§4. The hull and definability properties

Definition 4.1. *Let $\mathcal{M}$ be a premouse and $X \subseteq \mathcal{M}$. Then*

$$a \in H^{\mathcal{M}}(X) \quad \Leftrightarrow \quad \text{for some } s \in X^{<\omega} \text{ and formula } \varphi, \\ a = \text{unique } v \text{ such that } \mathcal{M} \models \varphi[v, s].$$

Notice here that $H^{\mathcal{M}}(X)$ in the *uncollapsed* hull of X inside $\mathcal{M}$.

Definition 4.2. *Suppose Ω is S-thick in $\mathcal{M}$, and let $\alpha < \Omega$. We say that $\mathcal{M}$ has the S-hull property at α iff whenever Γ is S-thick in $\mathcal{M}$*

$$P(\alpha)^{\mathcal{M}} \subseteq \textit{transitive collapse of} \quad H^{\mathcal{M}}(\alpha \cup \Gamma).$$

In his work on the core model for sequences of measures, Mitchell makes heavy use of a lemma which states (translated into our context) that if Ω is S-thick in $\mathcal{M}$ then $\mathcal{M}$ has the S-hull property at all $\alpha < \Omega$. This will fail as soon as we get past sequences of measures, as the following example shows.

Example 4.3. Suppose Ω is S-thick in $\mathcal{M}$. Let E be an extender from the $\mathcal{M}$ sequence which is total on $\mathcal{M}$, and $\kappa = \text{crit } E$. Suppose E has a generator $> \kappa$, and let ξ be the least such. (So $\kappa^{+^{\mathcal{M}}} < \xi$, and $E \restriction \xi = \dot{E}^{\mathcal{M}}_{\xi}$.) Now let $\mathcal{N} = \text{Ult}_0(\mathcal{M}, E) = \text{Ult}_{\omega}(\mathcal{M}, E)$. Then Ω is S-thick in $\mathcal{N}$. We claim that $\mathcal{N}$ fails to have the hull property at ξ. For let $i : \mathcal{M} \to \mathcal{N}$ be the canonical embedding and $\Gamma = \text{ran } i$. Thus Γ is S-thick in $\mathcal{N}$. Moreover we can factor i as follows:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "i"] \arrow[dr, "j"'] & \text{Ult}_0(\mathcal{M}, E) \\ & \text{Ult}_0(\mathcal{M}, E \restriction \xi) \arrow[u, "k"'] \end{tikzcd}$$

where $k([a, f]) = i(f)(a)$. We have $\xi = \text{crit } k$, and $\text{ran}(k) = H^{\mathcal{N}}(\xi \cup \Gamma)$, and so $Ult(\mathcal{M}, E \restriction \xi)$ is the transitive collapse of $H^{\mathcal{N}}(\xi \cup \Gamma)$. On the other hand, by coherence $\dot{E}^{\mathcal{N}}_{\xi} = \dot{E}^{\mathcal{M}}_{\xi} = E \restriction \xi$, so $E \restriction \xi \in \mathcal{N}$. As $E \restriction \xi$ is essentially a subset of ξ (in fact, of $(\kappa^+)^{\mathcal{N}}$) and $E \restriction \xi \notin \text{Ult}(\mathcal{M}, E \restriction \xi)$, we are done.

Remark. If Ω is S-thick in $\mathcal{M}$, and $\mathcal{P} = \mathcal{M}^{\mathcal{T}}_{\alpha}$ where $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ and there is no dropping along $[0, \alpha]_T$ and $\alpha \leq \Omega$, and $\mathcal{M}$ has the S-hull property at all ξ, then $\mathcal{P}$ has the S-hull property at ξ iff for no $\beta + 1 \in [0, \alpha]_T$ do we have $(\kappa^+)^{\mathcal{M}_{\beta}} \leq \xi < \nu$, where $\nu = \nu(E^{\mathcal{T}}_{\beta})$ and $\kappa = \text{crit}(E^{\mathcal{T}}_{\beta})$. So we can recover from $\mathcal{P}$, using the hull property, the pairs (κ, ν) such that some extender with critical point κ and sup of generators $= \nu > (\kappa^+)^{\mathcal{P}}$ is used on the branch from $\mathcal{M}$ to $\mathcal{P}$. Notice also that $\mathcal{P}$ will have the S-hull property at club many $\xi < \Omega$.

