§8. Embeddings of K

In this section we prove some general theorems concerning embeddings of K. We also use these theorems and the ideas behind them to give another proof that if there is a strongly compact cardinal, then there is an inner model with a Woodin cardinal.

In his work on the core model for sequences of measures ([M1] , [M ?]), Mitchell has shown that if there is no inner model satisfying $\exists\kappa(o(\kappa) = \kappa^{++})$, then for any universal weasel M there is an elementary $j : K \to M$; moreover, for any weasel M and elementary $j : K \to M$, j is the iteration map associated to some (linear) iteration of K. Thus the class of embeddings of K is precisely the class of iteration maps, and the class of range models for such embeddings is precisely the class of universal weasels. It follows at once that if there is no inner model satisfying $\exists\kappa(o(\kappa) = \kappa^{++})$, then any $j : K \to K$ is the identity; that is, K is "rigid".

Mitchell's results extend the original Dodd-Jensen theorem ([DJ1]) that if there is no inner model with a measurable cardinal, then whenever $j : K \to M$ is elementary, $M = K$ and $j =$ identity. The authors of [DJKM] strengthen Mitchell's results by weakening their non-large-cardinal hypothesis to "There is no inner model with a strong cardinal". We shall also prove such a strengthening of Mitchell's results, in Theorem 8.13 below.

The situation becomes more complicated once one gets past strong cardinals. We shall see that it is consistent with "There is no inner model having two strong cardinals" that there is a universal weasel which is not an iterate of K, and an elementary $j : K \to M$ which is not an iteration map. Assuming only that there is no inner model with a Woodin cardinal, however, we can still show that K is rigid. Using this fact, we can characterize K as the unique universal weasel which is elementarily embeddable in all universal weasels. We shall also show that if $j : K \to M$, where M is $\Omega+1$ iterable, and $\mu = \text{crit}(j)$, then $P(\mu)^K = P(\mu)^M$. We shall assume throughout this section that K^c satisfies "There are no Woodin cardinals", so that Ω is A_0-thick in K^c and $K = \text{Def}(K^c, A_0)$. Since we need only consider S-thick sets for $S = A_0$, we make the following definition.

Definition 8.1. *We say Γ is thick in W iff Γ is A_0-thick in W. Similarly, W has the hull (resp. definability) property at α iff W has the A_0-hull (resp. definability) property at α. Finally, $\text{Def}(W) = \text{Def}(W, A_0)$.*

We begin by showing that for any $\alpha < \Omega$, one can generate a witness that $\mathcal{J}_\alpha^K$ is A_0-sound from K by taking ultrapowers by the order zero measures at each measurable cardinal κ of K such that $\alpha < \kappa < \Omega$. The key to this result is the following.

Lemma 8.2. *Let W be an $\Omega+1$ iterable weasel which has the hull property at all $\alpha < \Omega$; then there is an iteration tree $\mathcal{T}$ on K with last model $\mathcal{M}_\theta^\mathcal{T} = W$, and such that*

(1) $\forall\alpha(\alpha+1<\theta \Rightarrow E^{\mathcal{T}}_\alpha$ *is a normal measure (i.e., has only one generator), so that* T *is linear*;

(2) $i^{\mathcal{T}}_{0,\theta}{}''K = \mathit{Def}(W)$.

Proof. Let $\mathcal{T}$ on K and $\mathcal{U}$ on W be the iteration trees resulting from a successful coiteration of K with W determined by the unique $\Omega+1$ iteration strategies on the two weasels. We show first that W never moves; i.e., $\mathcal{U}$ is trivial.

Claim 1. $lh\,\mathcal{U} = 1$.

Proof. Assume not, so that $E^{\mathcal{U}}_0$ exists. Let $\alpha < \Omega$ be inaccessible and such that $lh(E^{\mathcal{U}}_0) < \alpha$. Let R be an $\Omega+1$ iterable weasel which witnesses that $\mathcal{J}^K_\alpha$ is A_0-sound. Let $\mathcal{S}$ on R and $\mathcal{V}$ on W be the iteration trees resulting from a successful coiteration of R with W. Let $Q = \mathcal{M}^{\mathcal{S}}_\gamma = \mathcal{M}^{\mathcal{V}}_\delta$ be the common last model of the two trees, and $i = i^{\mathcal{S}}_{0\gamma}$ and $j = i^{\mathcal{V}}_{0\delta}$ be the iteration maps. Let $\mu = \mathrm{crit}(j)$.

Since $\mathcal{J}^R_\alpha = \mathcal{J}^K_\alpha$ and $lh(E^{\mathcal{U}}_0) < \alpha$, we have $E^{\mathcal{V}}_0 = E^{\mathcal{U}}_0$. But then $\mu < \nu(E^{\mathcal{V}}_0) < \alpha$. Since $j : W \to Q$ is an iteration map and W has the hull property everywhere, Q has the hull property at all $\xi \leq \mu$. Let

$$\theta = \text{least } \eta \in [0,\gamma]_S \text{ such that } \eta = \gamma \text{ or } \mathrm{crit}(i^{\mathcal{S}}_{\eta\gamma}) \geq \mu\,.$$

From example 4.3 and the remark following it, we see that whenever $\eta + 1 \in [0,\theta]_S$, then $E^{\mathcal{S}}_\eta$ is a normal measure, that is, has $\mathrm{crit}(E^{\mathcal{S}}_\eta)$ as its only generator. (Otherwise, Q would fail to have the hull property at $(\kappa^+)^Q$, where $\kappa = \mathrm{crit}(E^{\mathcal{S}}_\eta) < \mu$. Since $(\kappa^+)^Q \leq \mu$, this is impossible.) But now in any normal iteration tree, a normal measure can only be applied to the model from which it is taken. It follows that $\mathcal{S} \restriction \theta+1$ is just a linear iteration of the normal measures $E^{\mathcal{S}}_\eta$ for $\eta+1 \leq \theta$.

We claim that $\mathcal{M}^{\mathcal{S}}_\theta$ has the definability property at μ. For let Γ be thick in $\mathcal{M}^{\mathcal{S}}_\theta$; we want $a \in (\mu \cup \Gamma)^{<\omega}$ and a term τ such that $\mu = \tau^{\mathcal{M}^{\mathcal{S}}_\theta}[a]$. Let Δ be the thick class of fixed points of $i^{\mathcal{S}}_{0,\theta}$. Suppose first $i^{\mathcal{S}}_{0,\theta}(\mu) = \mu$; then since R witnesses that $\mathcal{J}^K_\alpha$ is A_0-sound, and $\mu < \alpha$, we can find $a \in (\Gamma \cap \Delta)^{<\omega}$ and τ such that $\tau^R[a] = \mu$. But then $\tau^{\mathcal{M}^{\mathcal{S}}_\theta}[a] = \mu$, as desired. Suppose next that $i^{\mathcal{S}}_{0,\theta}(\mu) > \mu$. The usual representation of iterated ultrapowers gives us a function $f : [\mu]^{<\omega} \to \mu$ and an $a \in [\mu]^{<\omega}$ such that $i^{\mathcal{S}}_{0,\theta}(f)(a) = \mu$. Since R witnesses that $\mathcal{J}^K_\alpha$ is A_0-sound, and $f \in \mathcal{J}^K_\alpha$, we have $b \in (\Gamma \cap \Delta)^{<\omega}$ and τ such that $f = \tau^R[b]$. But then $\mu = \tau^{\mathcal{M}^{\mathcal{S}}_\theta}[b](a)$, which gives the desired definition of μ.

Since $\mu = \mathrm{crit}(j)$, Q does not have the definability property at μ. It follows that $\theta < \gamma$ and $\mathrm{crit}(i^{\mathcal{S}}_{0,\theta}) = \mu$. But now the ancient Kunen argument yields a contradiction: let $A \subseteq \mu$ and $A \in \mathcal{M}^{\mathcal{S}}_\theta$. Since $\mathcal{M}^{\mathcal{S}}_\theta$ has the hull property at μ, we can write $A \cap \mu = \tau^{\mathcal{M}^{\mathcal{S}}_\theta}[a]$, where $i^{\mathcal{S}}_{0,\theta}(a) = a = j(a)$. It follows that $i^{\mathcal{S}}_{0,\theta}(A) \cap \nu = j(A) \cap \nu$, where $\nu = \inf\,(i^{\mathcal{S}}_{0,\theta}(\mu), j(\mu))$. This implies that the first extender used in $\mathcal{S}$ on $[\theta, \gamma]_S$ and the first extender used in $\mathcal{V}$ on $[0, \delta]_V$ are compatible, which is a contradiction. This proves claim 1. □

The proof of claim 1 also gives:

Claim 2. $\mathcal{T}$ is a linear iteration of normal measures.

Proof. If not, then we can find a weasel R which witnesses that $\mathcal{J}_\alpha^K$ is A_0-sound, where α is inaccessible and large enough that some extender with more than one generator used on $\mathcal{T}$ has length $< \alpha$. Again, let $\mathcal{S}$ on R and $\mathcal{V}$ on W come from coiteration, with $\mathcal{M}_\gamma^{\mathcal{S}} = \mathcal{M}_\delta^{\mathcal{V}}$. Since $\mathcal{J}_\alpha^K = \mathcal{J}_\alpha^R$, $\mathcal{S}$ and $\mathcal{T}$ have the same initial segment using extenders of length $< \alpha$. It follows that there is an $\eta + 1$ $S\gamma$ such that $\text{crit}(E_\eta^{\mathcal{S}}) < \alpha$ and $E_\eta^{\mathcal{S}}$ has more than one generator. Thus there is $\xi < \alpha$ such that $\mathcal{M}_\gamma^{\mathcal{S}}$ fails to have the hull property at ξ. On the other hand, the proof of claim 1 shows that $\text{crit}(i_{0,\delta}^{\mathcal{V}}) \geq \alpha$, and since W has the hull property everywhere, this means $\mathcal{M}_\delta^{\mathcal{V}}$ has the hull property at all $\xi \leq \alpha$. This is a contradiction. □

Let $lh\ \mathcal{T} = \theta + 1$, so that $W = \mathcal{M}_\theta^{\mathcal{T}}$.

