2. THE CENTER OF AN ALGEBRA

The study of groups has clearly shown that the properties of their direct decompositions depend to a large extent on those of their center or, in the case of groups with a set Ω of operators, on those of what is called the Ω-center.[9] This applies also to arbitrary algebras in the sense of 1.1; however, the definition of a center is in this case more involved. The *center* of an algebra will be defined (in 2.10) as the set-theoretical union of certain subalgebras which are referred to as *central subalgebras*.

Definition 2.1. *A subalgebra* C *of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

is called a *central subalgebra* *if it satisfies the following conditions:*

(i) *If* $c \varepsilon C$, *then there exists an element* $\bar{c} \varepsilon C$ *such that*

$$c + \bar{c} = 0;$$

(ii) *If* $a_1, a_2 \varepsilon A$ *and* $c_1, c_2 \varepsilon C$, *then*

$$(a_1 + c_1) + (a_2 + c_2) = (a_1 + a_2) + (c_1 + c_2);$$

(iii) *If* O_ξ *is a* μ-*ary operation, and if* $a_0, a_1, \ldots, a_\kappa, \ldots \varepsilon A$ *and* $c_0, c_1, \ldots, c_\kappa, \ldots \varepsilon C$ *for* $\kappa < \mu$, *then*

$$O_\xi(a_0 + c_0, a_1 + c_1, \ldots, a_\kappa + c_\kappa, \ldots) = O_\xi(a_0, a_1, \ldots, a_\kappa, \ldots) + O_\xi(c_0, c_1, \ldots, c_\kappa, \ldots).$$

Conditions (ii) and (iii) of this definition are closely related to conditions (iii) and (iv) of Definition 1.4; this circumstance will play an essential part in further developments. 2.1 (ii) can clearly be replaced by condition (ii) of Theorem 2.2 below. In case the rank μ of an operation O_ξ is finite, 2.1 (iii) is easily seen to be equivalent to each of the following conditions:

(iii') *If* $a_0, a_1, \ldots, a_\kappa, \ldots \varepsilon A$ *and* $c_0, c_1, \ldots, c_\kappa, \ldots \varepsilon C$ *for* $\kappa < \mu$, *then*

$$O_\xi(a_0 + c_0, a_1 + c_1, \ldots, a_\kappa + c_\kappa, \ldots) = O_\xi(a_0, a_1, \ldots, a_\kappa, \ldots) + \sum_{\kappa<\mu} O_\xi(0, 0, \ldots, 0, c_\kappa, 0, \ldots).$$

9. See, e.g., Speiser [1], p. 30, for groups without operators, and Kořinek [1], p. 273, for groups with operators.

(iii''). _If_ $a_0, a_1, \ldots, a_\kappa, \ldots \varepsilon A$ _for_ $\kappa < \mu$, $c \varepsilon C$, _and_ $\lambda < \mu$, _then_

$$O_\xi(a_0, a_1, \ldots, a_{\lambda-1}, a_\lambda + c, a_{\lambda+1}, \ldots) = O_\xi(a_0, a_1, \ldots, a_\lambda, \ldots) + O_\xi(0, 0, \ldots, 0, c, 0, \ldots).$$

Theorem 2.2. _Let_ C _be a central subalgebra of an algebra_

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle.$$

We then have:

(i) _If_ $a \varepsilon A$ _and_ $c \varepsilon C$, _then_ $a + c = c + a$.

(ii) _If_ $a_1, a_2 \varepsilon A$ _and_ $c \varepsilon C$, _then_

$$(a_1 + a_2) + c = a_1 + (a_2 + c) = (a_1 + c) + a_2.$$

(iii) _If_ $a_1, a_2 \varepsilon A$, $c \varepsilon C$, _and_ $a_1 + c = a_2 + c$, _then_ $a_1 = a_2$.

Proof: By 2.1 (ii) we have for $a_1, a_2 \varepsilon A$ and $c \varepsilon C$

$$(a_1 + c) + (a_1 + 0) = (a_1 + a_2) + (c + 0) \text{ and } (a_1 + 0) + (a_2 + c) = (a_1 + a_2) + (0 + c).$$

Hence (ii) follows by 1.1 (ii') and 1.2 (i). To prove (i) apply (ii) with $a_1 = 0$ and $a_2 = a$. (iii) follows from (ii) and 2.1 (i).

Theorem 2.3. _If_ C _is a central subalgebra of an algebra_

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle,$$

then

(i) C _is a subtractive subalgebra of_ $\underline{A}$;

(ii) C _is an Abelian group under the operation_ +.

