Contributions of K. Gödel to Relativity and Cosmology *

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Summary. K Gödel published two seminal papers on general relativity theory and its application to the study of cosmology. The first examined a non-expanding but rotating solution of the Einstein field equations, in which causality is violated; this lead to an in-depth examination of the concepts of causality and time in curved space-times. The second examined properties of a family of rotating and expanding spatially homogeneous solutions of the Einstein equations, which was a forerunner of many studies of such cosmologies. Together they stimulated examination of themes that were fundamental in the development of the Hawking-Penrose singularity theorems and in studies of cosmological dynamics. I review these two papers, and the developments that resulted from them.

1. Introduction

Gödel became interested in general relativity theory while he and Einstein were both on staff of the Institute for Advanced Studies in Princeton. Apparently they discussed the subject together often. His resultant two papers had a major impact:

> Curiously, the beginning of the modern studies of singularities in general relativity in many ways had its seeds in the presentation by Kurt Gödel (1949) of an exact solution of Einstein's equations for pressure-free matter, which could be thought of as a singularity-free, rotating but non-expanding cosmological model ... [this paper] was one of the papers presented in a special issue of *Reviews of Modern Physics* dedicated to Einstein on his 70th birthday. Gödel used this space-time as an example helping to clarify the nature of time in general relativity, for it is an exact solution of the Einstein equations in which there are closed timelike lines: an observer can travel into his own past, and (as an old man) stand alongside himself (as a young man). He shortly thereafter published a further paper (1952) discussing a family of exact solutions of Einstein's equations representing rotating and expanding spatially homogeneous universe models (and relying on the geometric results derived many decades earlier by Sophus Lie and Luigi Bianchi). As these permit non-zero redshifts, they could include realistic models of the observed universe.

* This paper is in its final form and no similar paper has been or is being submitted elsewhere

> These papers perhaps more than any other antecedents of later work particularly stimulated investigations leading to fruitful developments. (This may partly have been due to the enigmatic style in which they were written: literally for decades after, much effort was invested in giving proofs for some of the results stated without proof by Gödel).

(Tipler Clarke and Ellis 1980, pp. 111-112). This is what I will explore in the sequel. The discussion that follows cannot possibly consider all developments from these papers (according to the Science Citation Index, the first paper has been the subject of 220 citations between 1965 and 1993 and the second 47 citations in the same period); rather I will concentrate on main themes and arguments that have arisen.

2. Gödel's stationary universe

Gödel's paper of 1949 gave the first exact rotating fluid-filled cosmological solution of Einstein's gravitational field equations. It is uniquely characterized by its symmetry properties, for it is a highly symmetric space-time: it is the only perfect-fluid filled universe invariant under a G_5 of isometries multiply transitive on space-time, which is space-time homogeneous (there is a 4-dimensional subgroup of isometries simply transitive on space-time) and locally rotationally symmetric (there is a 1-dimensional isotropy group acting about each space-time point) (Ellis 1967). Thus every space-time point is equivalent to every other one, and the universe is axially symmetric about every event. However it is not spatially homogeneous, because there is no family of spatially homogeneous 3-surfaces in the space-time.

Because this universe is space-time homogeneous, the density μ and pressure p are the same everywhere, and hence (using the standard notation for kinematic variables, see Ehlers 1961, Ellis 1971) it does not expand ($\theta = 0$) and matter moves geodesically ($\dot{u}_a = 0$). It also has zero shear ($\sigma = 0$), so the matter velocity vector is a Killing vector field but is not hypersurface orthogonal:

$$u_{a;b} = u_{[a;b]} = \omega_{ab} \neq 0 \tag{2.1}$$

(i.e. it generates a timelike symmetry, making it stationary) and the only non-zero kinematic quantity is the vorticity ($\omega \neq 0$). The vorticity vector is covariantly constant:

$$\omega^a{}_{;c} = 0 \quad \Leftrightarrow \quad \omega_{ab;c} = 0\,. \tag{2.2}$$

The kinematic description (1),(2) uniquely characterizes these space-times (Ehlers 1961, Theorem 1.5.2 and 2.5.4). Thus the homogeneous substratum rotates uniformly relative to the local compass of inertia: ω^2 is constant everywhere.

