bounds. In the case of real coefficients the dependence of the parameters a, b, ω entering the bounds on the coefficients of H can be effectively estimated. For example, the value of b can be chosen arbitrarily close to the value $(4\|C\|)^{-1}$. Similar improvements can also be made in the present context by repetition of the arguments of [Rob]. Moreover in special cases such as polynomial groups the large t behaviour of the kernel can be related to the volume growth of the group. Again this improvement is possible under the current hypotheses.

It is also possible to derive estimates on the holomorphy sector of the semigroup S in terms of bounds on the coefficients of H . In particular one can prove that there is a sector of holomorphy which is universal for all isometric representations.

Finally one can also establish a useful identification of the closure of H in terms of the formal adjoint H^\dagger associated with the dual representation. It follows that $\bar{H} = H^{\dagger*}$ because $H^{\dagger*} = \overline{H^\dagger}$ is the generator of the adjoint of the semigroup generated by $\overline{H^\dagger}$ on \mathcal{F} . Moreover, $H^{\dagger*}$ extends \bar{H} . Since a semigroup generator has no proper extension it follows that $\bar{H} = H^{\dagger*}$. Similar identifications are known to hold for strongly elliptic operators.

3 Interpolation spaces

In the sequel we intend to describe regularity properties of the subelliptic operators H associated with a general continuous representation. These properties are all statements of the form 'if $x \in D(\bar{H})$ and $\bar{H}x$ is smooth then x is somewhat smoother'. Usually they can be expressed in terms of a scale of subspaces $(\mathcal{X}_\alpha, \|\cdot\|_\alpha)$, where the index α is a measure of smoothness, by inequalities of the form

$$\|x\|_\alpha \leq c(\|\bar{H}x\|_\beta + \|x\|_\beta)$$

for all $x \in D(\bar{H})$, i.e., they correspond to continuous embeddings of Banach subspaces characterized by smoothness properties of the representation. In order to make such statements more precise it is useful to introduce a variety of spaces adapted to the expression of smoothness properties. These spaces can be defined in various ways, by interpolation, by Lipschitz conditions on the representation, et cetera. But it is a key result that all the possible approaches are equivalent. We begin with the standard definitions of interpolation spaces related to the C^n -subspaces corresponding to an algebraic subbasis.

If $n_1, n_2 \in \mathbb{N}_0$, $n_1 < n_2$ and $x \in \mathcal{X}$ define the interpolation function $\kappa_x^{(n_1, n_2)}: (0, \infty) \rightarrow [0, \infty]$ by setting

$$\kappa_x^{(n_1, n_2)}(t) = \inf\{\|x_{n_1}\|'_{n_1} + t^{n_2-n_1}\|x_{n_2}\|'_{n_2} : x = x_{n_1} + x_{n_2}, x_{n_1} \in \mathcal{X}'_{n_1}, x_{n_2} \in \mathcal{X}'_{n_2}\} .$$

Then for $\gamma \in \langle 0, n_2 - n_1 \rangle$ define $\|\cdot\|_\gamma^{(n_1, n_2)}: \mathcal{X} \rightarrow [0, \infty]$ by

$$\|x\|_\gamma^{(n_1, n_2)} = \left(\int_0^1 dt t^{-1} \left(t^{-\gamma} \kappa_x^{(n_1, n_2)}(t) \right)^p \right)^{1/p}$$

where $p \in [1, \infty]$ and we adopt the convention $\infty^p = \infty^{1/p} = \infty$ for finite p . (Here and in the sequel there are obvious modifications which cover the case $p = \infty$ and which we will not explicitly record.) The interpolation space $(\mathcal{X}'_{n_1}, \mathcal{X}'_{n_2})_\gamma$ is defined by

$$(\mathcal{X}'_{n_1}, \mathcal{X}'_{n_2})_\gamma = \{x \in \mathcal{X} : \|x\|_\gamma^{(n_1, n_2)} < \infty\} .$$

It is a Banach space with respect to the norm $\|\cdot\|_\gamma^{(n_1, n_2)}$ and \mathcal{X}'_{n_2} is a norm dense subspace if $p \in [1, \infty)$ (see, for example, [BuB], §3.2).

There are at least three more interesting interpolation spaces. Let \mathcal{O} be a bounded open neighborhood of the identity e of G and $n \in \mathbb{N}$ then for each $\gamma \in \langle 0, n \rangle$ define $\|\cdot\|_\gamma^{n, U}: \mathcal{X} \rightarrow [0, \infty]$ by

$$\|x\|_\gamma^{n, U} = \|x\| + \left(\int_{\mathcal{O}^n} d\mu_n(\mathbf{g}) \left(|\mathbf{g}|^{-\gamma} \|(I - U(g_1)) \dots (I - U(g_n))x\| \right)^p \right)^{1/p} ,$$

where $\mathbf{g} = (g_1, \dots, g_n)$ and $|\mathbf{g}| = |g_1|' + \dots + |g_n|'$. Moreover, μ_n is the absolutely continuous measure with respect to the left invariant Haar measure on G^n with density $\mathbf{g} \mapsto |\mathbf{g}|^{-nD'}$ where D' is the local dimension corresponding to the algebraic basis $a_1, \dots, a_{d'}$. Then the Lipschitz space $\mathcal{X}_\gamma^n(U)$ is defined by

$$\mathcal{X}_\gamma^n(U) = \{x \in \mathcal{X} : \|x\|_\gamma^{n, U} < \infty\} .$$

It is a Banach space with respect to the norm $\|\cdot\|_\gamma^{n, U}$. Note that as p is fixed throughout we have suppressed it in the notation. Moreover, since the space is independent of the choice of \mathcal{O} , up to equivalence of norms, we have also omitted it from the notation.