Claim 3. $\text{Def}(W) = i_{0,\theta}^{\mathcal{T}}{}''K$.

Proof. Let $\alpha < \Omega$ be inaccessible and $i_{0,\theta}^{\mathcal{T}}{}''\alpha \subseteq \alpha$; we shall show that $\text{Def}(W) \cap \alpha = i_{0,\theta}^{\mathcal{T}}{}''\alpha$. Let R witness that $\mathcal{J}_\alpha^K$ is A_0-sound, and let $\mathcal{S}$ on R and $\mathcal{V}$ on W come from coiteration, with $\mathcal{M}_\gamma^{\mathcal{S}} = \mathcal{M}_\delta^{\mathcal{V}}$. Let η be least such that $\eta = \theta$ or $\text{crit}(i_{\eta\theta}^{\mathcal{T}}) \geq \alpha$. Since $\mathcal{J}_\alpha^K = \mathcal{J}_\alpha^R$, $\mathcal{S} \restriction \eta + 1 = \mathcal{T} \restriction \eta + 1$. We claim that $\eta + 1 \subseteq [0, \gamma]_S$; that is, the linear iteration giving us $\mathcal{T} \restriction \eta + 1$ is an initial segment of the branch of $\mathcal{S}$ leading to $\mathcal{M}_\gamma^{\mathcal{S}}$. For otherwise, letting $\beta{+}1 \in [0, \gamma]_S$ be least such that $lh\ E_\beta^{\mathcal{S}} \geq \alpha$, we have crit $E_\beta^{\mathcal{S}} < \alpha$. This implies that $\mathcal{M}_\gamma^{\mathcal{S}}$ has the hull property at all $\xi \leq \text{crit}(E_\beta^{\mathcal{S}})$, but not at $(\text{crit}(E_\beta^{\mathcal{S}})^+)^{\mathcal{M}_\gamma^{\mathcal{S}}}$. But then we get $\text{crit}(i_{0,\delta}^{\mathcal{V}}) = \text{crit}(E_\beta^{\mathcal{S}})$, and that $E_\beta^{\mathcal{S}}$ is compatible with the first extender used in $[0, \delta]_V$, as in the ancient Kunen argument. This is a contradiction.

Thus $\eta + 1 \subseteq [0, \gamma]_S$, and the argument also shows $\text{crit}(i_{\eta\gamma}^{\mathcal{S}}) \geq \alpha$. So

$$i_{0,\theta}^{\mathcal{T}} \restriction \alpha = i_{0,\eta}^{\mathcal{T}} \restriction \alpha = i_{0,\eta}^{\mathcal{S}} \restriction \alpha = i_{0,\gamma}^{\mathcal{S}} \restriction \alpha .$$

Finally, since $\mathcal{M}_\delta^{\mathcal{V}}$ has the hull property everywhere below α, $\text{crit}(i_{0\delta}^{\mathcal{V}}) \geq \alpha$. This gives

$$\begin{aligned} \text{Def}(W) \cap \alpha &= \text{Def}(\mathcal{M}_\delta^{\mathcal{V}}) \cap \alpha = Def(\mathcal{M}_\gamma^{\mathcal{S}}) \cap \alpha \\ &= i_{0\gamma}^{\mathcal{S}}{}''(\text{Def}(R) \cap \alpha) = i_{0\gamma}^{\mathcal{S}}{}''\alpha \\ &= i_{0,\theta}^{\mathcal{T}}{}''\alpha . \qquad \square \end{aligned}$$

Clearly, the claims yield 8.2. □

Lemma 8.3. *Suppose $K^c \models$ "There are no Woodin cardinals". Let α be a cardinal of K, and let W be the iterate of K obtained by taking ultrapowers by* *the order zero total measure at each measurable cardinal of K which is $\geq \alpha$. Then W witnesses that $\mathcal{J}_\alpha^K$ is A_0-sound; moreover, W has the hull property at all $\beta < \Omega$.*

Proof. Clearly W is universal, and no $\kappa \in A_0$ is measurable in W. It follows from 3.7(1) that Ω is thick in W, and it remains only to show that $\alpha \subseteq \mathrm{Def}(W)$ and that W has the hull property everywhere. Let $i : K \longrightarrow W$ be the iteration map.

Lemma 4.5 gives us an $\Omega{+}1$-iterable weasel M which has the hull property at all $\beta < \Omega$. Let α^* be the αth ordinal in $\mathrm{Def}(M)$, and for each $\beta < \alpha^*$ such that $\beta \notin \mathrm{Def}(M)$, pick a thick class Γ_β such that $\beta \notin H^M(\Gamma_\beta)$.

Let

$$R = \text{transitive collapse of } H^M(\bigcap\{\Gamma_\beta \mid \beta \in \alpha^* - \mathrm{Def}(M)\}).$$

Clearly, $\alpha \subseteq \mathrm{Def}(R)$, so R witnesses that $\mathcal{J}_\alpha^K$ is A_0-sound. It is easy to see that R has the hull property at all $\beta < \Omega$. [Let $A \subseteq \beta$, $A \in R$, and let Γ be thick in R. Let $\pi : R \longrightarrow M$ be the collapse map. Since $\pi''\Gamma$ is thick in M, and M has the hull property at $\pi(\beta)$, there is a term τ and $b \in \Gamma^{<\omega}$ and $a \in \pi(\beta)^{<\omega}$ such that $\pi(A) = \tau^M[a, \pi(b)] \cap \pi(\beta)$. The least a with this property is of the form $\pi(\bar{a})$ for some $\bar{a} \in \beta^{<\omega}$, and then $A = \tau^R[\bar{a}, b] \cap \beta$.]

Let $j : K \longrightarrow R$ be the iteration map associated to the linear iteration of normal measures given by 8.2. We have $j''K = \mathrm{Def}(R)$, and hence $\alpha \leq \mathrm{crit}(j)$.

Let $\mathcal{T}$ on R and $\mathcal{U}$ on W be the iteration trees associated to a successful coiteration of R with W, and let $\mathcal{M}_\gamma^\mathcal{T} = \mathcal{M}_\delta^\mathcal{U}$ be the common last model. Since $i_{0\gamma}^\mathcal{T} \circ j$ and $i_{0\delta}^\mathcal{U} \circ i$ are iteration maps, the Dodd-Jensen lemma gives $i_{0\gamma}^\mathcal{T} \circ j = i_{0\delta}^\mathcal{U} \circ i$. From this we get that $\mathrm{crit}(i_{0\gamma}^\mathcal{T}) \geq \alpha$ and $\mathrm{crit}(i_{0\delta}^\mathcal{U}) \geq \alpha$. For otherwise, since i and j have critical point $\geq \alpha$, we get $\kappa < \alpha$ such that $\kappa = \mathrm{crit}(i_{0\gamma}^\mathcal{T}) = \mathrm{crit}(i_{0\delta}^\mathcal{U})$. Also, for any $A \subseteq \kappa$ such that $A \in K$, $i(A) = j(A) = A$, so $i_{0\gamma}^\mathcal{T}(A) = i_{0\delta}^\mathcal{U}(A)$. This leads to the usual contradiction that the first extenders used in $[0, \gamma]_T$ and $[0, \delta]_U$ are compatible.

Now by 5.5, $i_{0\gamma}^\mathcal{T}{}''\mathrm{Def}(R) = \mathrm{Def}(\mathcal{M}_\gamma^\mathcal{T}) = \mathrm{Def}(\mathcal{M}_\delta^\mathcal{U}) = i_{0\delta}^\mathcal{U}{}''\mathrm{Def}(W)$. Since $\alpha \subseteq \mathrm{Def}(R)$ and $\mathrm{crit}(i_{0\gamma}^\mathcal{T}) \geq \alpha$, $\alpha \subseteq \mathrm{Def}(W)$.

It remains to show that W has the hull property everywhere. Let $\beta < \Omega$ be a cardinal of K, with $\alpha < \beta$. Let W^* be the iterate of K obtained by hitting the order zero measure on each measurable cardinal $\kappa \geq \beta$ of K exactly once, so that W^* witnesses that $\mathcal{J}_\beta^K$ is A_0-sound by what we have just shown. In particular, W^* has the hull property at all $\gamma < \beta$. Clearly, W is a linear iterate of W^* by normal measures (i.e., those of order zero on cardinals κ such that $\alpha \leq \kappa < \beta$), and therefore W has the hull property at all $\gamma < \beta$. Since β was arbitrary, W has the hull property everywhere. □

The reader may have noticed that 4.5 and 8.2 gave us a linear iterate by normal measures W of K satisfying the conclusion of 8.3, without much effort. (See paragraph 2 of the proof of 8.3.) What 8.3 gives, beyond this, is an iteration leading from K to W which is definable over K.

Our next lemma expresses a maximality property of K and its iterates.

Definition 8.4. *Let $\mathcal{M}$ be a premouse. We say that an extender F coheres with $\mathcal{M}$ just in case $(J_\alpha^{\mathcal{M}}, \in, \dot{E}^{\mathcal{M}} \restriction \alpha, \tilde{F})$ is a premouse, where $\alpha = lh\ F$.*

Of course, any extender on the $\mathcal{M}$-sequence coheres with $\mathcal{M}$. As another example: if $(\mathcal{T}, \mathcal{U})$ is a coiteration in which, at some intermediate stage, the current models are $\mathcal{M}_\alpha^\mathcal{T}$ and $\mathcal{M}_\beta^\mathcal{U}$, and $E_\beta^\mathcal{U}$ is part of the least disagreement at this stage, then $E_\beta^\mathcal{U}$ coheres with $\mathcal{M}_\alpha^\mathcal{T}$.

