SIMILARITIES OF ω-ACCRETIVE OPERATORS

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Abstract. Given a number $0 < \omega \leq \frac{\pi}{2}$, an ω-accretive operator is a sectorial operator A on Hilbert space whose numerical range lies in the closed sector of all $z \in \mathbb{C}$ such that $|\mathrm{Arg}(z)| \leq \omega$. It is easy to check that any such operator admits bounded imaginary powers, with $\|A^{it}\| \leq e^{\omega|t|}$ for any $t \in \mathbb{R}$. We show that conversely, A is similar to an ω-accretive operator if $\|A^{it}\| \leq e^{\omega|t|}$ for any $t \in \mathbb{R}$.

1. Introduction.

Let H be a Hilbert space and let A be a closed operator on H with dense domain $D(A)$. Given any $\omega \in (0, \pi)$, we let Σ_ω be the open sector of all complex numbers $z \in \mathbb{C}^*$ such that $|\mathrm{Arg}(z)| < \omega$, and we say that A is sectorial of type ω if its spectrum $\sigma(A)$ is included in the closure of Σ_ω and if for every $\theta \in (\omega, \pi)$, the set $\left\{z(z-A)^{-1} \,:\, z \notin \overline{\Sigma_\theta}\right\}$ is bounded.

Assume that $\omega \leq \frac{\pi}{2}$. We say that A is ω-accretive if it is sectorial of type ω and if

$$\langle A\xi, \xi\rangle \in \overline{\Sigma_\omega}, \qquad \xi \in D(A). \tag{1.1}$$

It is well-known that if the resolvent set $\rho(A)$ contains -1, say, then (1.1) implies that A is sectorial of type ω. Thus A is ω-accretive if and only if $-1 \in \rho(A)$ and (1.1) holds true. Note that with this terminology, $\frac{\pi}{2}$-accretivity coincides with maximal accretivity. The aim of this note is to give a characterization of injective ω-accretive operators up to similarity in terms of their imaginary powers.

If A is an injective maximal accretive operator on H, then we can define its imaginary powers and we have $\|A^{it}\| \leq e^{\frac{\pi}{2}|t|}$ for any real number $t \in \mathbb{R}$. Indeed this estimate is a consequence of von Neumann's inequality, see e.g. [1, Theorem G]. More generally, assume that A is an injective ω-accretive operator. Then $e^{i(\frac{\pi}{2}-\omega)}A$ and $e^{-i(\frac{\pi}{2}-\omega)}A$ are both maximal accretive hence for any $t \in \mathbb{R}$, we have $\|(e^{i(\frac{\pi}{2}-\omega)}A)^{it}\| \leq e^{\frac{\pi}{2}|t|}$ and $\|(e^{-i(\frac{\pi}{2}-\omega)A})^{it}\| \leq e^{\frac{\pi}{2}|t|}$. We easily deduce that

$$\|A^{it}\| \leq e^{\omega|t|}, \qquad t \in \mathbb{R}. \tag{1.2}$$

Our main result asserts that conversely, if A is an injective sectorial operator satisfying (1.2), then A is similar to an ω-accretive operator, that is, there exists a bounded and invertible operator $S\colon H \to H$ such that $S^{-1}AS$ is ω-accretive. We thus have the following characterization.

Theorem 1.1. *Let $\omega \in (0, \frac{\pi}{2}]$ be a number and let A be an injective sectorial operator on H. Then A is similar to an ω-accretive operator if and only if there exists a bounded and invertible operator $S\colon H \to H$ such that $\|S^{-1}A^{it}S\| \leq e^{\omega|t|}$ for any $t \in \mathbb{R}$.*

We wish to make three comments concerning this theorem. First, it complements a previous result of ours ([7]) saying that if A is an injective sectorial operator of type $< \frac{\pi}{2}$, then A is similar to a maximal accretive operator if and only if it admits bounded imaginary powers. Second, Simard's recent work ([12]) shows that our result is essentially optimal. Indeed on the one hand, [12, Theorem 1] implies that for any $\omega \leq \frac{\pi}{2}$, one can find A not similar to an ω-accretive operator whose imaginary powers satisfy an estimate $\|A^{it}\| \leq Ke^{\omega|t|}$ for some $K > 1$. On the other hand, [12, Theorem 4] shows that one can find A satisfying $\|A^{it}\| \leq e^{\frac{\pi}{2}|t|}$ for any $t \in \mathbb{R}$ without being maximal accretive. The third comment is that our proof heavily relies on some recent work of Crouzeix and Delyon ([5]) who established some remarkable estimates for the analytic functional calculus associated to an operator whose numerical range lies in a band of the complex plane.

