31 Borel metric spaces and lines in the plane

We give two applications of Harrington's technique of using Gandy forcing. First let us begin by isolating a principal which we call overflow. It is an easy consequence of the Separation Theorem.

Lemma 31.1 *(Overflow) Suppose $\theta(x_1, x_2, \ldots, x_n)$ is a Π^1_1 formula and A is a Σ^1_1 set such that*

$$\forall x_1, \ldots, x_n \in A \;\; \theta(x_1, \ldots, x_n).$$

Then there exists a Δ^1_1 set $D \supseteq A$ such that

$$\forall x_1, \ldots, x_n \in D \;\; \theta(x_1, \ldots, x_n).$$

proof:

For $n = 1$ this is just the Separation Theorem 27.5.

For $n = 2$ define

$$B = \{x : \forall y(y \in A \rightarrow \theta(x, y))\}.$$

Then B is Π^1_1 set which contains A. Hence by separation there exists a Δ^1_1 set E with $A \subseteq E \subseteq B$. Now define

$$C = \{x : \forall y(y \in E \rightarrow \theta(x, y)\}.$$

Then C is a Π^1_1 set which also contains A. By applying separation again we get a Δ^1_1 set F with $A \subseteq F \subseteq C$. Letting $D = E \cap F$ does the job. The proof for $n > 2$ is similar.

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We say that (B, δ) is a *Borel metric space* iff B is Borel, δ is a metric on B, and for every $\epsilon \in \mathbb{Q}$ the set

$$\{(x, y) \in B^2 : \delta(x, y) \leq \epsilon\}$$

is Borel.

Theorem 31.2 *(Harrington [39]) If (B, δ) is a Borel metric space, then either (B, δ) is separable (i.e., contains a countable dense set) or for some $\epsilon > 0$ there exists a perfect set $P \subseteq B$ such that $\delta(x, y) > \epsilon$ for every distinct $x, y \in P$.*

proof:

By relativizing the proof to an arbitrary parameter we may assume that B and the sets $\{(x, y) \in B^2 : \delta(x, y) \leq \epsilon\}$ are Δ^1_1.

Lemma 31.3 *For any $\epsilon \in \mathbb{Q}^+$ if $A \subseteq B$ is Σ^1_1 and the diameter of A is less than ϵ, then there exists a Δ^1_1 set D with diameter less than ϵ and $A \subseteq D \subseteq B$.*

proof:

This follows from Lemma 31.1, since

$$\theta(x,y) \quad \text{iff} \quad \delta(x,y) < \epsilon \text{ and } x,y \in B$$

is a Π^1_1 formula.

■

For any $\epsilon \in \mathbb{Q}^+$ look at

$$Q_\epsilon = \bigcup\{D \in \Delta^1_1 : D \subseteq B \text{ and } \mathrm{diam}(D) < \epsilon\}.$$

Note that Q is a Π^1_1 set. If for every $\epsilon \in \mathbb{Q}^+$ $Q_\epsilon = B$, then since there are only countably many Δ^1_1 sets, (B,δ) is separable and we are done. On the other hand suppose for some $\epsilon \in \mathbb{Q}^+$ we have that

$$P_\epsilon = B \setminus Q_\epsilon \neq \emptyset.$$

Lemma 31.4 *For every $c \in V \cap B$*

$$P_\epsilon \Vdash \delta(\mathring{a}, \check{c}) > \epsilon/3$$

where $\Vdash$ is Gandy forcing and $\mathring{a}$ is a name for the generic real (see Lemma 30.2).

proof:

Suppose not. Then there exists $P \leq P_\epsilon$ such that

$$P \Vdash \delta(a,c) \leq \epsilon/3.$$

Since P is disjoint from Q_ϵ by Lemma 31.3 we know that the diameter of P is $\dot{\geq} \epsilon$. Let

$$R = \{(a_0,a_1) : a_0, a_1 \in P \text{ and } \delta(a_0,a_1) > (2/3)\epsilon\}.$$

Then R is in $\mathbb{P}$ and by Lemma 30.3, if a is $\mathbb{P}$-generic over V with $a \in R$, then a_0 and a_1 are each separately $\mathbb{P}$-generic over V. But $a_0 \in R$ and $a_1 \in R$ means that $\delta(a_0,c) \leq \epsilon/3$ and $\delta(a_1,c) \leq \epsilon/3$. But by absoluteness $\delta(a_0,a_1) > (2/3)\epsilon$. This contradicts the fact that δ must remain a metric by absoluteness.

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Using this lemma and Lemma 30.6 is now easy to get a perfect set $P \subseteq B$ such that $\delta(x,y) > \epsilon/3$ for each distinct $x,y \in P$. This proves Theorem 31.2.

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Theorem 31.5 *(van Engelen, Kunen, Miller [20]) For any Σ^1_1 set A in the plane, either A can be covered by countably many lines or there exists a perfect set $P \subseteq A$ such that no three points of P are collinear.*

proof:

This existence of this proof was pointed out to me by Dougherty, Jackson, and Kechris. The proof in [20] is more elementary.

By relativizing the proof we may as well assume that A is Σ^1_1.