We now show that if an extender E coheres with the last model $\mathcal{M}$ of an iteration tree $\mathcal{T}$ on K, and a certain iterability condition is satisfied, then E is on the $\mathcal{M}$-sequence. The iterability condition is that we can extend $\mathcal{T}$ one step using E as if it came from the $\mathcal{M}$-sequence, and then continue iterating in the normal fashion. The following definition enables us to make this condition precise. The reader should see 9.6 and 9.7 for the general notion of a phalanx, and the definition of $\Phi(\mathcal{T})$, the phalanx derived from an iteration tree $\mathcal{T}$.

Definition 8.5. *Let $\mathcal{T}$ be an iteration tree with last model $\mathcal{M}_\alpha^\mathcal{T}$, let E cohere with $\mathcal{M}_\alpha^\mathcal{T}$, and suppose $lh\ E > lh\ E_\beta^\mathcal{T}$ for all $\beta < \alpha$. Let γ be least such that $\gamma = \alpha$ or $crit(E) < \nu(E_\gamma^\mathcal{T})$, and let $\mathcal{P}$ be the longest initial segment $\mathcal{R}$ of $\mathcal{M}_\gamma^\mathcal{T}$ such that $P(\kappa) \cap \mathcal{R} = P(\kappa) \cap \mathcal{J}_{lhE}^{\mathcal{M}_\alpha^\mathcal{T}}$, where $\kappa = crit(E)$. Let $k < \omega$ be least such that $\rho_{k+1}(\mathcal{P}) \leq \kappa$, if there is such a $k < \omega$, and let $k = \omega$ otherwise. Suppose $\mathcal{N} = Ult_k(\mathcal{P}, E)$ is wellfounded. Then the E-extension of $\Phi(\mathcal{T})$ is the phalanx $\Phi(\mathcal{T})^\frown\langle \mathcal{N}, k, \nu, \lambda\rangle$, where $\nu = \nu(E)$ and λ is the least cardinal of $\mathcal{N}$ which is $\geq \nu$.*

Theorem 8.6. *Suppose $K^c \models$ There are no Woodin cardinals". Let $\mathcal{T}$ be a normal iteration tree on W of length $\alpha + 1 < \Omega$, and suppose E coheres with $\mathcal{M}_\alpha^\mathcal{T}$ and $lh\ E \geq lhE_\beta^\mathcal{T}$ for all $\beta < \alpha$. Suppose that W witnesses that $\mathcal{J}_\mu^K$ is A_0-sound, where μ is inaccessible and large enough that $\alpha < \mu$ and $E \in V_\mu$ and $\forall \beta < \alpha(E_\beta^\mathcal{T} \in V_\mu)$. The following are equivalent:*

(a) *E is on the $\mathcal{M}_\alpha^\mathcal{T}$ sequence,*

(b) *the E-extension of $\Phi(\mathcal{T})$ is $\Omega + 1$ iterable.*

Proof. (a) $\Rightarrow$ (b) is just a re-statement of the fact that W is $\Omega + 1$ iterable. Now suppose $\mathcal{B} = \Phi(\mathcal{T}^\frown\langle \mathcal{N}, k, \nu, \lambda\rangle$ is the E-extension of $\Phi(\mathcal{T})$, and that $\mathcal{B}$ is Ω+1 iterable. Let us form the natural coiteration of $\mathcal{B}$ with $\Phi(\mathcal{T})$: at successor steps we iterate the least disagreement, beginning with the least disagreement between the last models $\mathcal{N}$ of $\mathcal{B}$ and $\mathcal{M}_\alpha^\mathcal{T}$ of $\Phi(\mathcal{T})$; the rules for iteration trees on phalanxes determine the models to which we apply the extenders from the least disagreement. At limit steps we use the (unique) $\Omega + 1$ iteration strategies for $\mathcal{B}$ and $\Phi(\mathcal{T})$ to pick branches. The usual argument shows that this coiteration terminates successfully at some stage $\leq \Omega$. The iteration tree on $\Phi(\mathcal{T})$ which is produced can clearly be regarded as an extension of $\mathcal{T}$; let us call this extension $\mathcal{U}$. We note that $\mathcal{U}$ is normal. [This is clear, except perhaps for the increasing-length condition on its extenders; for that we need only show $lh\ E^{\mathcal{U}}_{\alpha} > lh\ E^{\mathcal{U}}_{\beta}$ for all $\beta < \alpha$. But since E coheres with $\mathcal{M}^{\mathcal{U}}_{\alpha} = \mathcal{M}^{\mathcal{T}}_{\alpha}$, $\mathcal{N}$ agrees with $\mathcal{M}^{\mathcal{U}}_{\alpha}$ below $lh(E)$, and the disagreement of which $E^{\mathcal{U}}_{\alpha}$ is a part must occur at a length $\geq lh(E)$.] The iteration tree on $\mathcal{B}$ which is produced can be regarded as a system $\mathcal{S}$ extending $\mathcal{T}$ which has all the properties of a normal iteration tree except that, since $E^{\mathcal{S}}_{\alpha} = E$, it may not be true that $E^{\mathcal{S}}_{\alpha}$ is on the $\mathcal{M}^{\mathcal{S}}_{\alpha}$ sequence. We shall use iteration tree terminology in connection with $\mathcal{S}$; its meaning should be clear.

Claim 1. If $E^{\mathcal{S}}_{\eta}$ is compatible with $E^{\mathcal{U}}_{\xi}$, then $\eta = \xi \leq \alpha$ and $E^{\mathcal{S}}_{\eta} = E^{\mathcal{U}}_{\xi}$.

Proof. If $\mathcal{V}$ is a normal iteration tree, then $lh\ E^{\mathcal{V}}_{\sigma}$ is a cardinal in all models $\mathcal{M}^{\mathcal{V}}_{\tau}$ for $\tau > \sigma$, so that $E^{\mathcal{V}}_{\sigma}$ is incompatible with any extender on the $\mathcal{M}^{\mathcal{V}}_{\tau}$-sequence, for $\tau > \sigma$, by the initial segment condition and the fact that $E^{\mathcal{V}}_{\sigma}$ collapses its length. The system $\mathcal{S}$ has this property of normal trees because $E^{\mathcal{S}}_{\alpha}$ coheres with $\mathcal{M}^{\mathcal{S}}_{\alpha}$. Since $\mathcal{S} \restriction \alpha = \mathcal{U} \restriction \alpha = \mathcal{T}$, this gives the claim immediately if $\eta < \alpha$ or $\xi < \alpha$.

It cannot happen that $\eta > \alpha$ and $\xi \geq \alpha$. For then $E^{\mathcal{S}}_{\eta}$ and $E^{\mathcal{U}}_{\xi}$ are each part of a disagreement in the coiteration of $\mathcal{B}$ with $\Phi(\mathcal{T})$. They cannot be part of the same disagreement since they are compatible. Thus $lh(E^{\mathcal{S}}_{\eta}) \neq lh(E^{\mathcal{U}}_{\xi})$. Suppose that $lh(E^{\mathcal{S}}_{\eta}) < lh(E^{\mathcal{U}}_{\xi})$; the other case leads to a similar contradiction. By the inital segment condition on premice, $E^{\mathcal{S}}_{\eta}$ is on the $\mathcal{M}^{\mathcal{U}}_{\xi}$ sequence or an ultrapower thereof. Letting $\mathcal{M}^{\mathcal{S}}_{\tau}$ be the model on $\mathcal{S}$ with which $\mathcal{M}^{\mathcal{U}}_{\xi}$ is being compared, we have $\eta < \tau$ and $E^{\mathcal{S}}_{\eta}$ on the $\mathcal{M}^{\mathcal{S}}_{\tau}$ sequence or an ultrapower thereof. This means $lh(E^{\mathcal{S}}_{\eta})$ is not a cardinal of $\mathcal{M}^{\mathcal{S}}_{\tau}$, a contradiction.

We are left with the possibility that $\eta = \alpha$ and $\xi \geq \alpha$. Thus $E^{\mathcal{S}}_{\eta} = E$. If $lh(E^{\mathcal{U}}_{\xi}) > lh(E)$, we get that E is on the $\mathcal{M}^{\mathcal{U}}_{\xi}$ sequence or an ultrapower thereof, and hence on the $\mathcal{N}$-sequence or an ultrapower thereof, which is impossible since E coheres with $\mathcal{M}^{\mathcal{T}}_{\alpha}$. Thus $E^{\mathcal{U}}_{\xi} = E^{\mathcal{S}}_{\eta}$. □

Now let $\mathcal{M}^{\mathcal{S}}_{\gamma}$ and $\mathcal{M}^{\mathcal{U}}_{\delta}$ be the last models of $\mathcal{S}$ and $\mathcal{U}$ respectively.

Claim 2. $\mathcal{M}^{\mathcal{S}}_{\gamma} = \mathcal{M}^{\mathcal{U}}_{\delta}$.

Proof. If $[0,\gamma]_S \cap D^{\mathcal{S}} = \phi$, then since Ω is thick in $W = \mathcal{M}^{\mathcal{S}}_0$, Ω is thick in $\mathcal{M}^{\mathcal{S}}_{\gamma}$. But then $[0,\delta]_U \cap D^{\mathcal{U}} = \phi$ and hence $\mathcal{M}^{\mathcal{U}}_{\delta} = \mathcal{M}^{\mathcal{S}}_{\gamma}$. If $[0,\gamma]_S \cap D^{\mathcal{S}} \neq \phi$, then $\mathcal{M}^{\mathcal{U}}_{\delta} \trianglelefteq \mathcal{M}^{\mathcal{S}}_{\gamma}$ because $\mathcal{M}^{\mathcal{S}}_{\gamma}$ is not ω-sound. But then $[0,\delta]_U \cap D^{\mathcal{U}} \neq \phi$, as otherwise Ω is thick in $\mathcal{M}^{\mathcal{U}}_{\delta}$ while $\mathcal{M}^{\mathcal{S}}_{\gamma}$ computes κ^+ incorrectly for a.e. $\kappa \in A_0$. Thus $\mathcal{M}^{\mathcal{S}}_{\gamma} \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\delta}$, as $\mathcal{M}^{\mathcal{U}}_{\delta}$ is not ω-sound. □

Now let $\theta \leq \alpha$ be largest such that $\theta S \gamma$ and $\theta U \delta$. Since $\mathcal{M}^{\mathcal{S}}_{\alpha+1} = \mathcal{N}$ exists, $\theta < \gamma$, so we can set $\eta + 1 =$ unique $\beta \in [0,\gamma]_S$ such that S-pred$(\beta) = \theta$.