We now give a consequence of Theorem 1.1 concerning fractional powers of ω-accretive operators. Let $0 < \omega \leq \frac{\pi}{2}$ and $\alpha \in (0, 1]$ be two numbers. It is well-known that if A is an ω-accretive operator, then A^α is $\alpha\omega$-accretive. Although the converse does not hold true (see e.g. the discussion at the end of [12]), Theorem 1.1 implies the following.

Corollary 1.2. *Let A be an ω-accretive operator for some $\omega \leq \frac{\pi}{2}$ and let $\alpha \geq \frac{2\omega}{\pi}$ be a number. Then $A^{\frac{1}{\alpha}}$ is similar to an $\frac{\omega}{\alpha}$-accretive operator.*

Proof. We may assume that A is injective and that $\alpha \leq 1$. Then our assumption of ω-accretivity implies (1.2). Since $(A^{\frac{1}{\alpha}})^{it} = A^{i\frac{t}{\alpha}}$, we thus have $\|(A^{\frac{1}{\alpha}})^{it}\| \leq e^{\frac{\omega}{\alpha}|t|}$ for any $t \in \mathbb{R}$. According to Theorem 1.1, this implies that $A^{\frac{1}{\alpha}}$ is similar to an $\frac{\omega}{\alpha}$-accretive operator, whence the result by taking α-th powers. □

The proof of Theorem 1.1 is given in Section 3. It uses both H^∞ functional calculus techniques (as introduced by McIntosh in [8]) and a theorem of Paulsen ([9]) reducing our proof to the study of the complete boundedness of an appropriate functional calculus. In Section 2 below, we provide some background on Paulsen's Theorem for the convenience of the reader.

2. Background on complete boundedness and Paulsen's Theorem.

We only give a brief account on complete boundedness and its connections with similarity problems. More information and details, as well as important developments and applications can be found in [10].

Given a Hilbert space H, we let $B(H)$ denote the C^*-algebra of all bounded linear operators on H. If $\mathcal{C}$ is a C^*-algebra and $n \geq 1$ is an integer, we let $M_n(\mathcal{C})$ denote the C^*-algebra of all $n \times n$ matrices with entries in $\mathcal{C}$. Let us describe the resulting norm in two important special cases. Assume first that $\mathcal{C} = B(H)$. Then the C^*-norm on $M_n(B(H))$ is obtained by regarding elements of $M_n(B(H))$ as operators on the Hilbertian direct sum $H \oplus \cdots \oplus H$ of n copies of H. Thus for any $[T_{jk}] \in M_n(B(H))$, we have

$$(2.1) \quad \big\|[T_{jk}]\big\| = \sup\biggl\{ \biggl(\sum_{j=1}^{n} \Bigl\| \sum_{k=1}^{n} T_{jk}\xi_k \Bigr\|^2 \biggr)^{\frac{1}{2}} \ :\ \xi_k \in H,\ \sum_{k=1}^{n} \|\xi_k\|^2 \leq 1 \biggr\}.$$

Now consider the case when $\mathcal{C} = C_b(\Omega)$ is the space of all bounded and continuous functions $g\colon \Omega \to \mathbb{C}$ on some topological space Ω, equipped with its sup norm. Then the C^*-norm on $M_n(C_b(\Omega))$ is obtained by identifying $M_n(C_b(\Omega))$ with the space $C_b(\Omega; M_n)$ of bounded and continuous functions from Ω into M_n. Thus for any $[g_{jk}] \in M_n(C_b(\Omega))$, we have

$$(2.2) \qquad \big\|[g_{jk}]\big\| = \sup\Bigl\{ \big\|[g_{jk}(\lambda)]\big\|_{M_n} \ :\ \lambda \in \Omega \Bigr\}.$$

Let H be a Hilbert space, let $\mathcal{C}$ be a C^*-algebra and let $E \subset \mathcal{C}$ be a (not necessarily closed) subspace of $\mathcal{C}$. Then the space $M_n(E)$ of $n \times n$ matrices with entries in E may be obviously regarded as embedded in $M_n(\mathcal{C})$. By definition, a linear mapping $u\colon E \to B(H)$ is completely bounded if there exists a constant $K \geq 0$ such that

$$\big\|[u(a_{jk})]\big\|_{M_n(B(H))} \leq K \big\|[a_{jk}]\big\|_{M_n(E)}$$

for any $n \geq 1$ and any $[a_{jk}] \in M_n(E)$. In that case, the least possible K is denoted by $\|u\|_{cb}$ and is called the completely bounded norm of u. If the latter is ≤ 1, then we say that u is completely contractive. Obviously any completely bounded mapping u is bounded, with $\|u\| \leq \|u\|_{cb}$.