Definition 4.4. *Let Ω be S-thick in $\mathcal{M}$, and $\alpha < \Omega$. We say $\mathcal{M}$ has the S-definability property at α iff whenever Γ is S-thick in $\mathcal{M}$, $\alpha \in H^{\mathcal{M}}(\alpha \cup \Gamma)$.*

Even at the sequences of measures level, it is possible that Ω is S-thick in $\mathcal{M}$, but $\mathcal{M}$ fails to have the S-definability property at some α. For let Ω be S-thick in $\mathcal{P}$, and $\mathcal{M} = \mathrm{Ult}(\mathcal{P},\mathcal{U})$ where $\mathcal{U}$ is total on $\mathcal{P}$ with critical point α.

In view of the previous examples, we cannot expect that K^c will have the A_0-hull or definability properties at all $\alpha < \Omega$. We shall show, however, that K^c has these properties at many $\alpha < \Omega$.

Lemma 4.5. *Let W be an $\Omega + 1$-iterable weasel, and let Ω be S-thick in W; then there is an elementary $\pi : M \to W$ such that ran π is S-thick in W, and M has the S-hull property at all $\alpha < \Omega$.*

Proof. Let us use "thick" to mean "S-thick", and "hull property" for "S-hull property". We shall define by induction on $\alpha \le \Omega$ classes $N_\alpha \prec W$ such that N_α is thick in W. We shall have $N_{\alpha+1} \subseteq N_\alpha$ for all α, and $N_\lambda = \bigcap_{\beta<\lambda} N_\beta$ if λ is a limit. We then take ran π to be N_Ω.

In order to avoid dealing with collapse maps, let us say that a class $N \prec W$ which is thick in W has the hull property at κ, where $\kappa \in N$, iff $\bar{N}$ has the hull property at $\sigma(\kappa)$, where $\sigma : N \cong \bar{N}$ is the transitive collapse. Equivalently, N has the hull property at κ iff whenever $\Gamma \subseteq N$ is thick in W, and $A \subseteq \kappa$ and $A \in N$, then there is a set $B \in H^W((N \cap \kappa) \cup \Gamma)$ such that $B \cap \kappa = A$.

As we define the N_α's we define κ_α for $\alpha < \Omega$. κ_α will be the αth infinite cardinal of N_Ω. We shall have $N_\alpha \cap (\kappa_\alpha + 1) = N_\beta \cap (\kappa_\alpha + 1)$ for all $\beta > \alpha$. We also maintain inductively that N_β has the hull property at κ_α, for all $\beta > \alpha$.

Base step:

$$\begin{aligned} N_0 &= W, \\ \kappa_0 &= \omega . \end{aligned}$$

Limit step:

$$\begin{aligned} N_\lambda &= \bigcap_{\beta<\lambda} N_\beta \\ \kappa_\lambda &= \text{least } \kappa \in N_\lambda \text{ such that} \\ &\quad\; \kappa_\beta < \kappa \text{ for all } \beta < \lambda . \end{aligned}$$

(By induction, κ_β is a cardinal of N_λ for all $\beta < \lambda$. So κ_λ is a cardinal of N_λ.)

Successor step: Suppose we are given N_α and κ_α, where $N_\alpha \models \kappa_\alpha$ is a cardinal. For each $A \subseteq \kappa_\alpha$ such that $A \in N_\alpha$ and A is a counterexample to N_α having the hull property at κ_α, pick a thick class Γ_A witnessing this. Let

$$\Gamma = \bigcap \{\Gamma_A \mid A \subseteq \kappa_\alpha \wedge A \in N_\alpha \wedge \Gamma_A \text{ exists}\} ,$$

where we set $\Gamma = N_\alpha$ if no Γ_A's exist, i.e. if N_α has the hull property at κ_α. Set

$$N_{\alpha+1} = H^W((N_\alpha \cap (\kappa_\alpha + 1)) \cup \Gamma).$$

(Each $\Gamma_A \subseteq N_\alpha$, so $N_{\alpha+1} \subseteq N_\alpha$.)