Claim 3. $\theta < \delta$.

Proof. If not, then since $\theta \leq \alpha \leq \delta$, we must have $\theta = \alpha = \delta$.

Suppose first that $D^{\mathcal{T}} \cap [0,\alpha]_T = \phi$. Since then $\mathcal{M}_\alpha^{\mathcal{T}} = \mathcal{M}_\gamma^{\mathcal{S}}$ is a universal weasel, $[0,\gamma]_S \cap D^{\mathcal{S}} = \phi$. Let $\kappa = \mathrm{crit}(E_\eta^{\mathcal{S}})$. Since $\kappa < \nu(E_\theta) < \mu$, and $\mu \subseteq \mathrm{Def}(W)$, $\mathcal{M}_\theta^{\mathcal{S}}$ has the definability property at κ. Since $\kappa = \mathrm{crit}\ i_{\theta,\gamma}^{\mathcal{S}}$, $\mathcal{M}_\gamma^{\mathcal{S}}$ does not have the definability property at κ. But $\mathcal{M}_\theta^{\mathcal{S}} = \mathcal{M}_\alpha^{\mathcal{T}} = \mathcal{M}_\delta^{\mathcal{U}} = \mathcal{M}_\delta^{\mathcal{S}}$, a contradiction.

By claim 3, we can let $\xi + 1 \in [0,\delta]_U$ be the unique β such that U-pred$(\beta) = \theta$.

Claim 4. $E_\eta^{\mathcal{S}}$ is compatible with $E_\xi^{\mathcal{U}}$.

Proof. We claim first that $D^{\mathcal{S}} \cap (\eta+1,\gamma]_S = \phi$, and $\deg^{\mathcal{S}}(\eta+1) = \deg^{\mathcal{S}}(\gamma)$. For otherwise, let $\sigma+1$ be the site of the last drop in model or degree along $(\eta+1,\gamma]_S$, and let $k = \deg^{\mathcal{S}}(\sigma+1)$. By standard arguments, $(\mathcal{M}_{\sigma+1}^*)^{\mathcal{S}} = \mathfrak{C}_{k+1}(\mathcal{M}_\gamma^{\mathcal{S}})$, k is least such that $\mathcal{M}_\gamma^{\mathcal{S}}$ is not $k+1$ sound, and $i_{\sigma+1,\gamma}^{\mathcal{S}} \circ (i_{\sigma+1}^*)^{\mathcal{S}}$ is the canonical core embedding from $\mathfrak{C}_{k+1}(\mathcal{M}_\gamma^{\mathcal{S}})$ into $\mathcal{M}_\gamma^{\mathcal{S}}$. Since $\mathcal{M}_\gamma^{\mathcal{S}} = \mathcal{M}_\delta^{\mathcal{U}}$, there is a last drop $\tau+1$ in model or degree along $[0,\delta]_U$, and $\deg^{\mathcal{U}}(\tau+1) = k$, $(\mathcal{M}_{\tau+1}^*)^{\mathcal{U}} = \mathfrak{C}_{k+1}(\mathcal{M}_\delta^{\mathcal{U}})$, and $i_{\tau+1,\delta}^{\mathcal{U}} \circ (i_{\tau+1}^*)^{\mathcal{U}}$ is the core embedding. This gives $E_\sigma^{\mathcal{S}}$ compatible with $E_\tau^{\mathcal{U}}$, so that $\sigma = \tau \leq \alpha$ by claim 1. But then $\sigma \leq \theta$ by the definition of θ, while $\theta \leq \eta < \sigma$ by the definition of σ.

A similar argument shows that $D^{\mathcal{U}} \cap (\xi+1,\delta)_U = \phi$ and $\deg^U(\xi+1) = \deg^U(\gamma)$.

Next, suppose $\eta+1 \in D^{\mathcal{S}}$ or $\deg^{\mathcal{S}}(\eta+1) \neq \deg^{\mathcal{S}}(\theta)$. Arguing as above, we get that $E_\eta^{\mathcal{S}}$ is compatible with $E_\tau^{\mathcal{U}}$, where $\tau+1$ is the site of the last drop in model or degree along $[0,\delta]_U$. We cannot have $\tau+1 \in [0,\theta]_U$, since then $E_\tau^{\mathcal{U}} = E_\tau^{\mathcal{S}}$, so $E_\tau^{\mathcal{S}}$ is compatible with $E_\eta^{\mathcal{S}}$, while $\tau \neq \eta$ because $\eta+1 \notin [0,\theta]_U$. The only other possibility is $\tau+1 = \xi+1$, which gives $E_\eta^{\mathcal{S}}$ compatible with $E_\xi^{\mathcal{U}}$, as desired.

Similarly, if $\xi+1 \in D^{\mathcal{U}}$ or $\deg^{\mathcal{U}}(\xi+1) \notin \deg^{\mathcal{U}}(\theta)$, then $E_\eta^{\mathcal{S}}$ is compatible with $E_\xi^{\mathcal{U}}$. So we may assume that $i_{\theta,\gamma}^{\mathcal{S}}$ and $i_{\theta,\delta}^{\mathcal{U}}$ exist, and each is a $\deg^{\mathcal{T}}(\theta)$ embedding.

Suppose that $D^{\mathcal{T}} \cap [0,\theta]_T = \phi$, and let $\nu = \sup\{\nu(E_\sigma^{\mathcal{T}}) \mid \sigma+1 \in [0,\theta]_T\}$. Since $\mu \subseteq \mathrm{Def}(W)$, we have $\mu \subseteq H^{\mathcal{M}_\theta^{\mathcal{T}}}(\nu \cup \Gamma)$ for all thick Γ. Taking Γ to be the class of common fixed points of $i_{\theta,\gamma}^{\mathcal{S}}$ and $i_{\theta,\delta}^{\mathcal{U}}$, and noting that $\nu < \mathrm{crit}(i_{\theta,\gamma}^{\mathcal{S}}) < \mu$ and $\nu < \mathrm{crit}(i_{\theta,\delta}^{\mathcal{U}}) < \mu$, we get that $i_{\theta,\gamma}^{\mathcal{S}}(A) = i_{\theta,\delta}^{\mathcal{U}}(A)$ for all $A \subseteq \kappa$, where κ is the common critical point of the two embeddings. This implies othat $E_\eta^{\mathcal{S}}$ is compatible with $E_\xi^{\mathcal{U}}$.

Finally, suppose that $D^{\mathcal{T}} \cap [0,\theta]_T \neq \phi$, and again let $\nu = \sup\{\nu(E_\sigma^{\mathcal{T}}) \mid \sigma+1 \in [0,\theta]_T\}$. Let $k = \deg^{\mathcal{T}}(\theta)$. Then $\mathcal{M}_\theta^{\mathcal{T}} = H_{k+1}^{\mathcal{M}_\theta^{\mathcal{T}}}(\nu \cup p)$, where $p = p_{k+1}(\mathcal{M}_\theta^{\mathcal{T}})$. Since $i_{\theta,\gamma}^{\mathcal{S}}(p) = p_{k+1}(\mathcal{M}_\gamma^{\mathcal{S}}) = i_{\theta,\delta}^{\mathcal{U}}(p)$, we again get $i_{\theta,\gamma}^{\mathcal{S}}(A) = i_{\theta,\delta}^{\mathcal{U}}(A)$ for all A contained in the common critical point of the two embeddings, and thus that $E_\eta^{\mathcal{S}}$ is compatible with $E_\xi^{\mathcal{U}}$. This proves claim 4. □

By claims 1 and 4, $\eta = \xi \leq \alpha$ and $E^{\mathcal{S}}_\eta = E^{\mathcal{U}}_\xi$. We cannot have $\eta < \alpha$, for then $\eta + 1 = \xi + 1 \leq \alpha$, which gives $\eta + 1 \leq \theta$, contrary to the definition of η. Thus $\eta = \xi = \alpha$, so that $E = E^{\mathcal{S}}_\eta = E^{\mathcal{U}}_\alpha$. Since $E^{\mathcal{U}}_\alpha$ is on the $\mathcal{M}^{\mathcal{T}}_\alpha$ sequence, we are done. □

Remark 8.7. (a) We believe that one can prove 8.6 with K replacing W. That is, suppose $K^c \models$ "There are no Woodin cardinals", and let $\mathcal{T}$ be a normal iteration tree of length $\alpha + 1 < \Omega$. suppose E coheres with $\mathcal{M}^{\mathcal{T}}_\alpha$, $lh(E) > lh(E^{\mathcal{T}}_\beta)$ for all $\beta < \alpha$, and the E-extension of $\Phi(\mathcal{T})$ is $\Omega + 1$-iterable. Then E is on the $\mathcal{M}^{\mathcal{T}}_\alpha$ sequence. For let μ be inaccessible and $\mathcal{T}$, $E \in V_\mu$, and let W come from K by iterating normal order zero measures above μ, as in 8.3. Let $\pi : K \to W$ be the iteration map, and let $\pi\mathcal{T}$ on W be the copied tree; it is enough to show E is on the $\mathcal{M}^{\pi\mathcal{T}}_\alpha$ sequence. By 8.6, it is enough for this to show that the E-extension of $\Phi(\pi\mathcal{T})$ is $\Omega + 1$-iterable. We believe that one can do this by chasing the proper diagrams, but haven't gone through the details.