Proof: by 1.15, 2.1, and 2.2 (i), (ii).

Theorem 2.4. _For every algebra_

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

we have:

(i) $\{0\}$ _is a central subalgebra of_ $\underline{A}$.

(ii) _If_ F _is a non-empty family of central subalgebras of_ $\underline{A}$, _then the intersection of all subalgebras_ $C \varepsilon F$ _is a central subalgebra of_ $\underline{A}$.

(iii) _If_ $C_0, C_1, \ldots, C_\kappa, \ldots$ _with_ $\kappa < \nu < \omega$ _are central subalgebras of_ $\underline{A}$, _then there exists a central subalgebra_ C _of_ $\underline{A}$ _such that_ $C_\kappa \subseteq C$ _for_ $\kappa < \nu$.

(iv) If C_0, $C_1, \ldots, C_\kappa, \ldots$ with $\kappa < \nu < \omega$ are central subalgebras of $\underline{A}$, and if $\bigsqcap_{\kappa<\nu} C_\kappa$ exists, then $\bigsqcap_{\kappa<\nu} C_\kappa$ is a central subalgebra of $\underline{A}$.

Proof: (i) is an immediate consequence of 2.1. (ii) follows from 2.1 and 2.3 (ii). (iii) follows from (i), 2.1, and 2.2 (i); we define C to be the set of all elements c of the form

$$c = \sum_{\kappa<\nu} c_\kappa$$

where

$$c_\kappa \varepsilon C \text{ for } \kappa < \nu.$$

If the hypothesis of (iv) is satisfied and if C is defined in the way just indicated, then

$$C = \bigsqcap_{\kappa<\nu} C_\kappa.$$

Hence (iv) holds.

Theorem 2.5. If B and C are central subalgebras of an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

or, more generally, if B is a subtractive subalgebra of $\underline{A}$ and C is a central subalgebra of $\underline{A}$, then

(i) $B \cap C$ is a central subalgebra of $\underline{A}$:

(ii) $B \times C$ exists if, and only if, $B \cap C = \{0\}$.

Proof: (i) Let us replace C and $B \cap C$ in 2.1. Condition 2.1 (i) is then satisfied by 1.14; conditions 2.1 (ii), (iii) obviously hold; and hence $B \cap C$ is a central subalgebra of $\underline{A}$. (ii) Assume that

(1) $$B \cap C = \{0\}.$$

Let D be the set of all elements d of the form

$$d = b + c \text{ where } b \varepsilon B \text{ and } c \varepsilon C.$$

By 2.1 (ii), (iii), D is a subalgebra of $\underline{A}$ and conditions 1.4 (i), (iii), (iv) are satisfied. It remains to show that 1.4 (ii) holds. Suppose that

(2) $$b_1 + c_1 = b_2 + c_2 \text{ with } b_1, b_2 \varepsilon B \text{ and } c_1, c_2 \varepsilon C.$$

By 2.1 (i),

(3) $$c_1 + \bar{c}_1 = 0 \text{ where } \bar{c}_1 \varepsilon C.$$

Hence, by (2) and 2.2 (ii),

$$b_1 = b_2 + (c_2 + \bar{c}_1).$$

Therefore, by (1), (2), and 1.15,

$$c_2 + \bar{c}_1 = 0.$$

Consequently, by 2.2 (iii), (3), and (2),

$$c_1 = c_2 \text{ and } b_1 = b_2.$$

Thus 1.4 (ii) holds, and we have

(4) $$D = B \times C.$$

Conversely, (4) implies (1) by 1.8 (i); and the proof of (ii) is complete.

Theorem 2.6. *Let* B *and* C *be subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle.$$

We then have:

(i) *If* $C \subseteq B$ *and* C *is a central subalgebra of* $\underline{A}$, *then* C *is a tral subalgebra of* B.

(ii) *If* $A = B \times B'$ *for some subalgebra* B', *and* C *is a central subalgebra of* B, *then* C *is a central subalgebra of* $\underline{A}$.

Proof: by 1.4, 1.6 (i), (ii), and 2.1.

Theorem 2.7. *Let* B *and* C *be subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

and let f *be a* B, C-*homomorphism*. *If* B' *is a central subalgebra of* B, *then* $f^*(B')$ *is a central subalgebra of* $f^*(B)$.

Proof: by 2.1.

