Let $T \subseteq S$ and $S^{\prime}=T U(S-T)$. Then $S$ and $S^{\prime}$ are congruent.
Proof. First we have $S^{\prime} \subseteq S$ because $T \subseteq S$ and $S-T \subseteq S$. Hence $7(E x)\left(x \in S \& x \notin S^{\prime}\right)$. Therefore it remains only to prove that $7(E x)(x \notin S \&$ $\left.\mathbf{x} \in \mathrm{S}^{\prime}\right)$. But this is equivalent to $7(E x)(\mathbf{x} \notin \mathrm{S} \&(\mathbf{x} \in \mathrm{~T} \cdot \mathrm{v} \cdot \mathbf{x} \in \mathrm{S} \& \mathbf{x} \notin \mathrm{~T}))$ which again is equivalent to $7(E x)((x \notin S \& x \in T) v(x \notin S \& x \in S \& x \in T))$ which is equivalent to ( $E x$ ) $(x \notin S \& x \in T)$ which follows from $(x)(x \in T \rightarrow x \in S)$.

Simple examples of detachable subspecies of the natural number sequence are given by the even or the odd numbers. The linear continuum can be shown to have no other detachable subspecies than itself and the null species.

A species is said to be finite if there is a 1-to-1 correspondence between it and an initial part $1, \ldots, n$ of the natural number series. It is called denumerable if there is such a correspondence between the species and the whole number series. A species is called numerable if it can be mapped onto a detachable subspecies of the sequence of natural numbers.

An important notion is "finitary spread" or, more briefly, 'fan'. A fan is a spread with such a spread law that there are only finitely many allowed first terms, and for every $n$ every admitted sequence with $n$ terms has only a finite number of sequences with $n+1$ terms as admitted continuations. Above all the so-called fan theorem is important here. It says that if $\phi(\sigma)$ is an integral-valued function of $\sigma, \sigma$ varying through the different elements of the fan, then the value of $\phi$ is already determined by a finite initial sequence of $\sigma$. Therefore, if $\phi\left(\sigma_{1}\right)=\mathrm{m}$, there exists an n such that $\phi\left(\sigma_{2}\right)=\mathrm{m}$ as often as $\sigma_{2}$ has the same first $n$ terms as $\sigma_{1}$. An important application of the fan theorem is the proof of the statement that every function which is continuous on a bounded and closed point species is uniformly continuous on the point species. Further, such covering theorems as that of Heine-Borel can be proved. However, not all of the theorems of classical analysis can be proved in intuitionist mathematics.

I must confine my exposition of intuitionism to these scattered remarks A more thorough exposition would require a more complete treatment of intuitionist logic, and that would take more space than I have at my disposal here.

## 17. Mathematics without quantifiers

In all the theories we have treated above we have made use of the logical quantifiers, the universal one and the existential one. We have used them without scruples even in the case of an infinite number of objects. There is now a way of developing mathematics, in particular arithmetic, without the use of these operations which, in the case of an infinite number of objects, may be considered as an extension or extrapolation of conjunction and disjunction in the finite case. If we shall really consider the infinite as something becoming, something not finished or finishable, one might argue that we ought to avoid the quantifiers extended over an infinite range. Such a theory is possible. I myself published in 1923 a first beginning of such a strict finitist mathematics. I treated arithmetic, showing that by the use of
free variables for general statements, basing the theory on the principles of definition by recursion and proof by complete induction, ordinary arithmetic could be developed in a very natural way. Later this theory, called Recursive Arithmetic, has been more perfectly formalized, first in Hilbert Bernays, "'Grundlagen der Mathematik", Vol. 1, 1934, § 7, later also by H. B. Curry (Amer. J. Math. Vol. 63, 1941, pp. 263-282). But the most complete exposition of this kind of mathematics has been given by R. L. Goodstein. He has extended the use of these purely finitist methods also to analysis. However, since this kind of mathematics rather avoids set theory in its proper sense than replaces it by a new form of it, I find no reason to pursue this subject further in these lectures on set theory.

## 18. The possibility of set theory based on many-valued logic

It is well known that it is possible to set forth logical calculi, both propositional calculi and predicate calculi as well, where the statements can have more than the two truth values in classical logic. It is then natural to ask if it should not perhaps be easier to obtain a consistent set theory by taking into account many-valued logics. One might think that it could then perhaps be possible to avoid the distinction of type (and order), even if we maintained a general axiom of comprehension allowing the greater number of truth values. I myself have investigated the possibility of using truth functions of the kind proposed by Łukasiewicz. My results are published in a paper "Bemerkungen zum Komprehensions axiom". (Zeitschr. f. math. Logik und Grundlagen d. Math., Bd. 3, S. 1-17 (1957).) The basic logic is as follows: The truth values are numbers between 0 and 1 . The values of $p \& q, p v q$, $7 p$ are respectively the $\min$ (value of $p$, value of $q$ ), $\max$ (value of $p$, value of $q$ ), 1 - value of $p$. Further the value of $(x) p(x)$ is the minimum of the values of $p(x)$ for the diverse $x$. The value of $(E x) p(x)$ is the maximum of the values of $p(x)$. In the case of finitely many truth values they are the diverse multiples of the least one $\neq 0$. Some of my results are: If we shall have an unrestricted axiom of comprehension, a consistent theory is impossible if the number of truth values is finite. On the other hand, it seems to be possible to obtain a consistent set theory with an unrestricted axiom of comprehension if all rational numbers $\geqq 0$ and $\leqq 1$ are allowed as truth values. I was able to prove that a rudimentary set theory, where the axiom of comprehension

$$
(E y)(x)(x \in y \leftrightarrow \phi(x))
$$

is only used in the case that $\phi(x)$ is built up from the atomic membership propositions by use of the logical connectives, \& $\mathrm{v}, 7$, alone, is consistent. It ought to be noticed, however, that in any set theory where we use quantifiers extended over the whole domain, the set introduced by the axiom of comprehension are defined relative to the total domain, so that the whole theory in that respect is circular. If we want to avoid circularity, we must accept a distinction of the objects we are dealing with into types, orders or layers, or

