Part III
Classical Separation Theorems

26 Souslin-Luzin Separation Theorem

Define $A \subseteq \omega^\omega$ to be κ-Souslin iff there exists a tree $T \subseteq \bigcup_{n<\omega}(\kappa^n \times \omega^n)$ such that

$$y \in A \text{ iff } \exists x \in \kappa^\omega\ \forall n < \omega\ \ (x \restriction n, y \restriction n) \in T.$$

In this case we write $A = p[T]$, the projection of the infinite branches of the tree T. Note that ω-Souslin is the same as $\mathbf{\Sigma}^1_1$.

Define the κ-Borel sets to be the smallest family of subsets of ω^ω containing the usual Borel sets and closed under intersections or unions of size κ and complements.

Theorem 26.1 *Suppose A and B are disjoint κ-Souslin subsets of ω^ω. Then there exists a κ-Borel set C which separates A and B, i.e., $A \subseteq C$ and $C \cap B = \emptyset$.*

proof:

Let $A = p[T_A]$ and $B = p[T_B]$. Given a tree $T \subseteq \bigcup_{n<\omega}(\kappa^n \times \omega^n)$, and $s \in \kappa^{<\omega}$, $t \in \omega^{<\omega}$ (possibly of different lengths), define

$$T^{s,t} = \{(\hat{s},\hat{t}) \in T : (s \subseteq \hat{s} \text{ or } \hat{s} \subseteq s) \text{ and } (t \subseteq \hat{t} \text{ or } \hat{t} \subseteq t)\}.$$

Lemma 26.2 *Suppose $p[T_A^{s,t}]$ cannot be separated from $p[T_B^{r,t}]$ by a κ-Borel set. Then for some $\alpha < \kappa$ the set*
$p[T_A^{s\frown\alpha,t}]$ cannot be separated from $p[T_B^{r,t}]$ by a κ-Borel set.

proof:

Note that $p[T_A^{s,t}] = \bigcup_{\alpha<\kappa} p[T_A^{s\frown\alpha,t}]$. If there were no such α, then for every α we would have a κ-Borel set C_α with

$$p[T_A^{s\frown\alpha,t}] \subseteq C_\alpha \text{ and } C_\alpha \cap p[T_B^{r,t}] = \emptyset.$$

But then $\bigcup_{\alpha<\kappa} C_\alpha$ is a κ-Borel set separating $p[T_A^{s,t}]$ and $p[T_B^{r,t}]$.
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Lemma 26.3 *Suppose $p[T_A^{s,t}]$ cannot be separated from $p[T_B^{r,t}]$ by a κ-Borel set. Then for some $\beta < \kappa$*
$p[T_A^{s,t}]$ cannot be separated from $p[T_B^{r\frown\beta,t}]$ by a κ-Borel set.

proof:

Since $p[T_B^{r,t}] = \bigcup_{\beta<\kappa} p[T_B^{r\frown\beta,t}]$, if there were no such β then for every β we would have κ-Borel set C_β with

$$p[T_A^{s,t}] \subseteq C_\beta \text{ and } C_\beta \cap p[T_B^{r\frown\beta,t}] = \emptyset.$$

But then $\bigcap_{\beta<\kappa} C_\beta$ is a κ-Borel set separating $p[T_A^{s,t}]$ and $p[T_B^{r,t}]$.
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Lemma 26.4 *Suppose $p[T_A^{s,t}]$ cannot be separated from $p[T_B^{r,t}]$ by a κ-Borel set. Then for some $n < \omega$*
$p[T_A^{s,t\hat{}n}]$ cannot be separated from $p[T_B^{r,t\hat{}n}]$ by a κ-Borel set.

proof:
Note that

$$p[T_A^{s,t\hat{}n}] = p[T_A^{s,t}] \cap [t\hat{}n]$$

and

$$p[T_B^{r,t\hat{}n}] = p[T_B^{r,t}] \cap [t\hat{}n].$$

Thus if $C_n \subseteq [t\hat{}n]$ were to separate $p[T_A^{s,t\hat{}n}]$ and $p[T_B^{r,t\hat{}n}]$ for each n, then $\bigcup_{n<\omega} C_n$ would separate $p[T_A^{s,t}]$ from $p[T_B^{r,t}]$.
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To prove the separation theorem apply the lemmas iteratively in rotation to obtain, $u, v \in \kappa^\omega$ and $x \in \omega^\omega$ so that for every n, $p[T_A^{u\restriction n, x\restriction n}]$ cannot be separated from $p[T_B^{v\restriction n, x\restriction n}]$. But necessarily, for every n

$$(u \restriction n, x \restriction n) \in T_A \text{ and } (v \restriction n, x \restriction n) \in T_B$$

otherwise either $p[T_A^{u\restriction n, x\restriction n}] = \emptyset$ or $p[T_B^{v\restriction n, x\restriction n}] = \emptyset$ and they could be separated. But this means that $x \in p[T_A] = A$ and $x \in p[T_B] = B$ contradicting the fact that they are disjoint.
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