Theorem 2.8. *Let* B, B_0, $B_1, \ldots, B_\kappa, \ldots$ *with* $\kappa < \nu < \omega$ *be subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

and let

$$B = \prod_{\kappa<\nu} B_\kappa.$$

If C is a central subalgebra of B, then there exist central subalgebras C_κ of B_κ for $\kappa < \nu$ such that

$$C \subseteq \prod_{\kappa<\nu} C_\kappa.$$

Proof: by 1.20, 2.6 (ii), and 2.7. (We put $C_\kappa = f^*_\kappa(C)$ for $\kappa < \nu$ where $f_0, f_1, \ldots, f_\kappa, \ldots$ are homomorphisms with the properties stated in 1.20.)

Theorem 2.9. If B, C, and $D_0, D_1, \ldots, D_\kappa, \ldots$ with $\kappa < \nu < \omega$ are subalgebras of an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

such that $B \times C$ exists and

$$B \times C = \prod_{\kappa<\nu} D_\kappa,$$

then there exists a central subalgebra B' of B such that

$$C \subseteq \prod_{\kappa<\nu} [(B' \times C) \cap D_\kappa].$$

Proof: Consider first the case $\nu = 2$. By 1.20 there exists a $B \times C$, D_0 - homomorphism f and a $B \times C$, D_0 - homomorphism g such that

(1) $a = f(a) + g(a)$ for $a \in B \times C$, $f^*(B\times C) = D_0$, and $g^*(B\times C) = D_1$.

Hence, by 1.4 (i),

(2) $$C \subseteq f^*(C) \times g^*(C).$$

By 1.20 there exists a $B \times C$, B - homomorphism h with the following property:

(3) For every element $a \in B \times C$ there is an element $c \in C$ such that $a = h(a) + c$.

Let

(4) $$B' = h^*[(f^*(C) \times g^*(C)].$$

We are going to show that all the conditions of 2.1 are satisfied if A and C are replaced by B and B'. We begin with condition 2.1 (ii). Let b_1 and b_2 be any elements of B, and b_1' and b_2' any elements of B'. By 1.4 (ii), (3), and (4) we have

(5) $b_1 = h(b_1)$, $b_2 = h(b_2)$, as well as $b_1' = h(d_1)$ and $b_2' = h(d_2)$ for some elements $d_1, d_2 \varepsilon f^*(C) \times g^*(C)$.

Hence

$$f(d_1),\ f(d_2)\ \varepsilon\, f^*(C).$$

Consequently,

(6) $f(d_1) = f(c_1)$ and $f(d_2) = f(c_2)$ for some $c_1, c_2 \varepsilon C$.

By 1.4 (iii) and the hypothesis,

$$(b_1 + c_1) + (b_2 + c_2) = (b_1 + b_2) + (c_1 + c_2);$$

therefore, by (6),

$$[f(b_1) + f(d_1)] + [f(b_2) + f(d_2)] = [f(b_1) + f(b_2)] + [f(d_1) + f(d_2)];$$

and hence

(7) $f[(b_1 + d_1) + (b_2 + d_2)] = f[(b_1 + b_2) + (d_1 + d_2)]$.

Similarly,

(8) $g[(b_1 + d_1) + (b_2 + d_2)] = g[(b_1 + b_2) + (d_1 + d_2)]$.

Formulas (1), (7), and (8) give

$$(b_1 + d_1) + (b_2 + d_2) = (b_1 + b_2) + (d_1 + d_2);$$

together with (5), this implies 1.4 (ii), i.e.,

(9) $(b_1 + b_1') + (b_2 + b_2') = (b_1 + b_2) + (b_1' + b_2')$ for any $b_1, b_2 \varepsilon B$ and $b_1', b_2' \varepsilon B'$.

To derive 2.1 (iii), we proceed analogously applying 1.4 (iv) instead of 1.4 (iii). If finally b is any element in B', we have by (4)

$$b = h(a) \text{ where } a\ \varepsilon\ f^*(C) \times g^*(C).$$

Therefore, for some elements $c, c' \varepsilon C$,

(10) $$b = hf(c) + hg(c').$$

By (1), (8), and 1.6 (ii),

(11) $$hf(c) + hg(c) = 0 \text{ and } hg(c') + hf(c') = 0.$$

Since, by (4), the elements $hf(c)$, $hg(c)$, $hg(c')$, and $hf(c')$ are in B' and hence also in B, we conclude from (9) and (11) that

$$[hf(c) + hg(c')] + [hg(c) + hf(c')] = 0.$$

Consequently, in view of (10),

$$b + \bar{b} = 0 \text{ where } \bar{b} = hg(c) + hf(c');$$

and condition 2.1(i) is shown to hold. Thus, B' proves to be a central subalgebra of B.