Paulsen's Theorem asserts that any completely bounded homomorphism on an operator algebra (= subalgebra of a C^*-algebra) is similar to a completely contractive one. More precisely, we have the following statement (see [9]), that we will use in the situation when $\mathcal{C} = C_b(\Omega)$ for some Ω.

Theorem 2.1. *(Paulsen) Let H be a Hilbert space, let $\mathcal{C}$ be a C^*-algebra, let $\mathcal{A} \subset \mathcal{C}$ be a subalgebra, and consider a linear homomorphism $u\colon \mathcal{A} \to B(H)$. If u is completely bounded, then there exists a bounded invertible operator $S\colon H \to H$ such that the linear homomorphism $u_S\colon \mathcal{A} \to B(H)$ defined by letting $u_S(a) = S^{-1}u(a)S$ for any $a \in \mathcal{A}$ is completely contractive. In particular, $\|S^{-1}u(a)S\| \leq \|a\|$ for any $a \in \mathcal{A}$.*

We finally recall for further use that for any $[\alpha_{jk}] \in M_n$ and for any vectors $\xi_1, \dots, \xi_n$ and $\eta_1, \dots, \eta_n$ in a Hilbert space H, we have

$$\left| \sum_{j,k=1}^{n} \alpha_{jk} \langle \xi_k, \eta_j \rangle \right| \leq \big\| [\alpha_{jk}] \big\|_{M_n} \left(\sum_{k=1}^{n} \|\xi_k\|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|\eta_j\|^2 \right)^{\frac{1}{2}}. \tag{2.3}$$

3. Proof of Theorem 1.1.

We first introduce some notation concerning H^∞ functional calculus associated to sectorial operators (in the sense of [8], [3]). For any $\theta \in (0, \pi)$, we recall that

$$\Sigma_\theta = \{ z \in \mathbb{C} \,:\, |\mathrm{Arg}(z)| < \theta \}$$

and we let Γ_θ be the counterclockwise oriented boundary of Σ_θ. Then we let $H_0^\infty(\Sigma_\theta)$ be the space of all bounded analytic functions $f\colon \Sigma_\theta \to \mathbb{C}$ for which there exist two positive numbers $c > 0$, $s > 0$, such that

$$|f(z)| \leq c\, \frac{|z|^s}{1 + |z|^{2s}}, \qquad z \in \Sigma_\theta.$$

We recall that if A is a sectorial operator of type $\omega \in (0, \pi)$ and if $f \in H_0^\infty(\Sigma_\theta)$ for some $\theta \in (\omega, \pi)$, then we may define $f(A) \in B(H)$ by letting

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} f(z)(z - A)^{-1}\, dz,$$

where $\gamma \in (\omega, \theta)$ is an intermediate angle. (The definition of $f(A)$ does not depend on γ by Cauchy's Theorem.)

We let A be an injective sectorial operator satisfying (1.2) for some $\omega \in (0, \frac{\pi}{2})$ and aim at proving that A is similar to an ω-accretive operator. Recall from [11, Theorem 2] that A is necessarily sectorial of type ω. Changing A into $A^{\frac{\pi}{2\omega}}$, we may assume that $\omega = \frac{\pi}{2}$. We fix some $\theta \in (\frac{\pi}{2}, \pi)$ and we let $\mathcal{A}_0 = H_0^\infty(\Sigma_\theta)$ that we regard (by taking restrictions) as a subalgebra of $C_b(\Sigma_{\frac{\pi}{2}})$. Then we let $\mathcal{A} \subset C_b(\Sigma_{\frac{\pi}{2}})$ be the subalgebra linearly spanned by $\mathcal{A}_0$, the function $f_0(z) = \frac{1}{1+z}$, and the constant function 1. We clearly define a homomorphism $u\colon \mathcal{A} \to B(H)$ by letting $u(f) = f(A)$ for $f \in \mathcal{A}_0$, $u(f_0) = (1 + A)^{-1}$, $u(1) = 1$, and then extending linearly. We will prove that

$$u\colon \mathcal{A} \longrightarrow B(H) \quad \text{is completely bounded.} \tag{3.1}$$

