$$z = \overline{\chi_P}(y) \leftrightarrow Seq(z) \;\&\; lh(z) = y \;\&\; (\forall i < y)((z)_i = \chi_P(i)).$$

Hence by the table, it will suffice to show that $w = \chi_P(i)$ is Σ^0_{n+1}. Since P is Π^0_n, this follows from

$$w = \chi_P(i) \leftrightarrow (w = 1 \;\&\; P(i)) \vee (w = 0 \;\&\; \neg P(i))$$

and the table. □

14.9. COROLLARY. A relation is Δ^0_{n+1} iff it is recursive in Π^0_n.

Proof. A relation R is Δ^0_{n+1} iff both R and $\neg R$ are Σ^0_{n+1}; hence, by Post's Theorem, iff both R and $\neg R$ are RE in Π^0_n. By the relativized version of 14.6, this holds iff R is recursive in Π^0_n. □

Since $\neg R$ is recursive in R and $R = \neg\neg R$ is recursive in $\neg R$, 12.4 and the table show that we can replace Π^0_n by Σ^0_n in both Post's Theorem and its corollary.

15. Degrees

If F and G are total functions, we let $F \le_R G$ mean that F is recursive in G. By 12.5,

$$(1) \qquad F \le_R F;$$

and by the Transitivity Theorem

$$(2) \qquad F \le_R G \;\&\; G \le_R H \rightarrow F \le_R H.$$

Let $F \equiv_R G$ mean $F \le_R G \;\&\; G \le_R F$. It follows from (1) and (2) that $\equiv_R$ is an equivalence relation. The equivalence class of F is called the <u>degree</u> of F and is designated by dg F. By a <u>degree</u>, we mean the degree of some total function. We use small boldface letters, usually **a**, **b**, **c**, and **d**, for degrees.

We let $\mathrm{dg}(F) \le \mathrm{dg}(G)$ if $F \le_R G$. By (2), this depends only on dg(F) and dg(G), not on the choice of F and G in these equivalence classes. It follows from (1) and (2) that $\le$ is a partial ordering of the degrees, i.e., that

$$\mathbf{a} \le \mathbf{a},$$

$$\mathbf{a} \le \mathbf{b} \;\&\; \mathbf{b} \le \mathbf{a} \rightarrow \mathbf{a} = \mathbf{b},$$

$$\mathbf{a} \leq \mathbf{b} \ \&\ \mathbf{b} \leq \mathbf{c} \rightarrow \mathbf{a} \leq \mathbf{c}.$$

The degree of F is, roughly speaking, a measure of the difficulty of computing F; and $\mathrm{dg}\ F \leq \mathrm{dg}\ G$ means that F is at least as easy to compute as G.

By the degree of a relation R, we mean the degree of χ_R; we will sometimes say (with an abuse of language) that R belongs to that degree. Every degree is the degree of a relation; for a total function has the same degree as its graph (because of the equivalences $\mathcal{G}_F(\vec{x},y) \leftrightarrow F(\vec{x}) = y$ and $F(\vec{x}) = \mu y\, \mathcal{G}_F(\vec{x},y)$.) A total function or relation has the same degree as its contraction by the contraction equations; so every degree contains a real and a set.

Let $\mathbf{0}$ be the class of recursive total functions. It is easy to see that $\mathbf{0}$ is a degree and that $\mathbf{0} \leq \mathbf{a}$ for every degree $\mathbf{a}$. Thus $\mathbf{0}$ is the smallest degree.

15.1. Proposition. Every finite class of degrees has a least upper bound.

Proof. Let the degrees in the class be $\mathrm{dg}(F_1)$, ... $\mathrm{dg}(F_k)$ where F_1, ..., F_k are reals. Set $G(x) = \langle F_1(x),\dots,F_k(x)\rangle$. Since $F_i(x) = (G(x))_{i-1}$, F_i is recursive in G; so $\mathrm{dg}\ G$ is an upper bound of the set. Now let $\mathrm{dg}(H)$ be any upper bound. Then each F_i is recursive in H. Since G is recursive in $F_1,\dots,F_k$, it is recursive in H; so $\mathrm{dg}\ G \leq \mathrm{dg}\ H$, as required. □

Most of the rest of degree theory consists of showing that the partial ordering of the degrees fails to have some nice properties. We shall illustrate the idea by proving that it is not a linear ordering.