By (3), (4), and the hypothesis,

$$f^*(C) \subseteq B' \times C \text{ and } g^*(C) \subseteq B' \times C.$$

Hence, by means of (1),

$$f^*(C) \subseteq (B' \times C) \cap D_0 \text{ and } g^*(C) \subseteq (B' \times C) \cap D_1.$$

Therefore, by (2),

$$C \subseteq [(B' \times C) \cap D_0] \times [(B' \times C) \cap D_1].$$

This completes the proof for $\nu = 2$.

If now ν is an arbitrary finite ordinal, we put

(12) $$D'_{\kappa-1} = \prod_{\lambda<\kappa-1} D_\lambda \times \prod_{\lambda<\nu-\kappa} D_{\kappa+\lambda} \text{ for } 0<\kappa<\nu.$$

We then have be hypothesis

(13) $$B \times C = D_\kappa \times D'_\kappa \text{ for } \kappa < \nu.$$

By 1.20 and the hypothesis, there exist $B \times C$, D_κ - homomorphisms f_κ with $\kappa < \nu$ such that

(14) $$a = \sum_{\kappa<\nu} f_\kappa(a) \text{ for } a \in B \times C.$$

By (13) and the first part of the proof, there are central subalgebras B_κ of B with

$$C \subseteq [(B_\varkappa \times C) \cap D_\varkappa] \times D'_\varkappa \quad \text{for every } \varkappa < \nu.$$

Hence, by (12) and (14),

(15) $$f_\varkappa(c) \in B_\varkappa \times C \text{ for } c \in C \text{ and } \varkappa < \nu.$$

By 2.4 (iii), there exists a central subalgebra B' of B such that

$$B_\varkappa \subseteq B' \text{ for } \varkappa < \nu.$$

Therefore, by (15),

$$f_\varkappa(c) \in (B' \times C) \cap D_\varkappa \text{ for } \varkappa < \nu;$$

and consequently, with the help of (14),

$$C \subseteq \bigcap_{\varkappa<\nu} [(B' \times C) \cap D_\varkappa].$$

Thus, our theorem holds for an arbitrary $\nu < \omega$.

Definition 2.10. *Let*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

be an algebra. The center of $\underline{A}$ (or A)--in symbols A^c--is the union of all central subalgebras C of $\underline{A}$.

The following examples will serve to illustrate this definition:

Example I. If

$$\underline{A} = \langle A, + \rangle$$

is a group, then A^c is its center in the ordinary sense, i.e. the set of all elements $c \in A$ such that

$$a + c = c + a \text{ for every } a \in A.^{9}$$

Example II. Let

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

be a group with operators, i.e., let A be a group under the operation +, and $O_0, O_1, \ldots, O_\xi, \ldots$ be unary operations such that

$$O_\xi(a_1 + a_2) = O_\xi(a_1) + O_\xi(a_2) \text{ for all } a_1, a_2 \varepsilon A.$$

Then A^c is what is usually called the Ω-center of $\underline{A}$, Ω being the set consisting of all the operations O_ξ.[9]

Example III. If

$$\underline{A} = \langle A, +, \cdot \rangle$$

is a ring, then A^c consists of all elements $c\varepsilon A$ such that

$$a \cdot c = c \cdot a = 0 \text{ for every } a\varepsilon A;$$[10] 10

Thus in particular, if $\underline{A}$ is a ring with a unit element, or a ring without divisions of zero, then $A^c = \{0\}$; more generally, this applies to all rings in which no element different from zero is nilpotent.

Example IV. If

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

is an algebra, and if $a_1 + a_2 = 0$ implies $a_2 = 0$ for all $a_1, a_2 \varepsilon A$, then $\{0\}$ is the only central subalgebra of $\underline{A}$, and therefore $A^c = \{0\}$. The class of algebras which satisfy this condition includes all lattices and Boolean algebras (with $x + y$ as the least upper bound of x and y)[11]; many instances can also be found among semigroups.