This finishes the construction. It is clear that if $\alpha < \beta < \Omega$, then $N_\alpha \cap (\kappa_\alpha+1) = N_\beta \cap (\kappa_\alpha+1)$, κ_α is a cardinal of N_β, and N_β has the hull property at κ_α. Moreover, $\langle \kappa_\gamma \mid \gamma \leq \beta \rangle$ is an initial segment of the cardinals of N_β. Moreover, N_β is thick.

Set $N_\Omega = \bigcap_{\alpha<\Omega} N_\alpha$. The assertions of the last paragraph are also obvious for $\beta = \Omega$, except that N_Ω is not obviously thick in W. The following claim is the key to showing this.

Claim. Let $\lambda < \Omega$ be a limit; then N_λ has the hull property at κ_λ, and therefore $N_{\lambda+1} = N_\lambda$.

Proof. Let M be the transitive collapse of N_λ, and κ the image of κ_λ under collapse. So κ is a limit cardinal of M, and M has the hull property at all $\alpha < \kappa$. We want to show that M has the hull property at κ. Notice Ω is thick in M.

Let Γ be thick in M, and $H =$ transitive collapse of $H^M(\kappa \cup \Gamma)$. We are to show $P(\kappa) \cap M \subseteq H$.

Let $\mathcal{T}$ on H and $\mathcal{U}$ on M be the iteration trees resulting from a coiteration of H with M determined by $\Omega + 1$ iteration strategies. (Notice that H and M are $\Omega + 1$-iterable because they are embeddable in W, and by 3.3 the comparison ends as a stage $\leq \Omega$.) Let $lh\ \mathcal{T} = \gamma + 1$ and $lh\ \mathcal{U} = \theta + 1$, where $\gamma, \theta \leq \Omega$ by 3.3.

Since H and M are both universal, $H_\gamma = M_\theta$ (where these are the final models or the two trees), and $i^{\mathcal{T}}_{0,\gamma}$ and $i^{\mathcal{U}}_{0,\theta}$ are both defined.

It is enough to see that crit $i^{\mathcal{U}}_{0,\theta} \geq \kappa$, as then $P(\kappa) \cap M = P(\kappa) \cap M_\theta = P(\kappa) \cap H_\gamma \subseteq H$. So suppose that crit $i^{\mathcal{U}}_{0,\theta} = \mu < \kappa$. Notice $(\mu^+)^M < \kappa$.

Let E be the first extender used along $[0,\theta]_U$; that is, $E = E^{\mathcal{U}}_\eta$ where $\eta + 1 \in [0,\theta]_U$ and $\mathcal{U}$-pred$(\eta + 1) = 0$. So crit $E = \mu$ and $lh\ E \geq \kappa$. The argument of example 4.3 shows that $E \restriction (\mu^+)^M$ witnesses that M_θ doesn't have the hull property at $(\mu^+)^M = (\mu^+)^{M_\theta}$. On the other hand, M and hence M_θ has the hull property at all ordinals $< (\mu^+)^M$.

If crit $i^{\mathcal{T}}_{0,\gamma} \geq (\mu^+)^M = (\mu^+)^H$, then $H_\theta = M_\theta$ has the hull property at $(\mu^+)^M$. Thus crit $i^{\mathcal{T}}_{0,\gamma} =$ crit $i^{\mathcal{U}}_{0,\theta} = \mu$.