15.2. Proposition (Kleene–Post). There are degrees $\mathbf{a}$ and $\mathbf{b}$ such that neither $\mathbf{a} \leq \mathbf{b}$ nor $\mathbf{b} \leq \mathbf{a}$.

Proof. It suffices to produce reals F and G such that neither $F \leq_{\mathrm{R}} G$ nor $G \leq_{\mathrm{R}} F$. We break this down into infinitely many conditions which we wish to satisfy. Condition C_{2e} is that $F \neq \{e\}^G$; condition C_{2e+1} is that $G \neq \{e\}^F$.

We define F and G in infinitely many steps, at each of which we define finitely many values of F and G. At step e we ensure that C_e is satisfied. We shall only describe step $2e$, since step $2e+1$ is similar.

Suppose we are at the beginning of step $2e$. Let x be the least number such that $F(x)$ is not yet defined. Suppose first that there is a z in Seq such that $T_{1,1}(e,x,z)$ and such that $(z)_i = G(i)$ for every $i < lh(z)$ such that $G(i)$ is defined. Define $G(i) = (z)_i$ for every $i < lh(z)$ such that $G(i)$ is not yet defined. Then $\overline{G}(lh(z)) = z$ and hence $T_{1,1}(e,x,\overline{G}(lh(z)))$. It follows that $\{e\}^G(x) = U(lh(z))$. We set $F(x) = 1 \dot{-} U(lh(z))$, so that $F(x) \neq \{e\}^G(x)$.

Now suppose that there is no such z. Then we know that $\{e\}^G(x)$ will be undefined, so that we will have $F \neq \{e\}^G$. We set $F(x) = 0$ and define no new values of G.

It remains to show that F and G are total. At step $2e$, we defined F at the smallest argument for which it was not already defined. It follows that F is total. Similarly, the action at step $2e+1$ makes G total. □

A degree **a** is RE if it contains an RE relation. Since the contraction of an RE relation is an RE set, every RE degree contains an RE set. (However, every RE degree other than **0** contains sets which are not RE.)

The degree of $W_e(x)$, considered as a relation of e and x, is designated by **0'**. This degree is RE; and, since every RE set is clearly recursive in this relation, it is the largest RE degree. Since there is an RE set which is not recursive, there is an RE degree other than **0**; so **0** < **0'**.

We shall show that there is a connection between **0'** and limits. An infinite sequence $\{F_s\}$ of reals is recursive if the function G defined by $G(s,x) \simeq F_s(x)$ is recursive. We say F is the limit of $\{F_s\}$ if for each x, there is an s_0 such that $F_s(x) = F(x)$ for all $s \geq s_0$.

15.3. Lemma. If $\{F_s\}$ is a recursive sequence of reals, then $\{e\}_s^{F_s}(\vec{x}) \simeq z$ and $\{e\}_s^{F_s}(x)$ *is defined* are recursive relations of $e,s,\vec{x},z$. If F is the limit of $\{F_s\}$ and $\{e\}^F(x) \simeq z$, then $\{e\}_s^{F_s}(x) \simeq z$ for all sufficiently large s.

Proof. Making use of (1) of §12,

$$\{e\}_s^{F_s}(\vec{x}) \simeq z \leftrightarrow (\exists y < s)(T_{k,m}(e,\vec{x},y,\overline{F_s}(y)) \ \&\ U(y) \simeq z),$$

$$\{e\}_s^{F_s}(\vec{x}) \text{ is defined} \leftrightarrow (\exists y < s)\, T_{k,m}(e,\vec{x},y,\overline{F_s}(y)).$$

If $G(s,x) = F_s(x)$, $\overline{G}(s,x) = \overline{F_s}(x)$; so $\overline{F_s}(x)$ is a recursive function of s and x. This proves the first part of the proposition. Suppose that $\{e\}^F(\vec{x}) \simeq z$ and that y is the computation number of $\{e\}^F(\vec{x})$. Again using (1) of §12, $\{e\}^{F_s}(\vec{x}) \simeq z$ if $\overline{F_s}(y) = \overline{F}(y)$, which is true for large s. But then $\{e\}_s^{F_s}(x) \simeq z$ for $s > y$. □

15.4. Limit Lemma. A real is the limit of a recursive sequence of reals iff it has degree $\leq \mathbf{0}'$.

Proof. Let F be the limit of the recursive sequence $\{F_s\}$, and define an RE R by

$$R(n,x) \leftrightarrow \exists m(m > n \ \&\ F_m(x) \neq F_n(x)).$$

Setting $H(x) \simeq \mu n \neg R(n,x)$, H is total and $F(x) \simeq F_{H(n)}(x)$. It follows that F is recursive in R; so $\text{dg } F \leq \text{dg } R \leq \mathbf{0}'$.