From these properties it follows (Ellis 1971 Section 5.2) that the electric part of the Weyl tensor is non-zero and is given by

$$E_{ab} = -\omega_a\omega_b + h_{ab}\frac{1}{3}\omega^2\,,\; E_{ab;c} = 0 \tag{2.3}$$

but the magnetic part of the Weyl tensor is zero: $H_{ab} = 0$. Because of the rotational symmetry, the Weyl tensor is Petrov type D.

The matter source in the original solution is pressure-free matter, but there is a cosmological constant of negative sign (the opposite sign to that usually encountered). More generally one can regard the matter source as being a perfect fluid. The only non-trivial covariant field equation is the Raychaudhuri equation, which with restrictions (1), (2) becomes

$$\Lambda + 2\omega^2 = \frac{1}{2}\kappa(\mu + 3p). \tag{2.4}$$

The Bianchi identities give

$$E_s{}^{(m}\omega^{t)s} = 0\,, \tag{2.5}$$

$$-3E^t{}_s\omega^s = \kappa(\mu + p)\omega^t\,, \tag{2.6}$$

the first of which is identically satisfied and the second of which leads to the relation

$$2\omega^2 = \kappa(\mu + p)\,. \tag{2.7}$$

With the Raychaudhuri equation (4) this gives

$$\Lambda = \frac{1}{2}\kappa(-\mu + p). \tag{2.8}$$

Hence in the pressure-free case we get

$$\Lambda = -\frac{1}{2}\kappa\mu = -\omega^2 < 0. \tag{2.9}$$

One can alternatively represent it as a fluid or scalar field with

$$\Lambda = 0 \Rightarrow p = \mu = \omega^2/\kappa\,. \tag{2.10}$$

There are many ways to construct this solution, because of its high symmetry. Gödel himself apparently used a deformation of a metric (Klein's fundamental quadric) along a family of timelike lines at constant distance ('Clifford parallels') generating a space invariant under a 4-parameter simply transitive group of isometries.

Gödel used this exact solution of the Einstein equations to examine properties of time and causality in general relativity. Using axially symmetric comoving coordinates centered on a chosen world line, the metric tensor is

$$ds^2 = 2\omega^{-2}(-dt^2 + dz^2 + dr^2 - (\sinh^4 r - \sinh^2 r)d\phi^2 + 2\sqrt{2}\sinh^2 r\, d\phi\, dt)\,.$$

The light cones tip over more and more the further one moves out (Figure 31 in Hawking and Ellis 1973), so that for large enough r, the circles $\{r, t, x\, const\}$ are closed timelike lines. This demonstrates *causality violation*: an observer traveling on this path from some event P will end up, after some proper time has elapsed, at the same space-time event P; thus she can. as an old woman, stand next to herself as a young woman. Various paradoxes ensue (the old person can kill the young one at event P, for example, but then there will be no older person at that event who can kill the young one, for she will not have survived - in which case the young one survives after all until arriving at P, and is then able as an old person to kill the young one ...). Furthermore by traveling far enough away, any observer can reach an arbitrarily distant event in the past on her own world line, and so influence events in her own past history at an arbitrary early proper time in that history.

The essential point demonstrated is that the Einstein Field Equations, determining space-time curvature from the matter present, are compatible with such causal violation. Until this solution was discovered, it had been taken for granted this could not occur. Furthermore, because the universe is space-time homogeneous, there are closed timelike curves through every event (hence the causal violation is not localized to some small region). It must be emphasized that this breakdown of causality does not occur because of any multiple- connectivity of the space-time, such as happens for example in a 2-dimensional torus universe (it is easy to construct space-times with closed timelike lines if one allows 'cutting and pasting'). Rather the Gödel universe is simply connected (indeed it is homeomorphic to R^4).