(b) We believe, but haven't checked carefully, that the methods of §9 show that if $\mathcal{T}$, E, and α are as described in the hypothesis of 8.6, and if $(\mathcal{M}^{\mathcal{T}}_\alpha, E)$ is countably certified (in the sense of 1.2), then the E-extension of $\Phi(\mathcal{T})$ is iterable. Taking $\mathcal{T} = \emptyset$, this means that every countably certified extender which coheres with K is on the K-sequence.

We now show that K is rigid.

Theorem 8.8. *Suppose that $K^c \models$ "There are no Woodin cardinals", and let $j : K \to K$ be elementary; then j = identity.*

Proof. Suppose otherwise, and let $\kappa = \text{crit}(j)$. Let $\mu < \Omega$ be inaccessible and such that $j(\kappa) < \mu$. Let W be the result of hitting each order zero measure of K with critical point $> \mu$ exactly once (in increasing order) as in 8.3. Thus W witnesses that $\mathcal{J}^K_\mu$ is A_0-sound. Let F be the length $j(\kappa)$ extender over K, or equivalently W, derived from j. It will be enough to show $F \restriction \rho \in W$ for all $\rho < j(\kappa)$, since then these initial segments of F witness that κ is Shelah in W. The proof of this is an induction on ρ organized as is the proof of Lemma 11.4 of [FSIT].

Lemma 8.9. *Let $(\kappa^+)^W \leq \rho < j(\kappa)$, and suppose ρ is the sup of the generators of $F \restriction \rho$. Let G be the trivial completion of $F \restriction \rho$, and $\gamma = lh\ G$. Then $\dot{E}^W_\gamma = G = F \restriction \gamma$ unless ρ is a limit ordinal greater than $(\kappa^+)^W$, and is itself a generator of F. In this case*

$$G = \begin{cases} E_\gamma & \text{if } \rho \notin dom\ \dot{E}^W, \\ i(\dot{E}^W)_\gamma & \text{if } \rho \in dom\ \dot{E}^W, \end{cases}$$

where $i : \mathcal{J}^W_\rho \to Ult_0(\mathcal{J}^W_\rho, \dot{E}^W_\rho)$ is the canonical embedding.

Proof. By induction on ρ. Suppose first that ρ is not a generator of F which is $> (\kappa^+)^W$. It follows that the natural embedding from $\text{Ult}(W, F \restriction \rho)$ into

$\mathrm{Ult}(W,F)$ has critical point at least $\gamma = (\rho^+)^{\mathrm{Ult}(W,F\restriction\rho)}$. From this we get that G coheres with W; $(J_\gamma^W, \in, \dot{E}^W, \tilde{G})$ satisfies the initial segment condition on premice by our induction hypothesis. We now apply 8.6 to the trivial iteration tree on W whose only model is W. By 8.6, we are done if we show that the phalanx $\mathcal{B} = (\langle (W,\omega), (\mathrm{Ult}(W,G),\omega)\rangle, \langle \rho, \gamma\rangle)$ is $\Omega + 1$ -iterable, for then G is on the W sequence.

In order to show that $\mathcal{B}$ is $\Omega+1$ iterable, it is enough to find an elementary $\pi : \mathrm{Ult}(W,G) \to W$ such that $\pi \restriction \rho =$ identity, since then we can copy iteration trees on $\mathcal{B}$ as ordinary iteration trees on W. Now W is definable over K from μ, hence $j : W \to j(W)$ is elementary, where $j(W)$ is defined over K from $j(\mu)$ as W was from μ. Since G is an initial segment of the extender derived from j, we have the diagram

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[dr, "i_G"'] & j(W) \\ & \mathrm{Ult}(W,G) \arrow[u, "\psi"'] \end{tikzcd}$$

where $\psi \restriction \rho =$ identity. So it is enough to find an embedding $\sigma : j(W) \to W$ which is the identity up to ρ.

Let $\mathcal{U}$ be the linear iteration leading from K to W, so that $K = \mathcal{M}_0^{\mathcal{U}}$ and $W = \mathcal{M}_\Omega^{\mathcal{U}}$. Let $\mathcal{S} = j(\mathcal{U})$ be the linear iteration leading from K to $j(W)$, so that $K = \mathcal{M}_0^{\mathcal{S}}$ and $j(W) = \mathcal{M}_\Omega^{\mathcal{S}}$. Let $\alpha < \Omega$ be least so that $\mathrm{crit}(i^{\mathcal{U}}_{\alpha,\Omega}) > j(\mu)$. We now define by induction on $\beta \geq 0$ maps $\tau_\beta : \mathcal{M}_\beta^{\mathcal{S}} \to \mathcal{M}^{\mathcal{U}}_{\alpha+\beta}$ such that $\tau_\gamma \circ i^{\mathcal{S}}_{\beta\gamma} = i^{\mathcal{U}}_{\alpha+\beta,\alpha+\gamma}$ for $\beta < \gamma$. We begin with $\tau_0 = i^{\mathcal{U}}_{0,\alpha}$. Given τ_β, set

$$\tau_{\beta+1}([f]_H) = [\tau_\beta(f)]_{\tau_\beta(H)},$$

where $H = E^{\mathcal{S}}_\beta$; notice here that $\tau_\beta(H) = E^{\mathcal{U}}_{\alpha+\beta}$ so that this works out. [Let Z be the $\alpha+\beta$th order zero total measure of K above μ. By definition of α, Z is the βth order zero total measure of K above $j(\mu)$. Thus $E^{\mathcal{U}}_{\alpha+\beta} = i^{\mathcal{U}}_{0,\alpha+\beta}(Z)$ and $E^{\mathcal{S}}_\beta = i^{\mathcal{S}}_{0\beta}(Z)$. But then $\tau_\beta(E^{\mathcal{S}}_\beta) = \tau_\beta(i^{\mathcal{S}}_{0\beta}(Z)) = i^{\mathcal{U}}_{\alpha,\alpha+\beta}(\tau_0(Z)) = i^{\mathcal{U}}_{0,\alpha+\beta}(Z) = E^{\mathcal{U}}_{\alpha+\beta}$.] We define τ_λ using commutativity, for λ a limit. Clearly $\tau_\Omega : j(W) \to W$, and it is easy to see that $\tau_\Omega \restriction \mu$ is the identity.

This finishes the proof of 8.9 in the case $\rho = (\kappa^+)^W$ or ρ is not a generator of F. If $\rho > (\kappa^+)^W$ and ρ is a generator of F, then the natural embedding $\psi : \mathrm{Ult}(W,G) \to j(W)$ has critical point ρ. It is therefore not obvious that G coheres with W. Nevertheless, we can apply the condensation Theorem 8.2 of [FSIT]. For notice that $\psi(\rho) < j(\kappa) < \mu$, and that W and $j(W)$ agree below μ, and that ρ is a cardinal of $\mathrm{Ult}(W,G)$. Further, $\gamma = (\rho^+)^{\mathrm{Ult}(W,G)}$, so there are arbitrarily large $\eta < \gamma$ such that $\rho_\omega(\mathcal{J}_\eta^{\mathrm{Ult}(W,G)}) = \rho$ and ψ maps

$\mathcal{J}_\eta^{\mathrm{Ult}(W,G)}$ elementarily into $\mathcal{J}_{\psi(\eta)}^{j(W)}$ which is a level of W. If $\rho \notin \mathrm{dom}\ \dot{E}^W$, then 8.2 of [FSIT] gives $\mathcal{J}_\eta^{\mathrm{Ult}(W,G)} = \mathcal{J}_\eta^W$ for all $\eta < \gamma$, so that G coheres with W. (Again, the initial segment condition is our induction hypothesis.) We can then finish the proof just as in the case ρ is not a generator of F. So assume $\rho \in \mathrm{dom}\ \dot{E}^W$.

In this case, 8.2 of [FSIT] implies that G coheres with $\mathrm{Ult}(W, \dot{E}_\rho^W)$. (Notice that $\mathrm{Ult}(W, \dot{E}_\rho^W)$ makes sense: letting $\lambda = \mathrm{crit}\ \dot{E}_\rho^W$, we have $(\lambda^+)^{\mathcal{J}_\rho^W} = (\lambda^+)^{\mathrm{Ult}(W,G)}$ because ρ is a cardinal of $\mathrm{Ult}(W,G)$, which agrees with W below ρ. Since $\rho = \mathrm{crit}(\psi)$, $(\lambda^+)^{\mathrm{Ult}(W,G)} = (\lambda^+)^{j(W)} = (\lambda^+)^W$. Thus $\dot{E}_\rho^W$ measures all sets in W.) The proof of 8.9 will be complete if we show that G is on the $\mathrm{Ult}(W, \dot{E}_\rho^W)$ sequence. To this end, we apply 8.6 to the iteration tree $\mathcal{T}$ on W of length 2 such that $E_0^{\mathcal{T}} = \dot{E}_\rho^W$. We are done if we can show that the G-extension of $\Phi(\mathcal{T})$ is $\Omega + 1$ iterable. Using the natural map from $\mathrm{Ult}(W,G)$ into $j(W)$ we can copy iteration trees on the G-extension of $\Phi(\mathcal{T})$ as iteration trees on $\mathcal{B}$, where $\mathcal{B}$ is the phalanx with models $\langle W, \mathrm{Ult}(W, \dot{E}_\rho^W), j(W)\rangle$ and "exchange ordinals" $\langle \nu(\dot{E}_\rho^W), \rho\rangle$. So it is enough to see that $\mathcal{B}$ is $\Omega+1$-iterable. But we can embed $\mathcal{B}$ into a K^c-generated phalanx as follows: let $\pi : K \to K^c$ with $\mathrm{ran}(\pi) = \mathrm{Def}(K^c)$. Since W is obtained from K by a K-definable iteration process with critical points above μ, we have $\pi : W \to R$, where R is obtained in the same way from K^c, the critical points being above $\pi(\mu)$. Similarly, $\pi : j(W) \to S$ where S is obtained from K^c by iterating above $\pi(j(\mu))$. Finally, let $\sigma : \mathrm{Ult}(W, \dot{E}_\rho^W) \to \mathrm{Ult}(R, \pi(\dot{E}_\rho^W))$ be the shift map induced by π. We have $\sigma \restriction \nu = \pi \restriction \nu$, where $\nu = \nu(\dot{E}_\rho^W)$. Now $\mathrm{Ult}(R, \pi(\dot{E}_\rho^W))$ is not quite the last model of an iteration tree on K^c, since the ultrapowers do not come in the increasing length order. But this is easy to fix: we can find an embedding $\psi : \mathrm{Ult}(R, \pi(\dot{E}_\rho^W)) \to Q$, where Q comes from forming $\mathrm{Ult}(K^c, \pi(\dot{E}_\rho^W))$, and then doing the images of the ultrapowers in the iteration from K^c to R. We have $\psi \restriction \pi(\mu) = $ identity. We have then that the phalanx $\mathcal{D}$ with models $\langle R, Q, S\rangle$ and exchange ordinals $\langle \pi(\nu), \pi(\rho)\rangle$ is K^c-generated, and therefore is $\Omega + 1$ iterable. (cf. 6.9) We can use the maps $\langle \pi, \psi \circ \sigma, \pi\rangle$ to reduce trees on $\mathcal{B}$ to trees on $\mathcal{D}$, and hence $\mathcal{B}$ is $\Omega + 1$ iterable.