Now let $A \subseteq \mu$ and $A \in M$. Let $\Gamma = \{\alpha \mid i^{\mathcal{T}}_{0,\gamma}(\alpha) = i^{\mathcal{U}}_{0,\theta}(\alpha) = \alpha\}$. By 3.9 and 3.11, Γ is thick (in H, M, and $H_\gamma = M_\theta$). So we can find a term τ such that

$$A = \tau^M[\bar{\beta}, \bar{c}] \cap \mu$$

where $\bar{\beta} \in \mu^{<\omega}$ and $\bar{c} \in \Gamma^{<\omega}$, using the hull property at μ in M. But then

$$i^{\mathcal{U}}_{0,\theta}(A) = \tau^{M_\theta}[\bar{\beta}, \bar{c}] \cap i^{\mathcal{U}}_{0,\theta}(\mu).$$

Now $\tau^H[\bar{\beta}, \bar{c}] \cap \mu = \tau^{H_\gamma}[\bar{\beta}, \bar{c}] \cap \mu = i^{\mathcal{U}}_{0,\theta}(A) \cap \mu = A$. Thus

$$i^{\mathcal{T}}_{0,\gamma}(A) = \tau^{H_\gamma}[\bar{\beta}, \bar{c}] \cap i^{\mathcal{T}}_{0,\gamma}(\mu).$$

It follows that the 1st extenders used along $[0,\theta]_U$ and $[0,\gamma]_T$ agree up to the inf of the sups of their generators. This is a contradiction, as in the proof of the comparison lemma. This proves the claim. □

The claim implies N_Ω is thick in W. For by Födor's theorem, for all but nonstationary many $\alpha \in S$, $\alpha = \kappa_\alpha$ and for all $\beta < \alpha$ there is an α-club $C_\beta \subseteq N_\beta \cap \alpha^+$. Fix such an α. Let $C = \bigcap_{\beta<\alpha} C_\beta$; then C is α-club and $C \subseteq N_\alpha \cap \alpha^+$. But the claim tells us $N_\alpha = N_{\alpha+1}$, and hence $\kappa_{\alpha+1} = \alpha^+$. Since $N_{\alpha+1} \cap (\kappa_{\alpha+1}+1) = N_\Omega \cap (\kappa_{\alpha+1}+1)$, $C \subseteq N_\Omega \cap \alpha^+$.

This completes the proof of 4.5. □

We note in passing that the proof of the claim in 4.5 almost shows that if Ω is S-thick in M, then $\{\alpha < \Omega \mid M$ has the S-hull property at $\alpha\}$ is closed in Ω. It falls a bit short, however, and we do not know whether this is in fact true.

Lemma 4.6. *Let W be an $\Omega+1$ iterable weasel such that Ω is S-thick in W. Then for μ_0- a.e. $\alpha < \Omega$, W has the S-hull property at α.*

Proof. Let M be as given by lemma 4.5. Let $(\mathcal{T},\mathcal{U})$ be a coiteration of M with W determined by $\Omega+1$ iteration strategies. We suppose $lh\ \mathcal{T} = lh\ \mathcal{U} = \Omega+1$, the contrary case being very similar and left to the reader.

As both M and W are universal, there is no dropping on either $[0,\Omega]_T$ or $[0,\Omega]_U$, and $M_\Omega = W_\Omega$ (where these are the last models of $\mathcal{T}$ and $\mathcal{U}$ respectively). Let α be such that α is inaccessible, α is a limit point of $[0,\Omega]_T$ and $[0,\Omega]_U$, and $\forall \beta < \alpha$ ($lh\ E^{\mathcal{T}}_\beta < \alpha$ and $lh\ E^{\mathcal{U}}_\beta < \alpha$). Since branches of an iteration tree must be closed below their sup, all but nonstationary many inaccessible $\alpha < \Omega$ have these properties. Notice that $i^{\mathcal{T}}_{0,\alpha}(\alpha) = i^{\mathcal{U}}_{0,\alpha}(\alpha) = \alpha$, and that $\mathrm{crit}(i^{\mathcal{T}}_{\alpha,\Omega}) \geq \alpha$ and $\mathrm{crit}(i^{\mathcal{U}}_{\alpha,\Omega}) \geq \alpha$.