Taking this for granted, the conclusion goes as follows. By Paulsen's Theorem, there exists an invertible $S \in B(H)$ such that $\|S^{-1}u(f)S\| \leq \|f\|_{C_b(\Sigma_{\frac{\pi}{2}})}$ for all $f \in \mathcal{A}$. Moreover the function $f(z) = \frac{1-z}{1+z}$ belongs to $\mathcal{A}$ and $u(f) = (1 - A)(1 + A)^{-1}$. Since we have

$$\|f\|_{C_b(\Sigma_{\frac{\pi}{2}})} = \sup\Bigl\{\Bigl|\frac{1-z}{1+z}\Bigr| \,:\, \mathrm{Re}(z) > 0\Bigr\} = 1,$$

we conclude that

$$S^{-1}(1-A)(1+A)^{-1}S = (1-S^{-1}AS)(1+S^{-1}AS)^{-1} \quad \text{is a contraction.}$$

This shows that $S^{-1}AS$ is maximal accretive.

To prove (3.1), we will change our sectorial functional calculus into a band sectorial functional calculus by means of the Log function. For any $\gamma > 0$, let

$$P_\gamma = \{\lambda \in \mathbb{C} \,:\, |\mathrm{Im}(\lambda)| < \gamma\}$$

and let Δ_γ denote its counterclockwise oriented boundary. Let iB be the generator of the c_0-group $(A^{it})_t$, so that B should be thought as being $\mathrm{Log}(A)$. Our assumption that $\|A^{it}\| \leq e^{\frac{\pi}{2}|t|}$ for any $t \in \mathbb{R}$ means that $iB - \frac{\pi}{2}$ and $-iB - \frac{\pi}{2}$ both generate contractive semigroups on H. Hence $\frac{\pi}{2} - iB$ and $\frac{\pi}{2} + iB$ are both maximal accretive, whence

$$\mathrm{Re}\,\bigl\langle(\frac{\pi}{2} - iB)\xi, \xi\bigr\rangle \geq 0 \quad \text{and} \quad \mathrm{Re}\,\bigl\langle(\frac{\pi}{2} + iB)\xi, \xi\bigr\rangle \geq 0, \qquad \xi \in D(B).$$

In turn this is equivalent to say that the numerical range of B lies into the closure of $P_{\frac{\pi}{2}}$, that is,

$$\langle B\xi, \xi\rangle \in \overline{P_{\frac{\pi}{2}}}, \qquad \xi \in D(B),\ \|\xi\| \leq 1. \tag{3.2}$$

Let $H_0^\infty(P_\theta)$ be the space of all bounded analytic functions $g\colon P_\theta \to \mathbb{C}$ for which there exist a constant $c > 0$ such that $|g(\lambda)| \leq \frac{c}{1+|\lambda|^2}$ for any $\lambda \in P_\theta$. Then let $\gamma \in (\frac{\pi}{2}, \theta)$ be an arbitrary number. Since $iB - \frac{\pi}{2}$ and $-iB - \frac{\pi}{2}$ both generate contractive semigroups, the function $\lambda \mapsto$

$(\lambda-B)^{-1}$ is well-defined and bounded on Δ_γ hence for any $g\in H_0^\infty(P_\theta)$ we may define $g(B)\in B(H)$ by letting

$$g(B)=\frac{1}{2\pi i}\int_{\Delta_\gamma} g(\lambda)(\lambda-B)^{-1}\,d\lambda.$$

It is easy to check (using Cauchy's Theorem) that this definition does not depend on the choice of γ and that the mapping $v\colon g\mapsto g(B)$ is a linear homomorphism from $H_0^\infty(P_\theta)$ into $B(H)$. Moreover the sectorial and band functional calculi are compatible in the sense that for any $f\in H_0^\infty(\Sigma_\theta)$, the function $\lambda\mapsto f(e^\lambda)$ belongs to $H_0^\infty(P_\theta)$ and

$$g(B)=f(A)\quad\text{if}\quad g(\lambda)=f(e^\lambda). \tag{3.3}$$

We refer the reader to [2] for various relationships between sectorial and band functional calculi, from which a proof of (3.3) can be extracted. However we give a direct argument for the sake of completeness. Let φ be the function defined by $\varphi(z)=z(1+z)^{-2}$, so that $\varphi(A)=A(1+A)^{-2}$. It is well-known that $\varphi(A)$ has a dense range, so that we only need to prove that $g(B)\varphi(A)=f(A)\varphi(A)$. We fix two parameters $\frac{\pi}{2}<\gamma_2<\gamma_1<\theta$. Let λ be a complex number with $\mathrm{Im}(\lambda)=\gamma_1$. Applying the Laplace formula to the semigroup $(A^{-it})_{t\geq 0}$, we have (in the strong sense)

$$(\lambda-B)^{-1}=i(i\lambda-iB)^{-1}=-i\int_0^\infty e^{i\lambda t}A^{-it}\,dt.$$