This completes the proof of Lemma 8.9, and hence of Theorem 8.8. □

Theorem 8.8 leads to the following characterization of K.

Theorem 8.10. *Suppose $K^c \models$ "There are no Woodin cardinals"; then K is the unique universal weasel which elementarily embeds into all universal weasels.*

Proof. Let M be a universal weasel. For $\alpha < \Omega$, we construct a linear iterate $\mathcal{P}_\alpha$ of M by hitting each total order zero measure above α once. (More precisely, let $\mathcal{U}_\nu$ be the νth total-on-M measure of order zero on the M-sequence which has critical point $\geq \alpha$. Let $\mathcal{T}$ be the linear iteration tree on M such that $E_\nu^{\mathcal{T}} = i_{0,\nu}^{\mathcal{T}}(\mathcal{U}_\nu)$ for all ν, and let $lh\ \mathcal{T} = \gamma+1 \leq \Omega+1$; then we set $P_\alpha = \mathcal{M}_\gamma^{\mathcal{T}}$.)

It is not hard to see that if $\alpha < \beta < \Omega$, then P_α is a linear iterate of P_β. For let $\mathcal{U}_\nu$ be the νth total measure of order zero on the M-sequence with critical point between α and β, and let $j : M \longrightarrow P_\beta$ be the iteration map. We define a linear iteration tree $\mathcal{T}$ on P_β by: $E^{\mathcal{T}}_\nu = i^{\mathcal{T}}_{0,\nu}(j(\mathcal{U}_\nu))$. Letting $\gamma+1 = lh\ \mathcal{T}$ (so that γ is the order type of $\{\nu \mid \mathcal{U}_\nu \text{ exists}\}$), we have $P_\alpha = \mathcal{M}^{\mathcal{T}}_\gamma$. (This comes down to the fact that if i and k are the embeddings associated to normal measures on distinct measurable cardinals, then $i(k) = k$.)

Since M is universal, $(\alpha^+)^M = \alpha^+$ for all but nonstationary many $\alpha \in A_0$, by 3.7 (1). The construction of P_β guarantees that no $\gamma \in A_0 - \beta$ is the critical point of a total-on-P_β extender from the P_β sequence. Therefore Ω is A_0-thick in P_β, for all $\beta < \Omega$. From 5.7 we then have that $K \cong \mathrm{Def}(P_\beta)$, for all $\beta < \Omega$. Let $\pi_\beta : K \longrightarrow P_\beta$ have range $\mathrm{Def}(P_\beta)$.

Let $i_\beta : P_\beta \longrightarrow P_0$ be the linear iteration map described above, so that $i''_\beta\, \mathrm{Def}(P_\beta) = \mathrm{Def}(P_0)$ by 5.6. Since $i_\beta(\pi_\beta(\mu)) = \pi_0(\mu)$, we have $\pi_\beta(\mu) \leq \pi_0(\mu)$ for all β and μ. Now let us define $\sigma : K \longrightarrow M$ as follows: for $x \in K$, pick any μ such that $x \in \mathcal{J}^K_\mu$, and let β be inaccessible and such that $\pi_0(\mu) < \beta$; then we set $\sigma(x) = \pi_\beta(x)$. Note here that $\pi_\beta(x) \in \mathcal{J}^{P_\beta}_{\pi_\beta(\mu)} \subseteq \mathcal{J}^{P_\beta}_\beta = \mathcal{J}^M_\beta$, so indeed $\sigma(x) \in M$. Also, if $\beta < \gamma$, then the iteration from P_γ to P_β has critical point $\geq \beta$, so that $Def(P_\gamma) \cap \mathcal{J}^{P_\gamma}_\beta = \mathrm{Def}(P_\beta) \cap \mathcal{J}^{P_\beta}_\beta$. This implies that σ is well-defined. Finally, if $K \models \varphi[x]$, then $P_\beta \models \varphi[\pi_\beta(x)]$, so $M \models \varphi[\sigma(x)]$ because the iteration from M to P_β has critical point $\geq \beta$ and $\pi_\beta(x) \in \mathcal{J}^M_\beta$. This shows that σ is elementary.

Finally, we show uniqueness. Let $j : M \longrightarrow K$ be elementary, where M is a universal weasel. Let $i : K \longrightarrow M$ be elementary. Then $j \circ i : K \longrightarrow K$ elementarily, so $j \circ i =$ identity by 8.8. It follows that $M = K$. □

We now consider the situation "below $0^{\mathbb{P}}$".

Definition 8.11. *A proper premouse $\mathcal{M}$ is below $0^{\mathbb{P}}$ iff whenever E is an extender on the $\mathcal{M}$-sequence, and $\kappa = crit(E)$, then $\mathcal{J}^{\mathcal{M}}_\kappa \models$ "There are no strong cardinals".*

One important way in which a premouse $\mathcal{M}$ below $0^{\mathbb{P}}$ is simple is that every normal iteration tree $\mathcal{T}$ on $\mathcal{M}$ is "almost linear". More precisely: if $\mathcal{T}$ is a normal tree on $\mathcal{M}$ and T-pred$(\beta+1) = \alpha$, then for some $n \in \omega$, $\beta = \alpha + n$ and $\mathrm{crit}(E^{\mathcal{T}}_\beta) = \mathrm{crit}(E^{\mathcal{T}}_{\alpha+k})$ for all $k \leq n$. Thus $\mathcal{T}$ misses being linear only in that it may hit the same critical point finitely many times in immediate succession (and thus branch finitely) before going on to critical points larger than the lengths of all preceding extenders. [Proof: It is enough to show that whenever $\alpha \leq \beta$ and $\mathrm{crit}(E^{\mathcal{T}}_\beta) \leq \nu(E^{\mathcal{T}}_\alpha)$, then $\mathrm{crit}(E^{\mathcal{T}}_\beta) = \mathrm{crit}(E^{\mathcal{T}}_\alpha)$. So let $\alpha \leq \beta$, $\kappa = \mathrm{crit}(E^{\mathcal{T}}_\alpha)$, $\mu = \mathrm{crit}(E^{\mathcal{T}}_\beta)$, and suppose $\mu \leq \nu(E^{\mathcal{T}}_\alpha)$. If $\kappa < \mu$, then there are arbitrarily large $\lambda < \mu$ such that $E^{\mathcal{T}}_\alpha \restriction \lambda$ is on the $\mathcal{M}^{\mathcal{T}}_\beta$ sequence, so $\mathcal{M}^{\mathcal{T}}_\beta$ is not below $0^{\mathbb{P}}$. If $\mu < \kappa$, then there are arbitrarily large $\lambda < \kappa$ such that $E_\beta^{\mathcal{T}} \restriction \lambda$ is on the $\mathcal{M}_\beta^{\mathcal{T}}$ sequence, so $\mathcal{M}_\beta^{\mathcal{T}}$ is not below $0^{\mathbb{P}}$. Both statements follow at once from the inital segment condition and the normality of $\mathcal{T}$.]

Lemma 8.12. *Suppose K^c is below $0^{\mathbb{P}}$. Let W and M be universal weasels, with Ω A_0-thick in M and W, and $\mu \subseteq Def(W)$. Let $i : W \to Q$ and $j : M \to Q$ come from coiteration, and $\pi : K \to M$ with ran $\pi = Def(M)$. Then $j \restriction sup(\pi''\mu) =$ identity.*

Proof. Let $\mathcal{T}$ be the iteration tree on W producing $i = i_{0\delta}^{\mathcal{T}}$, and $\mathcal{U}$ the iteration tree on M producing j. Since we are below $0^{\mathbb{P}}$, $\mathcal{T}$ and $\mathcal{U}$ are almost linear. Suppose toward contradiction that $\text{crit}(j) < \pi(\gamma)$, where $\gamma < \mu$. Let $\kappa = \text{crit}(j) = \text{crit}(E_0^{\mathcal{U}})$.

Since $i''\text{Def}(W) = \text{Def}(Q) = j''\text{Def}(M)$ by 5.6, we have $i(\gamma) = j(\pi(\gamma))$, and thus $\kappa < \sup(i''\mu)$. But now for any $\eta < \sup(i''\mu)$, either Q has the definability property at η or $\text{crit}(E_\alpha^{\mathcal{T}}) \leq \eta < \nu(E_\alpha^{\mathcal{T}})$ for some $\alpha T \delta$. (If $\eta < i(\xi)$ and the second disjunct fails, we can write $\eta = i(f)(\bar{a})$ where $\bar{a} \in [\eta]^{<\omega}$ and $f : \xi \to \xi$, so that $f \in \text{Def}(W)$ and hence $i(f) \in \text{Def}(Q)$.) Since $\kappa = \text{crit}(j)$, Q does not have the definability property at κ, and thus we can fix $\alpha+1 < lh\ \mathcal{T}$ such that $\text{crit}(E_\alpha^{\mathcal{T}}) \leq \kappa < \nu(E_\alpha^{\mathcal{T}})$.