A necessary condition that causal violation can occur is that there exist no cosmic time, that is, no time function which increases in the future direction along every (timelike) world line. Gödel demonstrated that no such time function exists in these models, indeed he showed there are no inextendible spacelike surfaces at all in this space-time (on attempting to extend them, they necessarily become null and then timelike). This is possible because of the cosmic rotation signaled by the non-zero vorticity (for if the vorticity were zero, there would be a potential function for the fluid flow vector field that would provide a cosmic time function). However not all rotating universes admit causal violation; it occurs here because of the uniform extent of the rotation (it does not die away at infinity).

Gödel did not describe the geodesic properties of this space-time, but may have investigated them (see pp.560-1 and footnote 11 in Gödel 1949a). Later investigations by Kundt (1956) and Chandrasekhar and Wright (1961) explicitly showed that there are no closed timelike geodesics in the Gödel universe. This is compatible with Gödel's results because the closed timelike lines he found are non-geodesic (some force would have to be exerted, for example by a rocket engine, for an observer to move on them and experience the violation of causality). The past null cone of each point on the coordinate axis, generated by the null geodesics through that point, diverges out from there to a maximum radius r_m where closed (non-geodesic) null lines occur and it experiences self-intersections, and then reconverges to the axis (Hawking and Ellis 1973). Thus the past light cone of each event is quite different than in flat space-time. No null geodesic every reach further from their starting point than r_m.

This study of geodesics also showed that these space-times are geodesically complete (and so singularity-free). This means that this universe is an example of an Anti-Mach metric. One of the still unsolved problems of gravitational theory was raised *inter alia* by Ernst Mach: what gives an explanation of the origin of inertia? and why is it that in the real universe, distant galaxies are apparently at rest in a local inertial rest-frame? This could simply be a coincidence, but cosmologists have sought for a causal explanation of this fact: hopefully in the form of a direct link of local inertial properties to the distribution of matter (Einstein 1949a).

The Gödel universe shows conclusively that this is not a necessary connection, for in that universe the metric and curvature are regular everywhere, and the space-time is complete (there is no boundary at finite distance from any space-time point), but the matter in the universe rotates relative to a local inertial rest- frame (because of the non-zero vorticity). Thus specification of matter by itself (the singularity-free condition is needed for a complete matter specification, as otherwise singularities can be regarded as limiting distributions of matter) does not guarantee the Machian property we observe in the real universe: some extra boundary conditions have to be imposed to guarantee this condition. The Gödel universe sparked considerable new discussion of this feature (e.g. Oszvath and Schücking 1962, Rindler 1977, Adler et al 1975).

At the end of his paper, Gödel related his solution to the rotation of galaxies, comparing observed rotation rates with the vorticity in his solution. He acknowledged that his solution was not a realistic universe model, in that it does not expand (and so could not explain the observed redshifts in the spectra of distant galaxies). Nevertheless it is interesting that he made some attempt to relate it to astrophysical observations of galactic rotation by Hubble, estimating ω from equation (9) and a value of 10^{-30} gm/cc for the density of matter, presumably obtained from Hubble's data (he specifically mentions Hubble's estimates of the rotation rates of galaxies). Gödel must have been led to these considerations and estimates by his conversations with Einstein. In any case this section clearly shows Gödel functioning in the mode of an applied mathematician (comparing observational data with model parameters to check the validity of a universe model).