Hence using Fubini's Theorem, we obtain

$$\begin{aligned}(\lambda-B)^{-1}\varphi(A)&=\frac{-1}{2\pi}\int_0^\infty e^{i\lambda t}\int_{\Gamma_{\gamma_2}} z^{-it}\varphi(z)(z-A)^{-1}\,dz\;dt\\ &=\frac{1}{2\pi i}\int_{\Gamma_{\gamma_2}}\left(-i\int_0^\infty e^{i\lambda t}z^{-it}\,dt\right)\varphi(z)(z-A)^{-1}\,dz\end{aligned}$$

whence

$$(\lambda-B)^{-1}\varphi(A)=\frac{1}{2\pi i}\int_{\Gamma_{\gamma_2}}\frac{1}{\lambda-Log(z)}\,\varphi(z)(z-A)^{-1}\,dz.$$

The latter idendity can be proved as well if $\mathrm{Im}(\lambda)=-\gamma_1$ hence holds true for any $\lambda\in\Delta_{\gamma_1}$. Using Fubini's Theorem again and Cauchy's

Theorem, we therefore deduce that

$$\begin{aligned} g(B)\varphi(A) &= \frac{1}{2\pi i}\int_{\Delta_{\gamma_1}} g(\lambda)(\lambda - B)^{-1}\varphi(A)\,d\lambda \\ &= \Big(\frac{1}{2\pi i}\Big)^2 \int_{\Delta_{\gamma_1}} g(\lambda) \int_{\Gamma_{\gamma_2}} \frac{1}{\lambda - Log(z)}\,\varphi(z)(z-A)^{-1}\,dz\,d\lambda \\ &= \Big(\frac{1}{2\pi i}\Big)^2 \int_{\Gamma_{\gamma_2}} \Big(\int_{\Delta_{\gamma_1}} g(\lambda)\frac{1}{\lambda - Log(z)}\,d\lambda\Big)\varphi(z)(z-A)^{-1}\,dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_{\gamma_2}} g(Log(z))\varphi(z)(z-A)^{-1}\,dz \\ &= f(A)\varphi(A), \end{aligned}$$

which concludes the proof of (3.3).

We let $\mathcal{B}$ be equal to $H_0^\infty(P_\theta)$ regarded as a subalgebra of $C_b(P_{\frac{\pi}{2}})$. To prove (3.1), it will suffice to show that

$$(3.4)\qquad v\colon \mathcal{B} \longrightarrow B(H) \quad \text{is completely bounded.}$$

Indeed since the exponential function is a holomorphic bijection from $P_{\frac{\pi}{2}}$ onto $\Sigma_{\frac{\pi}{2}}$, it follows from (3.3) and the definition of the matrix norms on $\mathcal{A}$ and $\mathcal{B}$ (see (2.2)) that if v is completely bounded, then $u_{|\mathcal{A}_0}$ is completely bounded as well, with $\|u_{|\mathcal{A}_0}\|_{cb} \leq \|v\|_{cb}$. However $\mathcal{A}_0$ has codimension 2 in $\mathcal{A}$ hence the complete boundednes of $u_{|\mathcal{A}_0}$ implies that of u on $\mathcal{A}$.