We claim that $\kappa = \text{crit}(E_\alpha^{\mathcal{T}})$. This is where we make real use of the fact that our mice are below $0^{\mathbb{P}}$. For otherwise, letting $\theta = \text{crit}(E_\alpha^{\mathcal{T}})$, we have that $\mathcal{J}_\kappa^M \models \theta$ is a strong cardinal. This is because there are arbitrarily large $\lambda < \kappa$ such that $E_\alpha^{\mathcal{T}} \restriction \lambda$ is on the M-sequence. (Proof: κ is inaccessible in Q, and hence in $\mathcal{J}_\xi^{\mathcal{M}_\alpha^{\mathcal{T}}}$, where $\xi = lh\ E_\alpha^{\mathcal{T}}$. The initial segment condition gives arbitrarily large $\lambda < \kappa$ such that $E_\alpha^{\mathcal{T}} \restriction \lambda$ is on the $\mathcal{M}_\alpha^{\mathcal{T}}$ sequence. But $\mathcal{J}_\kappa^M = \mathcal{J}_\kappa^{\mathcal{M}_\alpha^{\mathcal{T}}}$.)

Let $i_{\alpha\delta}^{\mathcal{T}} : \mathcal{M}_\alpha^{\mathcal{T}} \to Q$ be the remainder of the iteration. Since $\mathcal{M}_\alpha^{\mathcal{T}}$ has the hull property at κ and $\kappa = \text{crit}(i_{\alpha\delta}^{\mathcal{T}})$, Q has the hull property at κ, and hence so does M. But then $i_{\alpha\delta}^{\mathcal{T}}(A) \cap \nu = j(A) \cap \nu$ for all $A \in P(\kappa)^M$, where $\nu = \inf(\nu(E_\alpha^{\mathcal{T}}), \nu(E_0^{\mathcal{U}}))$. It follows that $E_\alpha^{\mathcal{T}}$ is compatible with $E_0^{\mathcal{U}}$, the usual contradiction. □

Theorem 8.13. *Suppose K^c is below $0^{\mathbb{P}}$; then any universal weasel is a normal iterate of K, and any $j : K \to M$ is the iteration map associated to a normal iteration of K.*

Proof. The second statement actually implies the first, via 8.10, but we give an independent proof. Let $(\mathcal{T},\mathcal{U})$ be the coiteration of K with M. Since K^c is below $0^{\mathbb{P}}$, both $\mathcal{T}$ and $\mathcal{U}$ are normal, "almost linear" iteration trees. We wish to see $lh\ \mathcal{U} = 1$. If not, then $E_0^{\mathcal{U}}$ exists; let $\mu < \Omega$ be inaccessible and such that $lh\ E_0^{\mathcal{U}} < \mu$. Let K^* and M^* come from K and M, respectively, by hitting each total order zero measure above μ. So Ω is A_0-thick in K^* and M^*, and $\mu \subseteq \text{Def}(K^*)$. If $(\mathcal{T}^*,\mathcal{U}^*)$ is the coiteration of K^* with M^*, then $E_0^{\mathcal{U}^*} = E_0^{\mathcal{U}}$. This contradicts Lemma 8.12.

Next, let $j : K \to M$ be elementary. By the Dodd-Jensen lemma, M is universal, and hence there is a normal iteration tree $\mathcal{T}$ on K such that $M = \mathcal{M}_\gamma^\mathcal{T}$, for some $\gamma \leq \Omega$. We wish to show that $j = i_{0\gamma}^\mathcal{T}$. So fix $\eta < \Omega$; we shall show that $j(\eta) = i_{0\gamma}^\mathcal{T}(\eta)$.

Let $\beta < \Omega$ be such that $j(\eta) < \beta$ and $\forall\alpha$ $(\mathrm{crit}(E_\alpha^\mathcal{T}) < \beta \Rightarrow lh\ (E_\alpha^\mathcal{T}) < \beta)$. Let F be the length β extender derived from j, and let $j' : K \to M'$ be the canonical embedding of K into $\mathrm{Ult}(K, F) = M'$. Clearly, M' agrees with M below β, and $j'(\eta) = j(\eta)$. Let $\mathcal{T}'$ be the normal iteration tree on K whose last model $\mathcal{M}_\delta^{\mathcal{T}'} = M'$. The agreement between M and M' implies that $\forall\alpha[(lh(E_\alpha^\mathcal{T}) < \beta \vee lh(E_\alpha^{\mathcal{T}'}) < \beta) \Rightarrow E_\alpha^\mathcal{T} = E_\alpha^{\mathcal{T}'}]$. It follows that if $i_{0\delta}^{\mathcal{T}'}(\eta) = j'(\eta)$, then $i_{0\gamma}^\mathcal{T}(\eta) = j(\eta)$, and we are done.

Let K^* come from K by hitting each total order zero measure with critical point above β once. Let M^* come from M' via the same process, using critical points above $j'(\beta)$. Clearly $j' : K^* \to M^*$, Ω is A_0-thick in K^* and M^*, and $\beta \subseteq \mathrm{Def}(K^*)$. Also, $\{\alpha \mid j'(\alpha) = \alpha\}$ is A_0 thick in K^* and M^*, which is why we switched from j to j'.

Now let $i : K^* \to Q$ and $k : M^* \to Q$ be the iteration maps coming from the coiteration of K^* with M^*. Since K and K^* agree below β, as do M' and M^*, it will be enough to show that $i(\eta) = j'(\eta)$, for then $i(\eta) = i_{0\delta}^{\mathcal{T}'}(\eta)$ and we are done.But since j' has a thick class of fixed points, the proof of 5.6 gives $i''\ \mathrm{Def}(K^*) = \mathrm{Def}(Q) = (k \circ j')''\ \mathrm{Def}(K^*)$. Thus $i(\eta) = k(j'(\eta))$. Since $k \restriction \beta =$ identity by 8.12, $i(\eta) = j'(\eta)$, as desired. □

We have developed the theory of K^c and K under the assumption that there is a measurable cardinal Ω, and so this is a tacit hypothesis in 8.13. The measurable cardinal is not needed for the theory of K "below $0^{\mathbb{P}}$", however. (See [DJKM].) Thus 8.13 does not require this tacit hypothesis.

We now sketch an argument which shows that the hypothesis of 8.13 that K^c is below $0^{\mathbb{P}}$ cannot be substantially weakened. The reason is that the conclusion of 8.13 implies, via work of Jensen and Mitchell, that $K \cap HC$ is Σ_5^1 in the codes. On the other hand, Woodin has shown that it is consistent that K^c is "below two strong cardinals", and yet $K \cap HC$ is not Σ_5^1 in the codes.

We sketch the definition of K "below $0^{\mathbb{P}}$" due to Mitchell and Jensen. Let us call α a closure point of a premouse $\mathcal{M}$ iff α is a limit of $\mathcal{M}$-cardinals and $\forall\beta < \alpha\exists\gamma < \alpha\forall\theta \in (\gamma, \alpha)$ $(\mathcal{J}_\theta^\mathcal{M}$ is active $\Rightarrow \mathrm{crit}(E_\theta^\mathcal{M}) > \beta)$. Let us call a premouse $\mathcal{M}$ α-good just in case $\mathcal{M}$ is iterable, $\rho_\omega(\mathcal{M}) = \alpha$, and either α is a closure point of $\mathcal{M}$ or there is a universal weasel W such that for some β: $\mathcal{M} = \mathcal{J}_\beta^W$, $\rho_\omega(\mathcal{J}_\gamma^W) \geq \alpha$ for all $\gamma \geq \beta$, and $\mathrm{crit}(\dot{E}_\gamma^W) > \beta$ for all $\gamma > \beta$. Clearly, if W witnesses that $\mathcal{M}$ is α-good, then α is a cardinal of W. It can be shown that the relation R is Π_3^1, where $R(x, y) \Leftrightarrow (x, y \in {}^\omega\omega \wedge x$ codes a premouse $\mathcal{M} \wedge y$ codes an ordinal $\alpha \wedge \mathcal{M}$ is α-good). [If $\mathcal{M}, \alpha \in HC$, and α is not a closure point of $\mathcal{M}$, then $\mathcal{M}$ is α-good iff $\forall\mathcal{N} \in HC$ [$\exists\beta$ ($\mathcal{M} = \mathcal{J}_\beta^\mathcal{N} \wedge \forall\gamma > \beta(\mathrm{crit}(\dot{E}_\gamma^\mathcal{N}) > \beta) \wedge$ "$\mathcal{N}$ is iterable via extenders with critical point $> \beta$") $\Rightarrow \mathcal{N}$ is iterable and $\rho_\omega(\mathcal{N}) \geq \alpha$.] This is Π^1_3 in the codes; the antecedent in the bracketed conditional is Π^1_2 because of the iterability assertion. The $\Rightarrow$ direction of this equivalence comes from the coiteration of a weasel W witnessing $\mathcal{M}$ is α-good with a potential counterexample $\mathcal{N}$ to the right hand side. For the $\Leftarrow$ direction, the weasel W which witnesses that $\mathcal{M}$ is α-good is "$K^c(\mathcal{M})$", the result of the construction of §1 modified so that it begins with $\mathcal{N}_0 = \mathcal{M}$ and only uses extenders with critical point $> \mathrm{OR}^{\mathcal{M}}$. We use here that α is not a closure point of $\mathcal{M}$. This, together with the other conditions, implies that $\beta+1$ is contained in every hull formed in the $K^c(\mathcal{M})$ construction, so that this construction does indeed produce the desired W. If α is a closure point, $\alpha \notin$ some hull is possible.]