Example V. Let

$$\underline{A} = \langle A, + \rangle$$

be an Abelian group, and let a be an element of A which is different from zero. For every infinite sequence of elements $a_0, a_1, \ldots a_\kappa, \ldots$ (with $\kappa < \omega$) of A, we put

$$\overline{O}(a_0, a_1, \ldots, a_\kappa, \ldots) = 0 \text{ or } \overline{O}(a_0, a_1, \ldots, a_\kappa, \ldots) = a,$$

10. In the theory of lattices and rings the term "center" is sometimes used in the literature with an entirely different meaning; cf. Birkhoff [1], pp. 23 f., and Jacobson [1], p. 22.

according as the sequence $a_0, a_1, \ldots, a_\kappa, \ldots$ has finitely many or infinitely many distinct terms. It is easily seen that a subgroup C of $\underline{A}$ is a central subalgebra of the algebra

$$\underline{\bar{A}} = \langle A, +, \bar{0} \rangle$$

if, and only if, C is finite. An element $c \varepsilon A$ generates a finite subgroup if, and only if, it is of a finite order. Hence, the center of $\underline{\bar{A}}$ is the set of all elements $c \varepsilon A$ which are of a finite order. Thus, if A has infinitely many elements of a finite order, then the center of $\underline{\bar{A}}$ is not a central subalgebra of $\underline{\bar{A}}$; and furthermore, if the element a is of an infinite order, then this center is not even a subalgebra of $\underline{\bar{A}}$ since it is not closed under the operation $\bar{0}$.

As is seen from the last example, the center of an algebra $\underline{A}$ is not always a central subalgebra of $\underline{A}$, and in fact it need not even be a subalgebra of $\underline{A}$. For our further purposes, however, the notion of center proves useful only in those cases in which A^c is a central subalgebra of $\underline{A}$. Some important particular cases in which this condition is satisfied are discussed in the following

Theorem 2.11. Let

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

be an algebra.

(i) If A is finite or, more generally, if A^c is finite, then A^c is a central subalgebra of $\underline{A}$.

(ii) The same conclusion holds in case all the operations $O_0, O_1, \ldots, O_\xi, \ldots$ are of a finite rank (independent of whether A and A^c are finite or not).

Proof: If A^c is finite, then the family F of all central subalgebras of $\underline{A}$ is finite; moreover, by 2.4 (i), F is non-empty. Hence the conclusion of the first part of the theorem follows immediately by 2.4 (iii) and 2.10. Similarly, the second part can easily be derived from 2.1 and 2.10 by means of 2.4 (iii).

Some important properties of the center will be established in the next three theorems, 2.12-2.14.

Theorem 2.12. Let $A_0, A_1, \ldots, A_\kappa, \ldots$ with $\kappa < \nu < \omega$ be subalgebras of an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

such that

$$A = \prod_{\kappa<\nu} A_\kappa .$$

Then

$$(A_\kappa)^c = A_\kappa \cap A^c \text{ for } \kappa < \nu .$$

Proof: Let κ be an arbitrary ordinal less than ν. By 2.6 (ii) and the hypothesis, every central subalgebra of A_κ is also a central subalgebra of A. Hence, by 2.10,

(1) $$(A_\kappa)^c \subseteq A_\kappa \cap A^c .$$

On the other hand, it is seen from 2.10 that $A_\kappa \cap A^c$ is the union of all sets $A_\kappa \cap C$ where C is a central subalgebra of A; while, by 1.16, 2.5 (i), and 2.6 (i), and in view of the hypothesis, every such set $A_\kappa \cap C$ is a central subalgebra of A_κ. Hence, again by 2.10,

(2) $$A_\kappa \cap A^c \subseteq (A_\kappa)^c .$$

The conclusion follows from (1) and (2).

Theorem 2.13. Let B, C, and $D_0, D_1, \ldots, D_\kappa, \ldots$ with $\kappa < \nu < \omega$ be subalgebras of an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

such that $B \times C$ exists and

$$B \times C = \prod_{\kappa<\nu} D_\kappa .$$

If B^c is a subalgebra of $\underline{A}$, then

$$B^c \times C = \prod_{\kappa<\nu} [(B^c \times C) \cap D_\kappa] .$$

Proof: By 2.9 and 2.10,

(1) $$C \subseteq \prod_{\kappa<\nu} [(B^c \times C) \cap D_\kappa] .$$

For any given central subalgebra B' of B there exist central subalgebras D_κ' of D_κ for $\kappa < \nu$ such that