Some while after the publication of this solution, Heckmann and Schücking showed there is an exact Newtonian analogue of the solutions (Heckmann and Schücking 1955), provided one adopts generalized boundary conditions (which are in fact needed for any Newtonian cosmology at all to be viable). In terms of suitably adapted coordinates the gravitational potential is $\Phi = \frac{1}{2}\omega^2(x_1^2 + x_2^2)$ which diverges at infinity, and generates a non-zero Newtonian tidal force field $E_{ab} = \Phi_{,ab} - \frac{1}{3}h_{ab}\Phi^a{}_{;a}$. In this case the only field equations to be satisfied is

$$2\omega^2 + \Lambda = \frac{1}{2}\rho, \tag{2.11}$$

correspondingt to (4). Thus the relation between these variables is less restricted than in EFE, when additionally (7) must be satisfied. Clearly there is in this case no implication of causal violation, but this now gives a Newtonian example of an anti-Mach metric where this effect results from the imposition of specific boundary conditions at infinity. The issue of boundary conditions for Newtonian cosmology is ongoing, and in a sense has still not been satisfactorily resolved; this solution provides a specific example that shows the significance of this issue.

In summary, Heckman and Schücking express the impact of the solution thus:

> From a theoretical point of view, Gödel's model is highly interesting in several respects. It shows that in an infinite space the matter can rotate absolutely. This is the first indication that Mach's ideas are not automatically contained in Einstein's theory of gravitation. On the other hand this model pointed out, as showed by Gödel, that there may arise considerable difficulties if one wants to introduce an absolute time coordinate into a model of the Universe. The existence of closed timelike lines in Gödel's model showed moreover that space-time structures in the large might be very complicated and that startling situations could arise, e.g. a person could travel into his own past.

(Heckmann and Schücking 1962)

3. Gödel's expanding universes

Gödel's stationary rotating universe is not a viable model of the real universe because in it the galaxies show no systematic redshifts (Gödel 1949). Apparently Gödel must now have put a great deal of effort into examining properties of more realistic universe models that both rotate and expand. The results were presented at an International Congress of Mathematics held at Cambridge (Massachusetts) from 30th August to 5th September 1950 (Gödel 1952). This represents the first explicit construction of spatially homogeneous expanding and rotating cosmological models. They are invariant under a non-abelian G_3 of isometries simply transitive on spacelike surfaces[1]. These are now called *Bianchi universes* (Heckmann and Schücking 1962, Ellis and MacCallum 1968, MacCallum 1980), because the classification of the 3-dimensional symmetry group transitive on the homogeneous 3-spaces is derived from that introduced much earlier by L Bianchi, which is based on an examination of the structure constants of the Lie algebra of the symmetry group.

The models examined by Gödel belong to the Bianchi IX family, invariant under the group $SO(3)$, and consequently with compact spacelike surfaces of homogeneity. Indeed this was his starting point. The matter content is taken to be pressure-free matter ('dust'). The space-times are rotating solutions ($\omega \neq 0$) with the usual space-time signature, satisfying the further conditions:
'I. The solution is to be homogeneous in space,
II. Space is to be finite,
III. The density is not constant'.
The last condition implies that the models are expanding. In order that vorticity be non-zero, the models are *tilted*, i.e. the matter flow lines are not orthogonal to the surfaces of homogeneity (King and Ellis 1973)[2]. The paper argues that these conditions allow only the Type IX group as the group of isometries, and introduces a decomposition of the metric tensor into projection tensors along and perpendicular to the fluid flow lines, that has become fundamental in later work, as well as the idea of an expansion quadric (what is now called the expansion tensor). Gödel stated, mainly without proof, a number of interesting properties of these space-times, which remain interesting cosmological models today.

On the one hand, he developed relations between vorticity and the local existence of time functions determining simultaneity for a family of observers:

> A necessary and sufficient condition for a spatially homogeneous universe to rotate is that the local simultaneity of the observers moving along with the matter be not integrable (i.e. do not define a simultaneity in the large).

Thus $\omega = 0$ implies the local existence of a time function defining simultaneity for all fundamental observers (Ehlers 1960, Ellis 1971), and so $\omega \neq 0$

[1] The much simpler Abelian case had been considered previously by Kasner (1921) and Lemaître (1933); but these cannot rotate, and their construction does not require explicit consideration of the group action and structure.