One can easily show that for any $\beta \in [0,\alpha]_T$, M_β has the hull property at η whenever $\sup\{lh\ E^{\mathcal{T}}_\gamma \mid \gamma+1 \in [0,\beta]_T\} \leq \eta$. (Proof: let Γ be thick in M_β and let $A \subseteq \eta$, $A \in M_\beta$. Let $\eta^* \leq \eta$ be least such that $\eta \leq i^{\mathcal{T}}_{0\beta}(\eta^*)$. There is a function $f \in M$, $f : [\eta^*]^{<\omega} \times \eta^* \to \{0,1\}$, and an $a \in [\eta]^{<\omega}$, such that the characteristic function χ_A of A is given by: for $\xi < \eta$, $\chi_A(\xi) = i^{\mathcal{T}}_{0\beta}(f)(a,\xi)$. By the hull property in M we can find $\bar{\xi} \in \Gamma^{<\omega}$ such that $i^{\mathcal{T}}_{0\beta}(\bar{\xi}) = \bar{\xi}$, $b \in [\eta^*]^{<\omega}$, and a term τ such that $f = \tau^M[b,\bar{\xi}] \restriction ([\eta^*]^{<\omega} \times \eta^*)$. So $i^{\mathcal{T}}_{0\beta}(f) = \tau^{M_\beta}[i^{\mathcal{T}}_{0\beta}(b),\bar{\xi}] \restriction ([i^{\mathcal{T}}_{0\beta}(\eta^*)]^{<\omega} \times [i_{0\beta}(\eta^*)]$. So for $\gamma < \eta$, $\chi_A(\gamma) = \tau^{M_\beta}[i^{\mathcal{T}}_{0\beta}(b),\bar{\xi}](a,\gamma)$. Since $i^{\mathcal{T}}_{0\beta}(b) \in [\eta]^{<\omega}$ by the leastness of η^*, A is in the collapse of $H^{M_\beta}(\eta \cup \Gamma)$.)

Thus M_α has the hull property at α. Since $\alpha \leq \mathrm{crit}\ i^{\mathcal{T}}_{\alpha,\Omega}$, $M_\Omega = W_\Omega$ has the hull property at α. Since $\alpha \leq \mathrm{crit}\ i^{\mathcal{U}}_{\alpha,\Omega}$, W_α has the hull property at α.

Now let Γ be thick in W and $A \subseteq \alpha$, $A \in W$. We can find $\bar{\xi} \in \Gamma^{<\omega}$ and $b \in \alpha^{<\omega}$ such that $i^{\mathcal{U}}_{0\alpha}(\bar{\xi}) = \bar{\xi}$ and

$$i^{\mathcal{U}}_{0\alpha}(A) = \tau^{W_\alpha}[b, \bar{\xi}] \cap \alpha$$

for some term τ. Letting b be least which works for $\bar{\xi}$, $i^{\mathcal{U}}_{0\alpha}(A)$, and $\alpha = i^{\mathcal{U}}_{0\alpha}(\alpha)$, b is definable from elements of ran $i^{\mathcal{U}}_{0\alpha}$, so $b = i^{\mathcal{U}}_{0\alpha}(c)$ where $c \in \alpha^{<\omega}$. Then $A = \tau^W[c, \bar{\xi}] \cap \alpha$, as desired.

Thus W has the hull property at all but nonstationary many inaccessible $\alpha < \Omega$. □

Corollary 4.7. *Suppose $K^c \models$ there are no Woodin cardinals; then K^c has the A_0-hull property at μ_0- a.e. $\alpha < \Omega$.*

Proof. This is immediate from 2.12, 3.12, and 4.6. □

One should not expect that 4.6 will hold in full generality for the definability property. For suppose that for μ_0-a.e. $\alpha < \Omega$, α is measurable in K^c. Let W be the iterate of K^c obtained by using one total-on-K^c order zero measure from each measurable cardinal of K^c once. Then Ω is A_0-thick in W, and W is $\Omega + 1$ iterable, but W does not have the A_0-definability property at μ_0-a.e. α. Nevertheless, one can get a positive result in the case $W = K^c$, and this result will be important in the construction of "true K".

Lemma 4.8. *Suppose $K^c \models$ there are no Woodin cardinals; then for μ_0-a.e. $\alpha < \Omega$, K^c has the A_0-definability property at α.*

Proof. Assume the lemma fails, and for μ_0-a.e. α pick Γ_α thick in K^c such that

$$\alpha \notin H^{K^c}(\alpha \cup \Gamma_\alpha)\,.$$

We can also arrange that $\alpha < \beta \Rightarrow \Gamma_\alpha \supseteq \Gamma_\beta$.