(2) $$B' \subseteq \prod_{\kappa<\nu} D_\kappa' ;$$

this follows from 2.6 (ii), 2.8, and the hypothesis. By 2.6 (ii), 2.4 (iv), and the hypothesis, $\prod_{\kappa<\nu} D'_\kappa$ is a central subalgebra of B × C. Hence, by 2.8, there are central subalgebras B" and C" of B and C, such that

$$\prod_{\kappa<\nu} D'_\kappa \subseteq B'' \times C''.$$

Therefore,

$$D'_\kappa \subseteq (B^c \times C) \cap D \text{ for } \kappa < \nu.$$

Thus, by (2),

$$B' \subseteq \prod_{\kappa<\nu} [(B^c \times C) \cap D_\kappa].$$

Since this holds for every central subalgebra B' of B, we infer by 2.10 that

$$B^c \subseteq \prod_{\kappa<\nu} [(B^c \times C) \cap D_\kappa].$$

Hence, by (1),

$$B^c \times C \subseteq \prod_{\kappa<\nu} [(B^c \times C) \cap D_\kappa].$$

The inclusion in the opposite direction is obvious, and the proof is complete.

<u>Theorem</u> 2.14. <u>Let</u> B, C, <u>and</u> D <u>be subalgebras of an algebra</u>

$$\underline{A}^* = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle.$$

<u>If</u> B × C <u>exists and</u> B^c <u>is a central subalgebra of</u> <u>A,</u> <u>then the formulas</u>

$$B \times C = B \times D \text{ and } B^c \times C = B^c \times D$$

<u>are equivalent</u>.

Proof: Assume that

(1) $$B \times C = B \times D.$$

Then, by 2.13,

(2) $$B^c \times C = [(B^c \times C) \cap B] \times [(B^c \times C) \cap D].$$

By 1.16, 1.17, and (1),

$$(B^c \times C) \cap B = B^c \times (B \cap C).$$

Hence, by (1) and 1.8 (i), (ii),

$$(B^c \times C) \cap B = B.$$

Consequently, by (2),

$$B^c \times C = B^c \times [(B^c \times C) \cap D].$$

Therefore, by 1.7 (iii),

$$B^c \times C \subseteq B^c \times D.$$

The inclusion in the opposite direction can be proved in a similar way; so that finally

(3) $$B^c \times C = B \times D.$$

Thus (1) implies (3).

Assume now, conversely, that (3) holds. We are going to show that all the conditions of 1.4 are satisfied if in them we replace C and D by D and $B \times C$, respectively. If $a \varepsilon B \times C$, we have

(4) $$a = b' + c \text{ for some } b' \varepsilon B \text{ and } c \varepsilon C.$$

Hence, by (3), c is an element of $B^c \times D$, and therefore

(5) $$c = b'' + d \text{ for some } b'' \varepsilon B^c \text{ and } d \varepsilon D.$$

By 2.6 (i), B^c is a central subalgebra of $B \times C$. By (3), (4), and (5), b' and d are elements of $B \times C$; therefore, by (4), (5), and 2.2 (i), (ii),

(6) $$a = b + d \text{ where } b = b' + b'' \varepsilon B \text{ and } d \varepsilon D.$$

Suppose, conversely, that

(7) $$a = b + d \text{ where } b \varepsilon B \text{ and } d \varepsilon D.$$

Then, by (3),

$$d = b' + c \text{ for some } b' \varepsilon B^c \text{ and } c \varepsilon C.$$

Hence, by (7) and 2.2 (i), (ii),

$$a = (b + b') + d.$$

Therefore, a is in $B \times C$. Thus, 1.4 (i) is shown to hold. Now suppose that

(8) $b_1, b_2 \varepsilon B$; $d_1, d_2 \varepsilon D$; and $b_1 + d_1 = b_2 + d_2$.

Then, by (3),

(9) $d_1 = b_1' + c_1$ and $d_2 = b_2' + c_2$ for some $b_1', b_2' \varepsilon B^c$ and $c_1, c_2 \varepsilon C$.

Hence, by 2.1 (i),

(10) $b_1' + b_1'' = 0$ and $b_2' + b_2'' = 0$ where $b_1'', b_2'' \varepsilon B^c$.

By (8), (9), and 2.2 (i), (ii),

$$b_1 + d_1 = (b_1 + b_1') + c_1 \text{ and } b_2 + d_2 = (b_2 + b_2') + c_2.$$

Therefore, by (8) and (9),

(11) $b_1 + b_1' = b_2 + b_2'$ and $c_1 = c_2$.

Conditions (9) and (10) give by 2.1 (i), (ii)

$$c_1 = b_1'' + d_1 \text{ and } c_2 = b_2'' + d_2.$$

Therefore, by (3), (8), (10), and (11),

(12) $b_1'' = b_2''$ and $d_1 = d_2$.

Hence, by (10) and 2.2 (iii), $b_1' = b_2'$, and further, by (11) and 2.2 (iii),

(13) $b_1 = b_2$.