[2] Tilt is always a necessary condition for rotation; in Bianchi IX universes, it implies rotation.

implies tilt. This led him to an important observation: in such models, there is necessarily an anisotropy in source number counts: "for sufficiently great distances, there must be more galaxies in one half of the sky than in the other half". He estimated the size of this anisotropy (which is proportional to the vorticity), and which must occur in any tilted universe model whether rotating or not (King and Ellis 1973). He went on to develop vorticity conservation relations[3], and gave the condition for the vorticity vector to be parallel propagated along the matter flow lines (it must be an eigenvector of the shear tensor, cf. Ehlers 1960), relating this to the axes of rotation galaxies: "The fact that the direction of ω need not be displaced parallel to itself might be the reason for the irregular distribution of the directions of the axes of rotation of the galaxies (which at first sight seems to contradict an explanation of the rotation of galaxies from a rotation of the universe [together with conservation of angular momentum]"

Further, he linked these local studies to the global topology and the existence of closed timelike lines: "The precise necessary and sufficient condition for the non-existence of closed timelike lines (provided that the one-parameter manifold of the spaces $\rho = const$ is not closed) is that the metric in the spaces of constant density be spacelike". That is, provided the matter flowlines themselves do not close up, spatial homogeneity precludes closed timelike lines, but if the surfaces of homogeneity are timelike then closed timelike lines will occur (because these surfaces are compact). It follows that

> The non-existence of closed timelike lines is equivalent with the existence of a 'world-time', where by a world-time we mean an assignment of a real number t to every space-time point so that t always increases if one moves along a timelike line in its positive direction[4].

This is because if the surfaces of constant density are spacelike, a world-time can be defined by taking these 3-spaces as surfaces of constant time (and this is the only world time invariant under the group of transformations of the solution).

On the other hand, he gave some dynamical results that are deeper in that they involve a detailed study of the Einstein field equations (rather than just the kinematic identities that are the basis of the vorticity conservation results, see Ehlers 1960). First, he considered the locally rotationally symmetric ('LRS') cases, showing there exist no LRS cases satisfying the conditions above. Second, he stated that

> Under the additional assumption that the universe contains no closed timelike lines, the quadric of expansion, at no moment of time,

[3] partly implied in previous work by Synge (1937).

[4] Note that there is no requirement relating this function to measurements of simultaneity.

can be rotationally symmetric around ω. In particular it can never be a sphere, i.e. the expansion is necessarily coupled with a deformation. This is true even for *all* solutions satisfying I-III, and gives another directly observable property of rotating universes of this type.

Gödel suggested that the result on rotational symmetry might be related to the spiral structure of galaxies. The somewhat convoluted last statement above means that there are no expanding and rotating spatially homogeneous type IX universes with vanishing shear. Third, he stated existence of stationary homogeneous rotating solutions with finite space, no closed timelike lines, and $\lambda > 0$, in particular such as differ arbitrarily little from Einstein's static universe; but that there exist no stationary homogeneous solutions with $\lambda = 0$. These results however are almost an afterthought; the reason is that such models are unrealistic, for they cannot expand on average.

Gödel gave only the briefest of hints as to how he proved the dynamic results. Because of the symmetry of these space-times, the Einstein Field equations reduce to a system of ordinary differential equations. He did not give those equations, but he gave a Lagrangean function from which they could be derived, and stated an existence theorem: "for any value of [the cosmological constant] λ (including 0), there exist ∞^8 rotating solutions satisfying all the conditions stated. The same is true if in addition it is required that a world-time should exist (or should not exist)". The latter is the requirement that initially the surfaces of homogeneity should be spacelike or timelike.

This paper by Gödel is enigmatic, because the proofs of some of the major results are only sketched in the briefest manner[5]; the material is presented in a somewhat random order; and it is sparse on references[6]. Nevertheless it was a profound contribution to theoretical cosmology.