Lemma 31.6 *Suppose B is a Σ^1_1 set lying on a line in the plane. Then there exists a Δ^1_1 set D with $B \subseteq D$ such that all points of D are collinear.*

proof:

This follows from Lemma 31.1 since

$$\theta(x, y, z) \quad \text{iff} \quad x, y, \text{ and } z \text{ are collinear}$$

is Π^1_1 (even Π^0_1).

■

Define

$$\sim P = \bigcup \{D \subseteq \mathbb{R}^2 : D \in \Delta^1_1 \text{ and all points of } D \text{ are collinear}\}.$$

It is clear that $\sim P$ is Π^1_1 and therefore P is Σ^1_1. If $P \cap A = \emptyset$, then A can be covered by countably many lines.

So assume that

$$Q = P \cap A \neq \emptyset.$$

For any two distinct points in the plane, p and q, let line(p, q) be the unique line on which they lie.

Lemma 31.7 *For any two distinct points in the plane, p and q, with $p, q \in V$*

$$Q \Vdash \mathring{a} \notin \text{line}(\check{p}, \check{q}).$$

proof:

Suppose for contradiction that there exists $R \leq Q$ such that

$$R \Vdash \mathring{a} \in \text{line}(\check{p}, \check{q}).$$

Since R is disjoint from

$$\bigcup \{D \subseteq \mathbb{R}^2 : D \in \Delta^1_1 \text{ and all points of } D \text{ are collinear}\}$$

it follows from Lemma 31.6 that not all triples of points from R are collinear. Define the nonempty Σ^1_1 set

$$S = \{a : a_0, a_1, a_2 \in R \text{ and } a_0, a_1, a_2 \text{ are not collinear}\}$$

where $a = (a_0, a_1, a_2)$ via some standard tripling function. Then $S \in \mathbb{P}$ and by the obvious generalization of Lemma 30.3 each of the a_i is $\mathbb{P}$-generic if a is. But this is a contradiction since all $a_i \in \text{line}(p, q)$ which makes them collinear.

■

The following Lemma is an easy generalization of Lemma 30.6 so we leave the proof to the reader.

Lemma 31.8 *Suppose M is a countable transitive model of ZFC* and $\mathbb{P}$ is a partially ordered set in M. Then there exists $\{G_x : x \in 2^\omega\}$, a "perfect" set of $\mathbb{P}$-filters, such that for every x, y, z distinct, we have that (G_x, G_y, G_z) is $\mathbb{P} \times \mathbb{P} \times \mathbb{P}$-generic over M.*

Using Lemma 31.7 and 31.8 it is easy to get (just as in the proof of Theorem 30.1) a perfect set of triply generic points in the plane, hence no three of which are collinear. This proves Theorem 31.5.
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Obvious generalizations of Theorem 31.5 are:

1. Any $\utilde{\Sigma}^1_1$ subset of $\mathbb{R}^n$ which cannot be covered by countably many lines contains a perfect set all of whose points are collinear.

2. Any $\utilde{\Sigma}^1_1$ subset of $\mathbb{R}^2$ which cannot be covered by countably many circles contains a perfect set which does not contain four points on the same circle.

3. Any $\utilde{\Sigma}^1_1$ subset of $\mathbb{R}^2$ which cannot be covered by countably many parabolas contains a perfect set which does not contain four points on the same parabola.

4. For any n any $\utilde{\Sigma}^1_1$ subset of $\mathbb{R}^2$ which cannot be covered by countably many polynomials of degree $< n$ contains a perfect set which does not contain $n+1$ points on the same polynomial of degree $< n$.

5. Higher dimensional version of the above involving spheres or other surfaces.

A very general statement of this type is due to Solecki [100]. Given any Polish space X, family of closed sets Q in X, and analytic $A \subseteq X$; either A can be covered by countably many elements of Q or there exists a G_δ set $B \subseteq A$ such that B cannot be covered by countably many elements of Q. Solecki deduces Theorem 31.5 from this.

Another result of this type is known as the Borel-Dilworth Theorem. It is due to Harrington [39]. It says that if $\mathbb{P}$ is a Borel partially ordered set, then either $\mathbb{P}$ is the union of countably many chains or there exist a perfect set P of pairwise incomparable elements. One of the early Lemmas from [39] is the following:

Lemma 31.9 *Suppose A is a Σ^1_1 chain in a Δ^1_1 poset $\mathbb{P}$. Then there exists a Δ^1_1 superset $D \supseteq A$ which is a chain.*

proof:

Suppose $\mathbb{P} = (P, \leq)$ where P and $\leq$ are Δ^1_1. Then

$$\theta(x,y) \quad \text{iff} \quad x,y \in P \text{ and } (x \leq y \text{ or } y \leq x)$$

is Π^1_1 and so the result follows by Lemma 31.1.
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For more on Borel linear orders, see Louveau [65]. Louveau [66] is a survey paper on Borel equivalence relations, linear orders, and partial orders.

Q.Feng [22] has shown that given an open partition of the two element subsets of ω^ω, that either ω^ω is the union of countably many $0-$homogenous sets or there exists a perfect $1-$homogeneous set. Todorcevic [109] has given an example showing that this is false for Borel partitions (even replacing open by closed).