We have thus shown that (8) implies (12) and (13), and that consequently 1.4 (ii) holds. Furthermore, for any elements b_1, b_2 in B and d_1, d_2 in D, we have (9), and hence

$$(b_1 + d_1) + (b_2 + d_2) = [b_1 + (b_1' + c_1)] + [b_2 + (b_2' + c_2)];$$

by applying 2.2 (i), (ii) and 1.4 (iii) several times, we arrive at

$$(b_1 + d_1) + (b_2 + d_2) = (b_1 + b_2) + (d_1 + d_2).$$

Thus, 1.4 (iii) is satisfied. 1.4 (iv) can be verified in a similar way. Therefore (3) implies (1), and the proof is now complete.

Like the notion of a center, the familiar notion of central isomorphism[11] can also be extended to arbitrary algebras in the sense of 1.1. We define:

Definition 2.15. *Let* B *and* C *be subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle.$$

A B, C-*isomorphism* f *is called a central* B, C-*isomorphism if there exists a central subalgebra* D *of* $\underline{A}$ *such that, for every element* b∊B, f(b) *can be represented in the form*

$$f(b) = b + d \text{ with } d \in D.$$

In case a central B, C-*isomorphism exists, the subalgebras* B *and* C *are called central-isomorphic, in symbols,*

$$B \cong C.$$

It should be pointed out that the notion of central isomorphism is relative to a "superalgebra" $\underline{A}$. Hence the question arises whether two subalgebras B and C which are central-isomorphic in $\underline{A}$ are also central-isomorphic in a subalgebra A' of $\underline{A}$ which includes both B and C. It turns out that, in general, the answer is negative; it is affirmative, however, in case A' is a subtractive subalgebra of $\underline{A}$.

Theorem 2.16. *For any subalgebras* B, C, *and* D *of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

we have:

(i) $$B \cong B.$$

(ii) *If* $B \cong C$, *then* $C \cong B$.

11. Cf. Speiser [1], p. 134.

(iii) *If* $B \cong C$ *and* $C \cong D$, *then* $B \cong D$.

(iv) *If* $B \cong C$, *then* $B \simeq C$.

(v) *Assuming that* $C \times D$ *exists, we have* $B \cong C \times D$ *if, and only if, there exist subalgebras* C' *and* D' *of* $\underline{A}$ *such that* $B = C' \times D'$, $C' \cong C$, *and* $D' \cong D$.

Proof: (i) is obvious by 2.4 (i) and 2.15. (ii) follows from 2.1 (i), 2.2 (ii), and 2.15. (iii) can easily be derived from 2.2 (ii), 2.4 (iii), and 2.15. (iv) is an immediate consequence of 2.15. Finally, (v) follows from 1.4, 2.1 (ii), 2.4 (ii), and 2.15.

Theorem 2.17.[12] *If* B, C, *and* D *are subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \dots, O_\xi, \dots \rangle$$

such that

$$A = B \times C = B \times D,$$

then

$$C \cong D.$$

Proof: For any given element $a \varepsilon A$ let $f(a)$ be the unique element in C and $g(a)$ the unique element in D such that

$$a = b' + f(a) = b'' + g(a) \text{ for some } b', b'' \varepsilon B.$$

By 1.20 and 1.4 (ii), f is a D,C-homomorphism and g is a C,D-homomorphism. We first show that g is a C,D-isomorphism, and then that this isomorphism is central.

For every element $c \varepsilon C$ we have

(1) $c = b_1 + g(c)$ and $g(c) = b_2 + fg(c)$ where $b_1, b_2 \varepsilon B$.

Therefore, by 1.6 (ii),

(2) $$c = (b_1 + b_2) + fg(c).$$

Since $c, fg(c) \varepsilon C$ and C is a subtractive subalgebra of $\underline{A}$, this implies that $b_1 + b_2 \varepsilon C$. But $b_1 + b_2 \varepsilon B$; hence, by 1.8 (i),

12. For groups this theorem is known; see Kurosh [1], p. 108.

$$b_1 + b_2 = 0.$$

Consequently, by (2),

$$fg(c) = c \text{ for every } c \varepsilon C. \tag{3}$$

Therefore

$$g(c_1) = g(c_2) \text{ implies } c_1 = c_2 \text{ for } c_1, c_2 \varepsilon C. \tag{4}$$

Similarly we obtain

$$gf(d) = d \text{ for } d \varepsilon D.$$

Hence, if $d \varepsilon D$, there is an element $c \varepsilon C$ such that $d = g(c)$. We thus conclude that

$$g^*(C) = D.$$

Therefore, by (4), g is a C, D-isomorphism.