4. Resulting studies of causality

Gödel's papers (1949,1952) lead to an in-depth reconsideration of the nature of time and causality in relativity theory. He had showed there were acausal simply connected exact solutions of EFE. One stream of development was looking at space-times that were not simply connected, for example Bass and Witten (1957) showed that a compact space-time was necessarily acausal; and various papers considered specific high-symmetry space-times where one

[5] E. Schücking asked Gödel how he had proved the statements made, and in essence the answer was by detailed calculation. Schücking suggests that Gödel did not give more details of the proofs because the method used was inelegent (private communication).

[6] Indeed the only reference is to his own paper, Gödel (1949).

could determine all possible connectivities. Perhaps most significant was the broad realization that one could not take either the topology or the causality of space-time for granted: one needed to consider multiply connected space-times and possibilities such as wormholes, for example, as well as the possibility of causal violations.

Resulting from this, a general analysis of the ideas of causality took place, developing also from two other directions. First, Zeeman's remarkable paper 'Causality implies the Lorentz group' (Zeeman 1964) showing that causal orderings induced by the metric of Minkowski space-time are preserved only by the Lorentz group and dilations. Second, the study of Causal domains and their boundaries by Penrose, arising out of work on the Cauchy development of initial data for space-time and the idea of global hyperbolicity (due to Leray and others) on the one hand, and studies of the conformal structure of space-time on the other.

A series of important ideas arose, developed particularly by Penrose, Carter, Geroch, and Hawking, that were crucial in the later studies of causality and singularities:

(1) the idea of causal domains: the domain of dependence of initial data, of Cauchy horizons bounding this domain of dependence, and of Cauchy surfaces (surfaces on which initial data determines the evolution of the entire space-time) in space-times where no such horizons exist. The latter case was shown to be equivalent to the condition of Global Hyperbolicity, and implied geodesic connectivity of the space-time (which is not true in general).

(2) a series of causality conditions of increasing strength (causality, future distinguishing, past distinguishing, strong causality) leading up to the strongest and physically most relevant, namely stable causality. The latter was shown to be true if and only if there is a cosmic time function, i.e. a function that increases along all timelike curves (which is not true in the Gödel stationary universe). This generalizes and completes Gödel's statement on the relation between time functions and causality.

(3) the broad idea of null boundaries of causal domains, and an understanding of their properties. These boundaries include Cauchy horizons, particle and event horizons, and causality horizons (where the nature of the causality conditions obeyed by space-time changes).

These ideas are discussed in broad outline in Tipler Clarke and Ellis (1980); they are presented in technical detail in Penrose (1972) and Hawking and Ellis (1973). I believe it is fair to say that Gödel's paper gave the impetus to a lot of this work by initiating a new round of questioning of the nature of time and causality in relativity theory.

5. Resulting studies of universe models

The papers also resulted in a series of studies that greatly expanded our understanding of the dynamics of universe models, extending and in many cases completing the work initiated by Gödel.

Firstly, they initiated systematic analysis of the family of Bianchi universe models. Taub (1951) gave an enlightening study of the equations and properties of empty Bianchi universes with arbitrary group type. This built on and extended the (largely unexplained) methods used by Gödel in his expanding universe study, and made the needed techniques accessible to workers in the field, based on local properties of symmetry groups and the classification of 3-d Lie algebras developed by Luigi Bianchi from Sophus Lie's work. Taub found some new 'anti-Mach' metrics, notably the remarkable space later known as Taub-NUT space (see Hawking and Ellis 1973 for a discussion).