Next we introduce a uniform version of the Lipschitz spaces. First, for each $x \in \mathcal{X}$ and $n \in \mathbb{N}_0$ define $\omega_x^{(n)}: \langle 0, \infty \rangle \rightarrow [0, \infty)$ by $\omega_x^{(0)}(t) = \|x\|$ and

$$\omega_x^{(n)}(t) = \sup_{\substack{g_1, \dots, g_n \in G \\ |g_j|' \leq t}} \|(I - U(g_1)) \dots (I - U(g_n))x\|$$

for $n \in \mathbb{N}$. Secondly, for $\gamma \in \langle 0, n \rangle$ define $\|\cdot\|_\gamma^{n, \omega}: \mathcal{X} \rightarrow [0, \infty]$ by

$$\|x\|_\gamma^{n, \omega} = \|x\| + \left(\int_0^1 dt t^{-1} \left(t^{-\gamma} \omega_x^{(n)}(t) \right)^p \right)^{1/p} .$$

Then the space

$$\mathcal{X}_\gamma^{n, \omega} = \{x \in \mathcal{X} : \|x\|_\gamma^{n, \omega} < \infty\}$$

is a Banach space with respect to the norm $\|\cdot\|_\gamma^{n, \omega}$.

Finally, if S is a continuous semigroup on \mathcal{X} we introduce the corresponding Lipschitz spaces as follows. For $n \in \mathbb{N}$ and $\gamma \in \langle 0, n \rangle$ define $\|\cdot\|_\gamma^{n, S}: \mathcal{X} \rightarrow [0, \infty]$ by

$$\|x\|_\gamma^{n, S} = \|x\| + \left(\int_0^1 dt t^{-1} \left(t^{-\gamma} \|(I - S_t)^n x\| \right)^p \right)^{1/p} .$$

Then

$$\mathcal{X}_\gamma^{n,S} = \{x \in \mathcal{X} : \|x\|_\gamma^{n,S} < \infty\}$$

is a Banach space with respect to the norm $\|\cdot\|_\gamma^{n,S}$.

The operators A_i , $i \in \{1, \dots, d'\}$ are also defined, by restriction, on the interpolation spaces and the corresponding C^n -subspaces are denoted by $(\mathcal{X}_{n_1}, \mathcal{X}_{n_2})_{\gamma;n}$ and the C^n -norms by $\|\cdot\|_{\gamma;n}^{(n_1, n_2)}$. Explicitly $(\mathcal{X}_{n_1}, \mathcal{X}_{n_2})_{\gamma;n}$ is the common domain of all n -th order monomials M_n in the $A_1, \dots, A_{d'}$ and

$$\|x\|_{\gamma;n}^{(n_1, n_2)} = \sup \|M_m x\|_\gamma^{(n_1, n_2)}$$

where the supremum is over all monomials of order $m \leq n$ with the convention $M_0 = I$.

We have the following relation between these spaces.

Theorem 3.1 *Let*

$$H = - \sum_{i,j=1}^{d'} c_{ij} A_i A_j + \sum_{i=1}^{d'} c_i A_i + c_0 I$$

where $C = (c_{ij})$ is a real-valued symmetric strictly positive-definite matrix and the other coefficients are complex, i.e., $c_0, c_1, \dots, c_{d'} \in \mathbf{C}$. Further let S be the continuous holomorphic semigroup generated by the closure \overline{H} of H .

I. For each $n \in \mathbf{N}$ and each $\gamma \in \langle 0, n \rangle$

$$(\mathcal{X}, \mathcal{X}'_n)_\gamma = \mathcal{X}_\gamma^{n,\omega} = \mathcal{X}_\gamma^n(U) = \mathcal{X}_{\gamma/2}^{n,S} .$$

II. If $k < n$ and $\gamma \in \langle k, n \rangle$ then

$$(\mathcal{X}, \mathcal{X}'_n)_\gamma = (\mathcal{X}'_k, \mathcal{X}'_n)_{\gamma-k} .$$

III. If $\gamma \in \langle 0, n_1 \wedge n_2 \rangle$ then

$$(\mathcal{X}, \mathcal{X}'_{n_1})_\gamma = (\mathcal{X}, \mathcal{X}'_{n_2})_\gamma .$$

IV. If $\gamma \in \langle 0, n \rangle$ and $k \in \mathbf{N}$ then

$$(\mathcal{X}, \mathcal{X}'_n)_{\gamma;k} = (\mathcal{X}, \mathcal{X}'_{n+k})_{\gamma+k} .$$

In all statements the equality means that the Banach spaces are equal up to equivalence of norms.