Let $V_1 = \mathrm{Ult}(V, \mu_0)$, and $j : V \to V_1$ be the canonical embedding. Let $V_2 = \mathrm{Ult}(V_1, j(\mu_0))$, and $j_1 : V_1 \to V_2$ the canonical embedding. Let $\Omega_1 = j(\Omega)$ and $\Omega_2 = j_1(\Omega_1)$. Let $K_1 = j(K^c)$ and $K_2 = j_1(K_1)$.

In V_2, we consider the map

$$\pi : H \cong H^{K_2}(\Omega \cup (\Gamma_\Omega)^{V_2}) \prec K_2$$

which inverts the collapse. Since $\Omega \notin H^{K_2}(\Omega \cup \Gamma^{V_2}_\Omega)$, crit $\pi = \Omega$. Since K_2 is satisfied to have the hull property at Ω in V_2, $P(\Omega)^{K_2} \subseteq H$. Let E_π be the length $\pi(\Omega)$ extender derived from π. So $E_\pi \in V_2$, and measures all sets in $P(\Omega)^{K_2}$. Not every $E_\pi \restriction \nu$, $\nu \prec \pi(\Omega)$, belongs to K_2, as otherwise Ω is Shelah in K_2.

Claim. $E_\pi = E_j \cap ([\pi(\Omega)]^{<\omega} \times P(\Omega)^{K_2})$, where E_j is the extender derived from j.

Granted this claim, we can just repeat the proof of the main claim in the proof of Theorem 1.4 to get a contradiction. The point is that V_2 has suitable "background certificates" for the relevant fragments of E_j, so working in V_2 we get that every $E_\pi \restriction \nu$ is "on" the K_2 sequence in the right sense of "on". (1-smallness in no barrier to putting them on, as $K_2 \models$ there are no Woodins.)

Aside. Why isn't this an outright contradiction? We don't get $E_j \cap ([j(\Omega)]^{<\omega} \times P(\Omega)^{K_2})$ as member of V_2 without our false hypotheses.

Proof of Claim. Let $A \subseteq \Omega$ and $A \in K_2$. We must show that $\pi(A) = j(A) \cap \pi(\Omega)$. (It is easy to see that $\Gamma_\Omega^{V_2} \cap \Omega_1 \neq \emptyset$, and therefore $\pi(\Omega) < \Omega_1$.)

By the hull property for K_1 at Ω in V_1, we can find a term τ such that

$$A = \tau^{K_1}[\bar{c}, \bar{d}] \cap \Omega\,,$$

for some $\bar{c} \in \Omega^{<\omega}$ and $\bar{d} \in (\Gamma_\Omega^{V_1})^{<\omega}$. It follows that

$$j(A) = \tau^{K_2}[\bar{c}, j(\bar{d})] \cap \Omega_1\,.$$

Here we use that $j \circ j = j_1 \circ j$, so that $j(K_1) = K_2$. This also implies that $j(\Gamma^{V_1}) = \Gamma^{V_2}$, so that $j(\Gamma_\Omega^{V_1}) = \Gamma_{\Omega_1}^{V_2}$. Thus $j(\bar{d}) \in (\Gamma_{\Omega_1}^{V_2})^{<\omega}$.

On the other hand $\Gamma_{\Omega_1}^{V_2} \subseteq \Gamma_\Omega^{V_2}$, and $j(A) \cap \Omega = A$, so

$$A = \tau^{K_2}[\bar{c}, j(\bar{d})] \cap \Omega\,,$$

where $\bar{c} \in \Omega^{<\omega}$ and $j(\bar{d}) \in (\Gamma_\Omega^{V_2})^{<\omega}$. Moreover, from the definition of π,

$$\pi(A) = \tau^{K_2}[\bar{c}, j(\bar{d})] \cap \pi(\Omega)\,.$$

As $\pi(\Omega) < \Omega_1$, $\pi(A) = j(A) \cap \pi(\Omega)$, as desired. □