Heckmann and Schücking (1962) extended the equations to a study of fluid-filled Bianchi models, initiating the systematic study of this class of models. This has become an important topic of study in terms of providing a parametrized set of alternative models to the standard Friedmann-Lemaître models of cosmology. The dynamical and observational properties of the Bianchi models have been extensively studied (see Ellis and MacCallum 1968, King and Ellis 1973, MacCallum 1980, 1993, Wainwright and Ellis 1996, and references therein). There is only space to mention here four aspects of this study. First, the 'mixmaster' universe studied by Misner (1968), which is in fact the same model studied by Gödel (1952), was shown by Misner and then by Lifshitz, Belinskii, and Khalatnikov to have complex oscillatory properties at early times, leading to chaotic-like behaviour. Whether or not the early epoch of this universe exhibits truly chaotic behaviour is still the object of investigation (Hobill et al, 1994). Second, the Hamiltonian methods introduced by Misner became the cornerstone of the Hamiltonian approach to cosmology (Ryan 1972), which in turn is the foundation of the study of quantum cosmology (Coleman et al, 1991). Third, a very interesting series of dynamical systems investigations of these universes has been undertaken, which is just coming to fruition and leading on to similar studies of inhomogeneous universe models (Wainwright and Ellis 1996). Fourth, these analyses were extended to the case of Newtonian cosmology by Heckmann and Schücking (1955,1956), see also Raychaudhuri (1957).

Secondly, the (observationally unrealistic) space-time homogeneous G_4 cases were studied and completely solved by Ozsvath, Schücking, Farnsworth, and Kerr, see the references in Ellis (1967).

Thirdly, the LRS cases were completely determined, see Ellis (1967) for the dust case, Stewart and Ellis (1968) for the fluid case, and van Elst and Ellis (1996) for a covariant approach to the fluid and dust cases. These include some of the simplest interesting anisotropic cosmological models with non-trivial properties (see e.g. Collins and Ellis 1979 for phase planes and the singularity structure of the tilted type V LRS models).

Fourthly, the local covariant analysis of dynamics of cosmological models developed from Gödel's second paper, utilising and extending his use of the projection tensors and his analyses of vorticity and the expansion tensor (Ehlers 1960, Ellis 1971). A proof of his theorem on shear-free motion was given for the general homogeneous case by Schücking (1957) and then extended to the general inhomogeneous dust case by Ellis (1967), who proved that in all cases, dust-filled solutions with vanishing shear must have either vanishing rotation or vanishing expansion. Extension of this result to various perfect fluid cases followed, see Collins (1986) for a summary.

Fifthly, and perhaps most significant of all, Gödel's paper seems to have been influential in the formulation of Raychaudhuri's fundamentally important equation, giving the rate of change of the volume expansion along fluid flow lines in terms of the fluid shear, rotation, and matter content (Raychaudhuri 1955, Ehlers 1961). This is the fundamental equation of gravitational attraction, playing a central role in the dynamics of all cosmological models. It underlies the instability of the Einstein static universe (Ellis 1971), and directly gives simple singularity theorems for both the dust case (Raychaudhuri 1955) and for perfect fluids (Ehlers 1960): neither anisotropy nor inhomogeneity can avoid a singularity in universe models where matter moves without rotation or acceleration. Together with its null analogue, obtained by Ehlers and Sachs[7], this equation is one of the pillars of the important Penrose-Hawking singularity theorems.

6. The singularity theorems

The point here is simple: Raychaudhuri's result shows that irrotational dust cannot avoid a singularity at the beginning of the universe. Can rotation or pressure avoid the singularity?

All efforts at a direct attack, based on the dynamical equations, failed. Many thought that it was only the symmetry of the FL models that led to the prediction of a start to the universe. A similar issue arose in the case of gravitational collapse. The resolution of this problem came in a brilliant paper by Roger Penrose (1965) who used a combination of arguments from the convergence properties implied by the null version of Raychaudhuri's equation and analysis of its implications for the boundaries of causal sets, to

[7] See Tipler Clarke and Ellis 1980 for a discussion, and Hawking and Ellis 1973 for a derivation.

show there must be a singularity (in the sense of existence of inextendible incomplete null geodesics) at the endpoint of realistic gravitational collapse. Stephen Hawking then extended this kind of argument to the cosmological case (the start of the universe), proving a series of theorems applicable in that context, and leading to the combined Hawking-Penrose theorem that applies in both cases (Hawking and Penrose 1970) and uses both the timelike and null versions of the Raychaudhuri equation.