Note that the assumptions on H are slightly more restrictive than in the previous section because the matrix C must now be symmetric. Then the second-order part of the operator can be re-expressed as a sum of squares and for operators of that kind there are good kernel bounds available in [VSC]. These can be used to prove the theorem for a second-order operator of this form and the general case can subsequently be deduced by using perturbation theory. For details we refer to [EIR].

It follows immediately from this theorem that if $\gamma \in \langle 0, n \rangle$ and $p \in [1, \infty)$ then \mathcal{X}_∞ is dense in $(\mathcal{X}, \mathcal{X}'_n)_\gamma$ and the restriction of S to $(\mathcal{X}, \mathcal{X}'_n)_\gamma$ is a strongly continuous holomorphic semigroup.

4 Regularity properties

We can use the interpolation spaces and Lipschitz spaces to derive regularity properties for the subelliptic operators

$$H = - \sum_{i,j=1}^{d'} c_{ij} A_i A_j + \sum_{i=1}^{d'} c_i A_i + c_0 I$$

where $C = (c_{ij})$ is a real-valued symmetric strictly positive-definite matrix and the other coefficients are complex, i.e., $c_0, c_1, \dots, c_{d'} \in \mathbb{C}$. There are two types of regularity property. The first is relative to the interpolation spaces between the C^n -subspaces \mathcal{X}'_n of the algebraic basis used to define H and the second is relative to the C^n -subspaces \mathcal{X}_n of a full vector space basis. In passing from one to the other the rank r of the algebraic basis is important.

Let S be the continuous holomorphic semigroup generated by the closure \overline{H} of H and suppose that S is exponentially decreasing. This latter property can of course always be arranged by choosing c_0 sufficiently large.

Theorem 4.1

I. If $0 < \gamma < n$ and $k \in \mathbb{N}$ then

$$(\mathcal{X}, \mathcal{X}'_n)_{\gamma; k} = \{x \in D(\overline{H}^{k/2}) : \overline{H}^{k/2} x \in (\mathcal{X}, \mathcal{X}'_n)_{\gamma}\} .$$

II. If $n, N \in \mathbb{N}$ satisfy $n < 2N/r$ then

$$D(\overline{H}^N) \subseteq \mathcal{X}_n .$$

Moreover, if $\alpha > 0$

$$D(\overline{H}^\alpha) \subseteq (\mathcal{X}_k, \mathcal{X}_{k+1})_{\gamma} = (\mathcal{X}, \mathcal{X}_1)_{\gamma; k}$$

where k is the largest integer strictly smaller than $2\alpha/r$ and $\gamma \in \langle 0, 2\alpha/r - k \rangle$. Both sets of embeddings are continuous.

III. If $p = \infty$ and $\alpha > 0$ then

$$D(\overline{H}^\alpha) \subseteq (\mathcal{X}_k, \mathcal{X}_{k+2})_{\gamma} = (\mathcal{X}, \mathcal{X}_2)_{\gamma; k}$$

where k is the largest integer strictly smaller than $2\alpha/r$ and $\gamma = 2\alpha/r - k$. The embeddings are again continuous.

IV. If $\gamma > 0$, $\alpha > 0$, $n > \gamma + 2\alpha/r$ and $x \in D(\overline{H}^\alpha)$ then

$$\overline{H}^\alpha x \in (\mathcal{X}, \mathcal{X}_n)_{\gamma}$$

implies

$$x \in (\mathcal{X}, \mathcal{X}_n)_{\gamma+2\alpha/r} .$$

Moreover, there exists a $c > 0$, depending on n, γ and α , such that

$$\|x\|_{\gamma+2\alpha/r}^{(0,n)} \leq c \left(\|\overline{H}^\alpha x\|_{\gamma}^{(0,n)} + \|x\|_{\gamma}^{(0,n)} \right)$$

for all $x \in D(H^\alpha)$.

For unitary representations the regularity properties can be greatly improved. Then the regularity can be expressed relative to the C^n -spaces \mathcal{X}'_n , just as for strongly elliptic operators [Rob].

Theorem 4.2 *Let (\mathcal{X}, G, U) be a unitary representation. Then the following are valid.*

I. H is closed.

II. \mathcal{X}_∞ is dense in \mathcal{X}'_n for all $n \in \mathbb{N}$.

III. There exist $\theta > 0$ and $\omega > 0$ such that S is holomorphic in $\{z \in \mathbb{C} : |\arg z| < \theta\}$ and $\|S_z\| \leq e^{\omega|z|}$ for all $z \in \mathbb{C}$ with $|\arg z| < \theta$.

IV. If $n \in \mathbb{N}$ then

$$\mathcal{X}'_n = D(H^{n/2})$$

as Banach spaces.

V. If $n > 2\gamma > 0$ and $p = 2$ then

$$D(H^\gamma) = (\mathcal{X}, \mathcal{X}'_n)_{2\gamma}$$

as Banach spaces.

Proofs of these theorems can be found in [EIR].

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