Notice that if α is a cardinal of K, and $\rho_\omega(\mathcal{J}^K_\beta) = \alpha$, then $\mathcal{J}^K_\beta$ is α-good. This is clear if α is a closure point of K. If α is not a closure point of K, then the weasel W which witnesses that $\mathcal{J}^K_\beta$ is α-good is $\mathrm{Ult}(K, \dot{E}^K_\gamma)$, where γ is least such that $\gamma > \beta$ and $\mathrm{crit}(\dot{E}^K_\gamma) < \beta$, unless there is no such γ, in which case $W = K$ is the witness. (If there is such a γ, then $\mathrm{crit}(\dot{E}^K_\gamma) < \alpha$ for the least such γ. For if $\mathrm{crit}(\dot{E}^K_\gamma) = \alpha$, then α is a limit cardinal of K, so since α is not a closure point, there is a $\kappa < \alpha$ which is the critical point for extenders on the K-sequence with indices unbounded in α. The initial segment condition gives ν such that $\beta < \nu < \gamma$ and $\mathrm{crit}(\dot{E}^{\mathcal{P}}_\nu) = \kappa$, where $\mathcal{P} = \mathrm{Ult}(\mathcal{J}^K_\gamma, \dot{E}^K_\gamma)$ But $\dot{E}^{\mathcal{P}}_\nu = \dot{E}^K_\nu$ by coherence, and this contradicts the minimality of γ. Thus $\mathrm{crit}(\dot{E}^K_\gamma) < \alpha$, and therefore $\mathrm{Ult}(K, \dot{E}^K_\gamma)$ makes sense.)

Suppose that, conversely, whenever α is a cardinal of K, $\mathcal{J}^{\mathcal{M}}_\alpha = \mathcal{J}^K_\alpha$, and $\mathcal{M}$ is α-good, then $\mathcal{M} = \mathcal{J}^K_\beta$ for some β. It would follow that

$$\begin{aligned} \exists\gamma < (\alpha^+)^K(\mathcal{P} &= \mathcal{J}^K_\gamma) \\ \Leftrightarrow \exists\mathcal{M}(\mathcal{J}^{\mathcal{M}}_\alpha &= \mathcal{J}^K_\alpha \wedge \mathcal{M} \text{ is } \alpha\text{-good } \wedge \exists\gamma(\mathcal{P} = \mathcal{J}^{\mathcal{M}}_\gamma)). \end{aligned}$$

This easily implies that the function $\alpha \mapsto \mathcal{J}^K_\alpha$, restricted to HC, is Π^1_4 in the codes, and hence that $K \cap HC$ is Σ^1_5 in the codes. It is easy to see that if $\mathcal{M}$ is α-good, $\mathcal{J}^{\mathcal{M}}_\alpha = \mathcal{J}^K_\alpha$, and α is a closure point of $\mathcal{M}$, then $\mathcal{M} = \mathcal{J}^K_\beta$ for some β. This is because all critical points in the coiteration of $\mathcal{M}$ with K are $\geq \alpha$. So if $K \cap HC$ is not Σ^1_5 in the codes, there is a cardinal α of K which is not a closure point of K, and an α-good $\mathcal{M}$ such that $\mathcal{J}^{\mathcal{M}}_\alpha = \mathcal{J}^K_\alpha$ but $\mathcal{M}$ is not an initial segment of K.

Let $\mathcal{M}$ be α-good, as witnessed by W, and $\mathcal{J}^{\mathcal{M}}_\alpha = \mathcal{J}^K_\alpha$. If W is an iterate of K, then the iteration must use extenders of length $> \alpha$ because $\mathcal{J}^W_\alpha = \mathcal{J}^K_\alpha$, and this implies $\mathcal{M}$ is an initial segment of K because $\rho_\omega(\mathcal{M}) = \alpha$. So if $K \cap HC$ is not Σ^1_5 in the codes, then there is a universal weasel W which is not an iterate of K. By 8.10, there is nevertheless an elementary $j : K \to W$, and of course this j cannot be an iteration map.

We shall not attempt to sketch Woodin's proof that it is consistent that K^c has no Woodin cardinals (and is in fact "below two strong cardinals") and yet $K \cap HC$ is not Σ^1_5 in the codes. See [H].

Although we do not have the decisive characterization of embeddings of K given by 8.13, once we get past $0^{\mathbb{P}}$, we can prove the following consequence of the characterization.

Theorem 8.14. *Suppose $K^c \models$ There are no Woodin cardinals, and let M be $\Omega + 1$ iterable, $j : K \to M$ be elementary, and $\kappa = crit(j)$. Then*

(1) $P(\kappa)^M = P(\kappa)^K$,

(2) the trivial completion of the $(\kappa, \kappa + 1)$ extender derived from j is on the K-sequence,

(3) if $K \models$ "γ is regular but not measurable", then $j(\gamma) = \sup(j''\gamma)$.

Proof. We simply trace through the proofs of 8.12 and 8.13, and see what we get.

Let M, j, κ, and γ be as in the statement of 8.14. Let $\beta < \Omega$ be inaccessible with κ, $\gamma < \beta$. Let F be the length β extender derived from j. Let K^* be the witness that $\mathcal{J}^K_\beta$ is A_0-sound which is obtained from K by hitting each order zero measure above β. Let $M^* = \mathrm{Ult}(K^*, F)$. Since M is universal by Dodd-Jensen, $(\alpha^+)^M = \alpha^+$ for all but nonstationary many $\alpha \in A_0$ by 3.7, and hence Ω is A_0 thick in M^*. Further, $\{\alpha \mid j^*(\alpha) = \alpha\}$ is A_0-thick in K^*, where $j^* : K^* \to M^*$ is the canonical embedding. It will be enough to show that $P(\kappa)^{M^*} = P(\kappa)^{K^*}$, that the $(\kappa, \kappa + 1)$ extender derived from j^* is on the K^* sequence, and that $j^*(\gamma) = \sup(j^{*''}\gamma)$.

Let $i : K^* \to Q$ and $k : M^* \to Q$ be the iteration maps coming from a coiteration of K^* with M^*. We have $\beta \subseteq \mathrm{Def}(K^*)$, and since j^* has a thick class of fixed points, $\kappa \subseteq \mathrm{Def}(M^*)$. Standard arguments then show that $i \restriction \kappa = (k \circ j^*) \restriction \kappa =$ identity, and therefore $P(\kappa)^{K^*} = P(\kappa)^Q = P(\kappa)^{M^*}$. Further, since $\kappa = \mathrm{crit}(k \circ j^*)$, $\kappa \notin \mathrm{Def}(Q)$, and therefore $\kappa = \mathrm{crit}(i)$. Since K^* and M^* have the hull property at κ, if $\mathrm{crit}(k) = \kappa$ then the usual argument shows that the first extenders used along the branches of the iteration trees producing i and k are compatible, which is a contradiction. Therefore $\mathrm{crit}(k) > \kappa$. But then the $(\kappa, \kappa + 1)$ extender derived from i is the same as the $(\kappa, \kappa + 1)$ extender derived from j^*. Since i is an iteration map, the $(\kappa, \kappa + 1)$ extender derived from i is on the K^*-sequence, and we have proved (1) and (2).

Since $K^* \models$ "γ is regular but not measurable", and i is an iteration map, $i(\gamma) = \sup(i''\gamma)$. Since $\beta \subseteq \mathrm{Def}(K^*)$, $i \restriction \beta = (k \circ j^*) \restriction \beta$, and therefore $(k \circ j^*)(\gamma) = \sup((k \circ j^*)''\gamma)$. It follows that $j^*(\gamma) = \sup(j^{*''}\gamma)$, which proves (3). □

Clearly, 8.14 can be pushed a little further, in that more of j is given by an iteration of K than its normal measure part.

We conclude this section by giving another proof that if there is a strongly compact cardinal, then there is an inner model with a Woodin cardinal. The proof we gave in §7 required an excursion into descriptive set theory, whereas this proof does not.

Lemma 8.15. *Suppose $K^c \models$ There are no Woodin cardinals, and let $\mu < \Omega$ be measurable; then $(\mu^+)^K = \mu^+$.*

Proof. Let $j : V \to M$ with $\text{crit}(j) = \mu$. Since μ is measurable, we can prove all the results of §1 - §6 with μ replacing Ω. (Let K^c_μ be K^c as constructed in V_μ. If $K^c_\mu \models$ There is a Woodin cardinal, then there is a model of height Ω having a Woodin cardinal $\delta < \Omega$. As we remarked in §2, this is impossible if $K^c \models$ There are no Woodin cardinals.) Let K_μ be the model constructed in §5, but "below μ". Let $\mathcal{U} = \{X \subseteq \mu \mid \mu \in j(X)\}$ be the normal ultrafilter generated by j. By 5.18, for $\mathcal{U}$ a.e. $\alpha < \mu$, $(\alpha^+)^{K_\mu} = \alpha^+$. But now the results of §6 give an inductive definition of K_μ which is precisely the same as that of $\mathcal{J}^K_\mu$, and therefore $K_\mu = \mathcal{J}^K_\mu$. It follows that $(\alpha^+)^K = \alpha^+$ for $\mathcal{U}$ a.e. $\alpha < \mu$.

But then $(\mu^+)^{K^M} = (\mu^+)^M = \mu^+$. Since $j : K \to K^M$, we have $(\mu^+)^K = (\mu^+)^{K^M}$ by 8.14. So $(\mu^+)^K = \mu^+$, as desired. □

Theorem 8.16. *Let Ω be measurable, and let $\mu < \Omega$ be μ^+-strongly compact; then $K^c \models$ There is a Woodin cardinal.*

Proof. Suppose otherwise. Let $j : V \to M$ come from the ultrapower of V by a fine, μ-complete ultrafilter on $P_\mu(\mu^+)$. It is well known that $j(\mu^+) > \sup(j''\mu^+)$. But $\mu^+ = (\mu^+)^K$ by 8.15, and j is continuous at $(\mu^+)^K$ by (3) of 8.14. This is a contradiction. □