By 2.9 and the hypothesis, there exists a central subalgebra B' of B for which

$$D \subseteq [(B' \times D) \cap B] \times C. \tag{5}$$

By 1.16 and the hypothesis, B is subtractive subalgebra of A. Consequently, by 1.17,

$$(B' \times D) \cap B = B' \times (B \cap D). \tag{6}$$

By 1.8 (i),

$$B \cap D = \{0\}.$$

Hence, by (5) and (6),

$$D \subseteq B' \times C. \tag{7}$$

By 2.6 (ii), B' is a central subalgebra of $\underline{A}$. If $c \varepsilon C$, then n
$g(c) \varepsilon D$ whence, by (7),

$$g(c) = b + c' \text{ with } b \varepsilon B' \text{ and } c' \varepsilon C.$$

Therefore, by (1), (3), 1.4 (ii), and the hypothesis,

(8) $$g(c) = b + c.$$

We have shown that, for every element $c \varepsilon C$, there exists an element $b \varepsilon B'$ which satisfies (8). Hence, by 2.2 (i) and 2.15, g is a central C, D-isomorphism, and consequently,

$$C \cong D.$$

This completes the proof.

Theorem 2.18. *If* B, C, *and* D *are subalgebras of an algebra*

$$\underline{A} = < A, +, O_0, O_1, \ldots, O_\xi, \ldots >$$

such that $B \times C$ *exists and*

$$B \times C = B \times D,$$

then

$$C \cong D.$$

Proof: We can repeat here the first part of the proof of 2.17 without any changes. We could also argue as follows: By 2.17, C and D are central-isomorphic in the algebra $B \times C$; hence we obtain the conclusion by 2.16 (iv).

We conclude this discussion with a rather special theorem, which will be used as a lemma in the next section (in the proof of 3.7).

Theorem 2.19. *If* B *and* C *are subalgebras of an algebra*

$$\underline{A} = < A, +, O_0, O_1, \ldots, O_\xi, \ldots >$$

such that $B \times C$ *exists and is a subtractive subalgebra of* $\underline{A}$, *and if* $B \cong C$, *then* B *and* C *are central subalgebras of* $B \times C$.

Proof: Let D be a central subalgebra of $\underline{A}$ and f a B, C-isomorphism which satisfy the conditions of 2.15. Then every element $b \varepsilon B$ can be represented in the form

(1) $$b = c + d \text{ with } c \varepsilon C \text{ and } d \varepsilon D.$$

We want to show that all the conditions of 2.1, with both A and C replaced by B, are satisfied. If, in fact, b_1, b_2, b_1', $b_2' \varepsilon B$, we have

(2) $b_1' = c_1 + d_1$ and $b_2' = c_2 + d_2$ where c_1, $c_2 \varepsilon C$ and d_1, $d_2 \varepsilon D$.

Therefore, by 2.2 (ii),

$$(b_1 + b_1') + (b_2 + b_2') = [(b_1 + c_1) + d_1] + [(b_2 + c_2) + d_2].$$

Hence, by 1.4 (iii), 2.1 (ii), 2.2 (ii), and (2),

$$(b_1 + b_1') + (b_2 + b_2') = (b_1 + b_2) + (b_1' + b_2').$$

Thus 2.1 (ii) holds. To derive 2.1 (iii) we proceed in a similar way.

Let now b be any element of B. We then have (1); and by applying 2.1 (i) to the subalgebra C = D, we obtain

(3) $$d + \overline{d} = 0 \text{ where } \overline{d} \varepsilon D.$$

We now apply 1.15, first to (1) and then to (3); and we conclude that

$$d, \overline{d} \varepsilon B \times C.$$

Hence, by 1.4 (i),

$$\overline{d} = \overline{b} + \overline{c} \text{ for some } \overline{b} \varepsilon B \text{ and } \overline{c} \varepsilon C.$$

Therefore, by 1.6 (ii), 2.2 (ii), (1), and (3),

$$(b + \overline{b}) + \overline{c} = c.$$

This gives, by 1.4 (ii),

$$b + \overline{b} = 0;$$

so that condition 2.1 (i) is also satisfied.

Thus, B is a central subalgebra of itself, and consequently, by 2.6 (ii), it is a central subalgebra of B × C. For similar reasons C is a central subalgebra of B × C, and the proof is complete.