Now suppose that $\text{dg } F \leq \mathbf{0}'$. Then F is recursive in an RE set. Say that $F = \{f\}^{W_e}$. Define $F_s(x) = \{f\}_s^{W_{e,s}}(x)$ if the right side is defined and $F_s(x) = 0$ otherwise. Since $\{W_{e,s}\}$ is recursive and has limit W_e, 15.3 shows that $\{F_s\}$ is recursive and has limit F. □

The problem of whether there is an RE degree other than $\mathbf{0}$ and $\mathbf{0}'$ is known as Post's Problem. We solve this problem by extending 15.2 to RE degrees. The method used is called the priority method.

By an RE construction, we mean a construction in which at each step, a finite number of numbers are put into a set A, and the construction is recursive (i.e., x is put into A at step s is a recursive function of x and s). Then A is RE, since

$$x \in A \leftrightarrow \exists s(x \text{ is put into } A \text{ at step } s).$$

15.5. PROPOSITION (FRIEDBERG–MUCHNIK). There are RE degrees $\mathbf{a}$ and $\mathbf{b}$ such that neither $\mathbf{a} \leq \mathbf{b}$ nor $\mathbf{b} \leq \mathbf{a}$.

Proof. We shall use an RE construction to construct RE sets A and B such that neither $A \leq_R B$ nor $B \leq_R A$. We shall use Church's Thesis to verify that the construction is recursive; this is clearly a non–essential use of Church's Thesis. We wish to satisfy the conditions C_e, where C_{2e} is $A \neq \{e\}^B$ and C_{2e+1} is $B \neq \{e\}^A$. We shall discuss only C_{2e}; C_{2e+1} is treated similarly but with A and B interchanged.

Let us consider first how we could make C_{2e} hold if we had no other conditions to worry about. Pick an x; we will make C_{2e} hold by insuring that $x \in A$ iff $\{e\}^B(x) \simeq 1$. Let B^s be the (finite) set of number put into B before step s. At step s, if x has not yet been put in A, we see if $\{e\}_s^{B^s}(x) \simeq 1$; we can do this effectively by 15.3. If $\{e\}_s^{B^s}(x) \simeq 1$, we put x into A, and agree not to put any numbers $< s$ into B at step s or later; otherwise, we do nothing at step s. If $\{e\}^B(x) \simeq 1$, then $\{e\}_s^{B^s}(x) \simeq 1$ for all sufficiently large s by 15.3; so $x \in A$. Suppose that $x \in A$; say x is put into A at step s. Then $\{e\}^{B^s}(x) \simeq 1$ with computation number $z < s$. Since no number $< s$ is put into B at step s or later, $\overline{\chi_{B^s}}(z) = \overline{\chi_B}(z)$; so $\{e\}^B(x) \simeq 1$.

When we try to treat all of the conditions at once, we run into conflicts; it may happen that C_e wants to put a number x into A and C_f wants to keep x out of A. We resolve this conflict be giving priority to the lower numbered condition. Thus if $e < f$, we put x into A; if $f < e$, we keep x out of A. (No condition will conflict with itself.)

Let $r_s(e)$ be the largest $t < s$ such that a number is put into A or B by C_e at step t (or 0 if there is no such t). Let $R_s(e)$ be the maximum of the $r_s(f)$ for $f < e$, and let $x_s(e) = \langle e, R_s(e)\rangle$. Then $x_s(e)$ is known at the beginning of step s.

(The numbers $< r_s(e)$ are the numbers C_e wants to keep out of A and B at step s. Hence C_e can put $x_s(e) > R_s(e)$ into A or B at s without violating our priorities.)

Now we describe step s. Let $f = (s)_0$ and $x = x_s(f)$. If $f = 2e$ and $\{e\}_s^{B^s}(x) \simeq 1$, we put x in A (unless it has been put in A earlier); if $f = 2e+1$ and $\{e\}_s^{A^s}(x) \simeq 1$, we put x in B.