We now come to the heart of the proof, which consists in showing that for an operator B whose spectrum is included in $\overline{P_{\frac{\pi}{2}}}$, the condition (3.2) implies (3.4). That (3.2) implies the boundedness of v is a recent result of Crouzeix and Delyon ([5]) and our proof of the complete boundedness of v will essentially be a repetition of their arguments, up to some adequate matrix norm manipulations. Before embarking into computations, we notice that (3.2) is equivalent to the following real/imaginary parts decomposition for B:

$$(3.5)\qquad B = C + iD, \quad \text{with} \quad C = C^*,\ D = D^*,\ \|D\| \leq \frac{\pi}{2}.$$

In this decomposition, C is a possibly unbounded self-adjoint operator with $D(C) = D(B)$. Let $(E(s))_s$ be the resolution of the identity for C and for any integer $m \geq 1$, let

$$C_m = \int_{(-m,m)} s\,dE(s) \qquad \text{and} \qquad B_m = C_m + iD.$$

Then C_m is a bounded self-adjoint operator hence B_m is a bounded operator whose numerical range lies in $\overline{P_{\frac{\pi}{2}}}$. Moreover for any $\lambda \notin \overline{P_{\frac{\pi}{2}}}$, we have

$$(3.6) \qquad (\lambda - B_m)^{-1} \longrightarrow (\lambda - B)^{-1} \quad \text{strongly.}$$

Indeed, $(\lambda - B_m)^{-1} - (\lambda - B)^{-1} = (\lambda - B_m)^{-1}(C_m - C)(\lambda - B)^{-1}$, we have $C_m\xi \to C\xi$ for any $\xi \in D(B) = D(C)$, and since the operators $\frac{\pi}{2} \pm iB_m$ are maximal accretive, we have a uniform estimate

$$(3.7) \qquad \|(\lambda - B_m)^{-1}\| \leq d(\lambda, P_{\frac{\pi}{2}}), \qquad m \geq 1.$$

Next, by Lebesgue's Theorem, it follows from (3.6) and (3.7) that $g(B_m) \to g(B)$ strongly for any $g \in \mathcal{B}$. Thus for any $n \geq 1$ and any $[g_{jk}] \in M_n(\mathcal{B})$, we have

$$\big\|[g_{jk}(B)]\big\| \leq \limsup_m \big\|[g_{jk}(B_m)]\big\|$$

To prove the complete boundedness of v, it therefore suffices to prove that the mappings $g \mapsto g(B_m)$ are uniformly completely bounded. To achieve this goal we shall now assume that B is bounded and shall prove that

$$(3.8) \qquad \|v\|_{cb} \leq \frac{2}{\sqrt{3}} + 2.$$

Let $\gamma \in (\frac{\pi}{2}, \theta)$ be an arbitrary intermediate angle. Then according to [5, (5)] (and its proof), we may write

$$v(g) = g(B) = v_1^\gamma(g) + v_2^\gamma(g)$$

for any $g \in \mathcal{B} = H_0^\infty(P_\theta)$, with

$$v_1^\gamma(g) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} g(x)\big((x + 2\gamma i - B^*)^{-1} - (x - 2\gamma i - B^*)^{-1}\big)\, dx;$$
$$v_2^\gamma(g) = \frac{1}{2\pi i}\int_{\Delta_\gamma} g(\lambda)\big((\lambda - B)^{-1} - (\overline{\lambda} - B^*)^{-1}\big)\, d\lambda.$$

Moreover it is easy to check that for any $x \in \mathbb{R}$, one has

$$\begin{aligned}(x + 2\gamma i - B^*)^{-1} &- (x - 2\gamma i - B^*)^{-1} \\ &= -4\gamma i\,(x + 2\gamma i - B^*)^{-1}(x - 2\gamma i - B^*)^{-1} \\ &= -4\gamma i\,\big(M(x) - iN(x)\big)^{-1},\end{aligned}$$

where $M(x)$ and $N(x)$ are self-adjoint operators defined by

$$M(x) = (x - C)^2 - D^2 + 4\gamma^2 \quad \text{and} \quad N(x) = CD + DC - 2xD.$$

(The boundedness of C allows this real/imaginary parts decomposition.) It follows from (3.5) that

$$M(x) \geq (x-C)^2 + 3\Big(\frac{\pi}{2}\Big)^2. \tag{3.9}$$

In particular, $M(x)$ is invertible and with $Q(x) = M(x)^{-\frac{1}{2}}N(x)M(x)^{-\frac{1}{2}}$, we may write

$$(x+2\gamma i-B^*)^{-1}-(x-2\gamma i-B^*)^{-1} = -4\gamma i\, M(x)^{-\frac{1}{2}}\big(1-iQ(x)\big)^{-1}M(x)^{-\frac{1}{2}}.$$

Let $n \geq 1$ be an integer and let $[g_{jk}]$ be an element of $M_n(\mathcal{B})$ with norm ≤ 1. According to (2.2), this simply means that

$$\big\|[g_{jk}(\lambda)]\big\|_{M_n} \leq 1, \qquad \lambda \in P_{\frac{\pi}{2}}. \tag{3.10}$$