The nature of these arguments, given in depth in Hawking and Ellis (1973), is summarized in Tipler Clarke and Ellis (1980), Section 3. The implication is that, classically considered, space-time has a beginning at the start of the universe. More realistically, a modern view would be that we cannot avoid a quantum gravity regime at the beginning of the universe (if gravity is indeed quantized), or in any case a quantum-field dominated era where energy violations take place. The issue of the nature of the beginning of the universe is still the subject of intense debate; the Hawking-Penrose theorems have set the parameters within which the discussion takes place. Those theorems owe much to Gödel's papers both in terms of the foundations they laid for analysis of causality in general relativity, and the initiation of dynamical analyses that clarified the role and nature of vorticity and led to the timelike and null versions of the Raychaudhuri equation.

7. Gödel's dialogue with Einstein

Because most of the interaction between Einstein and Gödel took place during their talks in the Institute, little is written down of that debate. However there is a brief public interchange between them resulting from Gödel's work. It is printed in the book edited by P A Schilpp (1949), produced for the occasionof Einstein's 70th birthday on 14th March 1949.

In his contribution to that book, Gödel (1949a) explains there are world models in which there exists no objective lapse of time. He then comments:

> It might be asked: Of what use is it if such conditions prevail in *possible* worlds? Does that mean anything for the question interesting us whether in *our* world there exists an objective lapse of time? I think it does. For (1) Our world, it is true, can hardly be represented by the particular solutions referred to above (because these solutions are static and therefore yield no redshift for distant objects); there exist however also *expanding* rotating solutions. In such a universe an absolute time might also fail to exist, and it is not impossible that our world is a universe of this kind. (2) The mere compatibility with the laws of nature of worlds in which there is no distinguished absolute time, and therefore no objective lapse of time can exist, throws some light on the meaning of time also in those worlds where

> an absolute time *can* be defined. For, if someone asserts that this absolute time is lapsing, he accepts as a consequence that, whether or not an objective lapse of time exists (i.e. whether or not time in the ordinary sense of the word exists), depends on the particular way in which matter and its motion are arranged in the world. This is not a straightforward contradiction; nevertheless, a philosophical view leading to such consequences can hardly be considered as satisfactory.

(Gödel 1949a, p.562). This article shows how Gödel was primarily concerned with the "non-objectivity of the present", and only secondarily with closed timelike lines. Einstein replies,

> Kurt Gödel's essay constitutes, in my opinion, an important contribution to the general theory of relativity, especially the analysis of the concept of time.... if [causal violations exist] the distinction 'earlier - later' is abandoned for world points which lie far apart in the cosmological sense, and those paradoxes, regarding the direction of the causal connections, arise, of which Mr Gödel has spoken. Such cosmological solutions of the gravitation equations (with not vanishing cosmological constant) have been found by Mr Gödel. It will be interesting to weigh whether these are not to be excluded on physical grounds.

(Einstein 1949). Later various causality assumptions were introduced to specifically exclude causal violations (see Hawking and Ellis 1973, Tipler Clarke and Ellis 1980), with a general assumption that this was necessary for physical resonableness of solutions; but that assumption has been challenged from time to time.

This debate has been renewed with vigour in the past couple of years, with the discovery of closed timelike lines associated with moving cosmic strings and 'wormholes', and the introduction by Hawking of the 'chronology protection conjecture'. An illuminating presentation of this new discussion may be found in Kip Thorne's splendid book *Black Holes and Time Warps* (Thorne 1994). The debate is not yet ended.

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