We now have to prove that the construction works, i.e., that all of the conditions are satisfied. We shall first prove that for each e, there are only finitely many numbers $x_s(e)$. The proof is by induction on e. It is clearly sufficient to prove that for each $f < e$, C_f puts only finitely numbers into A and B. But any number put into A or B by C_f is $x_s(f)$ for some some f; and there are only finitely many $x_s(f)$ by the induction hypothesis.

Now we show that C_{2e} is satisfied. Let x be the largest of the numbers $x_s(2e)$; we show that $x \in A$ iff $\{e\}^B(x) \simeq 1$. Since $x_s(2e)$ is increasing in s, $x_s(2e) = x$ for all sufficiently large s. If $\{e\}^B(x) \simeq 1$, then $\{e\}_s^{B^s}(x) \simeq 1$ for all large s. Choosing s this large with $(s)_0 = 2e$, we see that x is put into A. Suppose that x is put into A at step s. Since x is of the form $<2e,z>$, it is put into A by (C_{2e}); so $\{e\}_s^{B^s}(x) \simeq 1$. Hence it is enough to show that no number $< s$ is put into B after step s. Suppose that $x_t(f) < s$ is put into B at step $t > s$. Then f is odd; so $f \neq 2e$. If $2e < f$, then $x_t(f) > R_t(f) \geq r_t(2e) \geq s$, a contradiction. Thus $f < 2e$; so $R_{t+1}(2e) \geq r_{t+1}(f) \geq t > s \geq R_s(2e)$ and hence $x_{t+1}(2e) > x_s(2e) = x$, contradicting the choice of x. A similar proof shows that C_{2e+1} is satisfied. □

If $F \equiv_R G$, then by 12.4 the same functions are recursive in F and G. Hence if we have relativized some property Q, then <u>Q in F</u> is equivalent to <u>Q in G</u>. Thus if $\mathbf{a} = \text{dg } F$, we may as well say <u>$Q$ in $\mathbf{a}$</u> for <u>Q in F</u>. We define a degree $\mathbf{b}$ to be <u>RE in $\mathbf{a}$</u> if $\mathbf{b}$ contains a relation RE in $\mathbf{a}$. The relativization of the Friedberg–Muchnik Theorem tells us that for each $\mathbf{a}$, there are degrees $\mathbf{b}$ and $\mathbf{c}$

RE in **a** such that neither **b** $\leq$ **c** nor **c** $\leq$ **b**. The method used to define **0'**, when relativized, shows that there is a largest degree RE in **a**. It is called the jump of **a**, and is designated by **a'**. The relativized Limit Lemma shows that a real is a limit of a recursive in **a** sequence of reals iff it has degree $\leq$ **a'**.

16. Evaluation of Degrees

We shall now show how to evaluate the degrees of certain explicitly given relations.

Let Φ be a class of relations. We say a relation R is Φ complete if R is in Φ and every relation in Φ is reducible to R (where reducible is defined before 13.3). It follows that R has the largest degree of any relation in Φ; so any two Φ complete relations have the same degree. (Caution: Some authors use complete in a somewhat different way.)

EXAMPLE. If F is total, $W_e^F(x)$ is RE in F complete; its degree is the jump of dg F. Hence any RE in F complete relation has degree (dg F)'.

The degree obtained by applying the jump n times to **0** is designated by $\mathbf{0}^n$.

16.1. PROPOSITION. For every n, there is a Σ_n^0 complete set of degree $\mathbf{0}^n$ and a Π_n^0 complete set of degree $\mathbf{0}^n$.

Proof. We use induction on n. If $n = 1$, let P be a recursive set; if $n > 1$, let P be a Π_{n-1}^0 complete set of degree $\mathbf{0}^{n-1}$. Then $W_e^P(x)$ has degree $\mathbf{0}^n$ by the example. By Post's Theorem, Σ_n^0 is the class of relation RE in P; so $W_e^P(x)$ is Σ_n^0 complete. Then $\neg W_e^P(x)$ is of degree $\mathbf{0}^n$ and is Π_n^0 complete. □

16.2. COROLLARY. Every Σ_n^0 complete or Π_n^0 complete relation has degree $\mathbf{0}^n$. □

If Φ is a class of RE sets, then the set of indices of sets in Φ is called the index set of Φ.

16.3. PROPOSITION (RICE). If Φ is a non-empty class of RE sets which is