We let $\xi_1, \ldots, \xi_n$, $\eta_1, \ldots, \eta_n$ be arbitrary elements of H. Then

$$\begin{aligned}
&\sum_{j,k=1}^{n} \big\langle v_1^\gamma(g_{jk})\xi_k, \eta_j\big\rangle \\
&= \sum_{j,k=1}^{n} \Big(\frac{-2\gamma}{\pi}\Big) \int_{-\infty}^{+\infty} g_{jk}(x) \big\langle M(x)^{-\frac{1}{2}}\big(1-iQ(x)\big)^{-1}M(x)^{-\frac{1}{2}}\xi_k, \eta_j\big\rangle\, dx \\
&= \Big(\frac{-2\gamma}{\pi}\Big) \int_{-\infty}^{+\infty} \sum_{j,k=1}^{n} g_{jk}(x) \big\langle \big(1-iQ(x)\big)^{-1}M(x)^{-\frac{1}{2}}\xi_k, M(x)^{-\frac{1}{2}}\eta_j\big\rangle\, dx\,.
\end{aligned}$$

Applying (2.3) and (3.10), we obtain that

$$\begin{aligned}
&\Big|\sum_{j,k=1}^{n} \big\langle v_1^\gamma(g_{jk})\xi_k, \eta_j\big\rangle\Big| \\
&\leq \frac{2\gamma}{\pi} \int_{-\infty}^{+\infty} \Big(\sum_k \big\|\big(1-iQ(x)\big)^{-1}M(x)^{-\frac{1}{2}}\xi_k\big\|^2\Big)^{\frac{1}{2}} \Big(\sum_j \big\|M(x)^{-\frac{1}{2}}\eta_j\big\|^2\Big)^{\frac{1}{2}} dx.
\end{aligned}$$

Since $Q(x)$ is self-adjoint, the operator $\big(1-iQ(x)\big)^{-1}$ is a contraction for any $x \in \mathbb{R}$ hence applying Cauchy-Schwarz, we finally obtain that

$$\begin{aligned}
&\Big|\sum_{j,k=1}^{n} \big\langle v_1^\gamma(g_{jk})\xi_k, \eta_j\big\rangle\Big| \\
&\leq \frac{2\gamma}{\pi} \Big(\sum_k \int_{-\infty}^{+\infty} \big\|M(x)^{-\frac{1}{2}}\xi_k\big\|^2\, dx\Big)^{\frac{1}{2}} \Big(\sum_j \int_{-\infty}^{+\infty} \big\|M(x)^{-\frac{1}{2}}\eta_j\big\|^2\, dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

Now observe that for any $\xi \in H$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \big\|M(x)^{-\frac{1}{2}}\xi\big\|^2\,dx &= \int_{-\infty}^{+\infty} \langle M(x)^{-1}\xi,\xi\rangle\,dx\\ &\leq \int_{-\infty}^{+\infty} \Big\langle \Big((x-C)^2 + 3\big(\frac{\pi}{2}\big)^2\Big)^{-1}\xi,\xi\Big\rangle\,dx\end{aligned}$$

by (3.9). Moreover using the spectral representation of C we see that the latter integral is equal to

$$\int_{-\infty}^{+\infty} \frac{\|\xi\|^2}{x^2+3\big(\frac{\pi}{2}\big)^2}\,dx = \frac{2}{\sqrt{3}}\,\|\xi\|^2.$$

Combining with the previous estimate, this yields

$$\Big|\sum_{j,k=1}^{n}\big\langle v_1^\gamma(g_{jk})\xi_k,\eta_j\big\rangle\Big| \leq \frac{4\gamma}{\pi\sqrt{3}}\Big(\sum_k \|\xi_k\|^2\Big)^{\frac{1}{2}}\Big(\sum_j \|\eta_j\|^2\Big)^{\frac{1}{2}}.$$

In view of the definition of matrix norms on $B(H)$ (see (2.1)), we deduce

$$\big\|[v_1^\gamma(g_{jk})]\big\| \leq \frac{4\gamma}{\pi\sqrt{3}}. \tag{3.11}$$

We now turn to an estimate for v_2^γ. We rewrite the definition of the latter mapping as

$$v_2^\gamma(g) = \int_{\Delta_\gamma} g(\lambda)T(\lambda)\,|d\lambda|,$$

where $T(\lambda)$ is equal to $\frac{1}{2\pi i}\big((\lambda-B)^{-1}-(\overline{\lambda}-B^*)^{-1}\big)$ if $\mathrm{Im}(\lambda)=-\gamma$ and is equal to its opposite if $\mathrm{Im}(\lambda)=\gamma$. The key point is that $T(\lambda)$ is a nonnegative operator for any $\lambda\in\Delta_\gamma$. Indeed assume for example that $\mathrm{Im}(\lambda)=-\gamma$. Then

$$\begin{aligned}&\frac{1}{2\pi i}\big((\lambda-B)^{-1}-(\overline{\lambda}-B^*)^{-1}\big)\\ &\qquad= \frac{1}{2\pi i}(\lambda-B)^{-1}(2i\gamma+B-B^*)(\overline{\lambda}-B^*)^{-1}\\ &\qquad= \frac{1}{\pi}(\lambda-B)^{-1}(\gamma+D)(\overline{\lambda}-B^*)^{-1},\end{aligned}$$

which is nonnegative by (3.5). Then arguing as above, we obtain that for any vectors $\xi_1,\dots,\xi_n$, and $\eta_1,\dots,\eta_n\in H$, we have

$$\Big|\sum_{j,k=1}^{n}\big\langle v_2^\gamma(g_{jk})\xi_k,\eta_j\big\rangle\Big|\leq\sup_{\lambda\in P_\gamma}\Big\{\big\|[g_{jk}(\lambda)]\big\|\Big\}\Big(\sum_k\int_{\Delta_\gamma}\big\|T(\lambda)\xi_k\big\|^2\,|d\lambda|\Big)^{\frac{1}{2}}$$
$$\times\Big(\sum_j\int_{\Delta_\gamma}\big\|T(\lambda)\eta_j\big\|^2\,|d\lambda|\Big)^{\frac{1}{2}}.$$

Now observe that since B is bounded, the function $\lambda\mapsto(\lambda-B)^{-1}-(\overline{\lambda}-B^*)^{-1}$ is integrable on Δ_γ and that $\frac{1}{2\pi i}\int_{\Delta_\gamma}(\lambda-B)^{-1}-(\overline{\lambda}-B^*)^{-1}\,d\lambda=2$ by Cauchy's Theorem. Hence for any $\xi\in H$, we have

$$\int_{\Delta_\gamma}\big\|T(\lambda)\xi\big\|^2\,|d\lambda|=\frac{1}{2\pi i}\int_{\Delta_\gamma}\big\langle\big((\lambda-B)^{-1}-(\overline{\lambda}-B^*)^{-1}\big)\xi,\xi\big\rangle\,d\lambda=2\|\xi\|^2.$$

Combining with the above estimate, we obtain that

$$\big\|[v_2^\gamma(g_{jk})]\big\|\leq 2\sup_{\lambda\in P_\gamma}\Big\{\big\|[g_{jk}(\lambda)]\big\|\Big\}.$$

Since

$$\lim_{\gamma\to\frac{\pi}{2}}\Big(\sup_{\lambda\in P_\gamma}\Big\{\big\|[g_{jk}(\lambda)]\big\|\Big\}\Big)=\sup_{\lambda\in P_{\frac{\pi}{2}}}\Big\{\big\|[g_{jk}(\lambda)]\big\|\Big\}\leq 1,$$

we finally deduce that

$$\big\|[v(g_{jk})]\big\|\leq\inf_{\gamma>\frac{\pi}{2}}\Big\{\big\|[v_1^\gamma(g_{jk})]\big\|+\big\|[v_2^\gamma(g_{jk})]\big\|\Big\}\leq\frac{2}{\sqrt{3}}+2,$$

which concludes our proof of (3.8).

Remark 3.1. Two results analogous to the one in [5] appear in [6] and [4]. On the one hand, it is shown in [6] that if $\Omega\subset\mathbb{C}$ is bounded and convex and if B is a bounded operator on H whose numerical range lies in Ω, then the analytic functional calculus associated to B is bounded with respect to the norm induced by $C_b(\Omega)$. On the other hand, it is shown in [4] that if A is an ω-accretive operator on H, then its analytic functional calculus is bounded with respect to the norm induced by $C_b(\Sigma_\omega)$. In the two cases, it it actually possible to show that these bounded functional calculi are completely bounded. If we apply Paulsen's Theorem to the functional calculus considered in [4] (sectorial case), we recover Corollary 1.2.

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